# Analytic Number Theory Class 3

Reminder: Final class: Wednesday $27^{th}$ (Zoe)

Deadline: Sat $23^{rd}$.

**SECOND PUBLIC EXAMINATION**

**Honour School of Mathematics Part C: Paper C3.8**
**Honour School of Mathematics and Computer Science Part C: Paper C3.8**
**Honour School of Mathematics and Philosophy Part C: Paper C3.8**

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# ANALYTIC NUMBER THEORY

## Trinity Term 2018

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**MONDAY, 4 JUNE 2018, 2.30pm to 4.15pm**

*You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.*

*You should ensure that you:*

- *start a new answer booklet for each question which you attempt.*
- *indicate on the front page of the answer booklet which question you have attempted in that booklet.*
- *cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.*
- *hand in your answers in numerical order.*

*If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.*

**Do not turn this page until you are told that you may do so**

3. (a) [2 marks] Show that

$$\zeta(\sigma) \leqslant 1 + \frac{1}{\sigma - 1}$$

for all real $\sigma > 1$.

(b) [5 marks] Show that if $\Re s > 1$ then

$$\frac{\zeta(s)^4}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^s},$$

where $\tau(n)$ denotes the number of divisors of $n$. [*You need not rigorously justify the convergence of the series, or any rearrangements you make.*]

(c) [6 marks] By choosing $s = 1 + \frac{1}{\log X}$, or otherwise, show that

$$\sum_{n \leqslant X} \tau(n)^2 \ll X \log^4 X$$

uniformly for $X \geqslant 2$.

(d) [3 marks] Let $W : \mathbb{R} \to \mathbb{R}$ be smooth with compact support in $[1, 10]$ and $\int_{\mathbb{R}} W dx = 1$. Let $\sigma > 1$. Show that

$$\sum_{n} \tau(n)^2 W(n/X) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\zeta(s)^4}{\zeta(2s)} \tilde{W}(s) X^s ds.$$

[*Results about the Mellin transform may be stated without proof, but they should be clearly stated*]

(e) [6 marks] Show that the residue of $\zeta(s)^4 X^s \tilde{W}(s)/\zeta(2s)$ at $s = 1$ is $cX \log^3 X + O(X \log^2 X)$, where $c$ is a constant you should specify. [*You may use the fact that* $\zeta(2) = \pi^2/6$.]

(f) [3 marks] Very briefly, and without details, explain why one expects that, as $X \to \infty$,

$$\sum_{n} \tau(n)^2 W(n/X) \sim cX \log^3 X.$$

3. (a) [2 marks] Show that

$$\zeta(\sigma) \leqslant 1 + \frac{1}{\sigma - 1}$$

for all real $\sigma > 1$.

for $\mathrm{Re}(s) > 1$

$$|\zeta(s)| = \left|\sum_{n=1}^{\infty} \frac{1}{n^s}\right| \leqslant \sum_{n=1}^{\infty} \frac{1}{n^{\mathrm{Re}(s)}}$$

$$\leqslant 1 + \int_1^{\infty} \frac{dt}{t^{\mathrm{Re}(s)}} \qquad \text{since } \frac{1}{n^\sigma} \leqslant \int_{n-1}^{n} \frac{dt}{t^\sigma}$$

$$= 1 + \frac{1}{\sigma - 1}.$$

(b) [5 marks] Show that if $\Re s > 1$ then

$$\frac{\zeta(s)^4}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^s},$$

where $\tau(n)$ denotes the number of divisors of $n$. *[You need not rigorously justify the convergence of the series, or any rearrangements you make.]*

$$\frac{\zeta(s)^4}{\zeta(2s)} = \prod_p \frac{(1-p^{-2s})}{(1-p^{-s})^4} = \prod_p \frac{(1+p^{-s})}{(1-p^{-s})^3} \quad (\text{for } \mathrm{Re}(s)>1.)$$

$$\sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^s} = \prod_p \left( \sum_{j=0}^{\infty} \frac{\tau(p^j)^2}{p^{js}} \right) = \prod_p \left( \sum_{j=0}^{\infty} \frac{(j+1)^2}{p^{js}} \right) \quad (\tau^2 \text{ multiplicative})$$

$\therefore$ Want to show $\displaystyle\sum_{j=0}^{\infty} \frac{(j+1)^2}{p^{js}} = \frac{1+p^{-s}}{(1-p^{-s})^3}$.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \Rightarrow \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2} \quad \left(\text{Diff w.r.t. } x \text{ and multiply by } x\right).$$

$$\Rightarrow \sum_{n=0}^{\infty} n^2 x^n = \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3} = \frac{x(1+x)}{(1-x)^3}.$$

$$\therefore \sum_{j=0}^{\infty} \frac{(j+1)^2}{p^{js}} = \frac{1+p^{-s}}{(1-p^{-s})^3}. \qquad \square.$$

(c) [6 marks] By choosing $s = 1 + \frac{1}{\log X}$, or otherwise, show that

$$\sum_{n \leqslant X} \tau(n)^2 \ll X \log^4 X$$

uniformly for $X \geqslant 2$.

Note $n \leqslant X \Rightarrow \frac{1}{n^{1+1/\log x}} \geqslant \frac{1}{e x}$.

$$\therefore \quad \frac{1}{eX} \sum_{n \leqslant X} \tau(n)^2 \leqslant \sum_{n \leqslant X} \frac{\tau(n)^2}{n^s} \leqslant \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^s}$$

$$= \frac{\zeta(s)^4}{\zeta(2s)} \qquad \text{(by b)}$$

$$\leqslant \frac{(1 + \log x)^4}{\zeta(2 + 2/\log x)} \qquad \text{(by a)}.$$

but $\zeta(2 + 2/\log x) \geqslant \zeta(3) \gg 1$.

$$\therefore \quad \sum_{n \leqslant X} \tau(n)^2 \ll X (\log x)^4 \quad \text{for } x \geqslant 2.$$

(d) [3 marks] Let $W : \mathbb{R} \to \mathbb{R}$ be smooth with compact support in $[1, 10]$ and $\int_{\mathbb{R}} W dx = 1$. Let $\sigma > 1$. Show that

$$\sum_n \tau(n)^2 W(n/X) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\zeta(s)^4}{\zeta(2s)} \tilde{W}(s) X^s ds.$$

*[Results about the Mellin transform may be stated without proof, but they should be clearly stated]*

$$W(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \tilde{W}(s) x^{-s} ds.$$

$$\therefore \quad \sum_{n \geq 1} \tau(n)^2 W\left(\frac{n}{X}\right) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\sum_{n \geq 1} \tau(n)^2 n^{-s}\right) \tilde{W}(s) X^s ds$$

(since converges absolutely).

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\zeta(s)^4}{\zeta(2s)} \tilde{W}(s) X^s ds.$$

(e) [6 marks] Show that the residue of $\zeta(s)^4 X^s \tilde{W}(s)/\zeta(2s)$ at $s = 1$ is $cX \log^3 X + O(X \log^2 X)$, where $c$ is a constant you should specify. [*You may use the fact that* $\zeta(2) = \pi^2/6$.]

(f) [3 marks] Very briefly, and without details, explain why one expects that, as $X \to \infty$,

$$\sum_n \tau(n)^2 W(n/X) \sim cX \log^3 X.$$

e): Note: $\frac{\zeta(s)^6 X^s \tilde{W}(s)}{\zeta(2s)}$ has a pole of order 4 at $s=1$

since $\zeta(s)$ has a simple pole
$\tilde{W}(s), \zeta(2s), X^s$ analytic at $s=1$.
(no zero/pole).

Recall: Residue is the coefficient of $\frac{1}{s-1}$ in Laurent expansion around $s=1$.

$\zeta(2s) = \zeta(2) + 2(s-1)\zeta'(2) + \dots$

$\zeta(s) = \frac{1}{s-1} + c_0 + c_1(s-1) + \dots$

$X^{s-1} = 1 + (s-1)\log X + \frac{(s-1)^2}{2}(\log X)^2 + \frac{(s-1)^3}{6}(\log X)^3 + \dots$

$\tilde{W}(s) = \tilde{W}(1) + (s-1)\tilde{W}'(1) + \frac{(s-1)^2}{2}\tilde{W}''(1) + \dots$

Note: all these terms are constants except $\log X$ factors.

$\therefore$ Coefficient of $\frac{1}{s-1}$ in $\frac{\zeta(s)^4}{\zeta(2s)} X\, X^{s-1}\tilde{W}(s)$

is $X \times$ Polynomial of degree 3 in $\log X$.

Leading term is $\frac{1}{(s-1)^4} \cdot \frac{(s-1)^3}{6}(\log X)^3 \cdot \tilde{W}(1) \cdot X = \frac{1}{s-1} \frac{X\tilde{W}(1)}{6\zeta(2)}(\log X)^3$

$\therefore \operatorname{Res}_{s=1} \frac{\zeta(s)^4 \tilde{W}(s) X^s}{\zeta(2s)} = X\frac{\tilde{W}(1)(\log X)^3}{6\zeta(2)} + O((\log X)^2 X)$.

$\tilde{W}(1) = \int_{-\infty}^{\infty} W(t)\,dt = 1$, $\zeta(2) = \pi^2/6$ $\therefore c = \frac{1}{\pi^2}$.

f) $\sum_n \tau(n)^2 W\left(\frac{n}{X}\right) = \int_{c-i\infty}^{c+i\infty} \tilde{W}(s) X^s \frac{\zeta(s)^4}{\zeta(2s)} ds$

Truncate integral to $|\mathrm{Im}(s)| \leq T$.

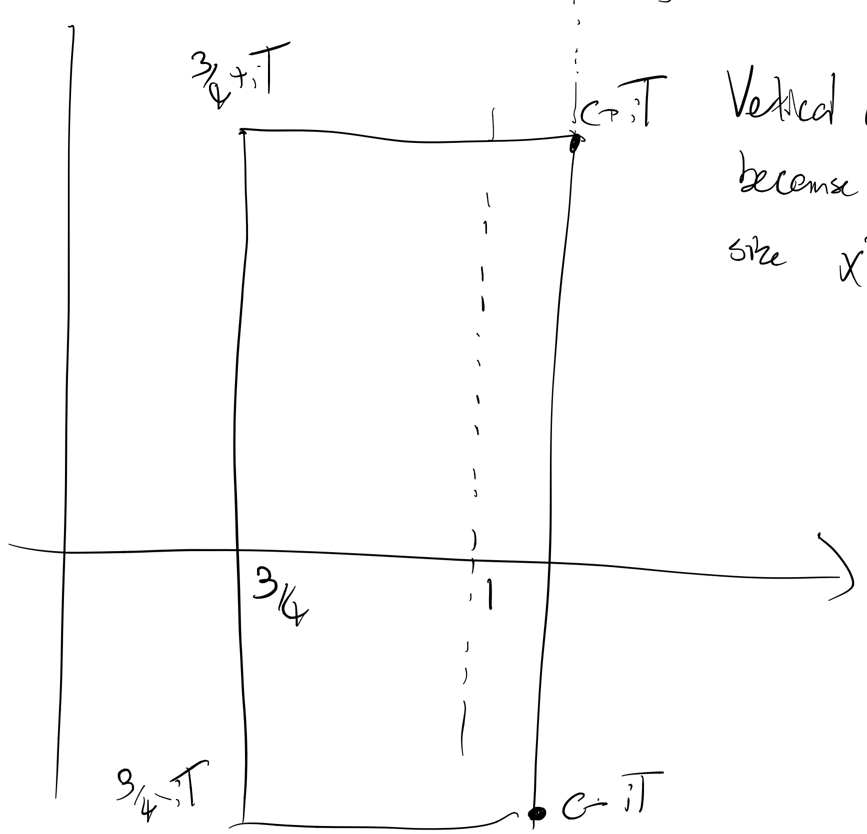
Move line of integration to $\mathrm{Re}(s) = 3/4$.

Pick up a pole at $s=1$,

contributes $\mathrm{Res}_{s=1} = (1+o(1)) \frac{X (\log X)^3}{\pi^2}$

Need to show horizontal / vertical contours are small.

Horizontal contours small because $\tilde{W}(s)$ decays rapidly with $|\mathrm{Im}(s)|$ large.




Vertical contour small because $X^s$ is of size $X^{3/4}$ (small).

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**SECOND PUBLIC EXAMINATION**

**Honour School of Mathematics Part C: Paper C3.8**
**Honour School of Mathematic and Philosophy Part C: Paper C3.8**
**Honour School of Mathematics and Statistics Part C: Paper C3.8**

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# ANALYTIC NUMBER THEORY

## Trinity Term 2017

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**MONDAY, 5 JUNE 2017, 9.30am to 11.15am**

*You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.*

*You should ensure that you:*

- *start a new answer booklet for each question which you attempt.*
- *indicate on the front page of the answer booklet which question you have attempted in that booklet.*
- *cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.*
- *hand in your answers in numerical order.*

*If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.*

**Do not turn this page until you are told that you may do so**

2. *You may assume that $\zeta$ extends to a meromorphic function, holomorphic except for a simple pole at 1, and not vanishing at 0. You may assume the Euler product for $\zeta$ and that $\sum |\rho|^{-2}$ is finite, where the sum is over all zeros $\rho$, trivial and non-trivial, of $\zeta$. You may use bounds on the Mellin transform of smooth functions without proof, provided they are stated correctly.*

   (a) [2 marks] State the partial fraction expansion for $\zeta'/\zeta$.

   (b) [10 marks] Show that $\zeta'(s)/\zeta(s) = O(|s|^3)$ uniformly for all $s \in \mathbb{C}$ with $|s| \geqslant 2$ and at distance at least 1 from any zero of $\zeta$.

   (c) [3 marks] Show that this bound may be improved to $O(1)$ uniformly for $\Re s \geqslant 2$.

   (d) [10 marks] Show that $\zeta''(s)/\zeta(s) = O(1)$ uniformly for $\Re s \geqslant 2$.

3. *You may assume that $\zeta$ extends to a meromorphic function, holomorphic except for a simple pole at 1, and not vanishing at 0. You may assume the Euler product for $\zeta$ and that $\sum |\rho|^{-2}$ is finite, where the sum is over all zeros $\rho$, trivial and non-trivial, of $\zeta$. You may use bounds on the Mellin transform of smooth functions without proof, provided they are stated correctly.*

   (a) [2 marks] State the explicit formula, giving an expression for $\sum_n \Lambda(n)W(\frac{n}{X})$ where $W$ is a smooth, compactly support function supported on $[c,\infty)$.

   (b) [10 marks] Show that $\zeta(s)$ does not vanish on the line $\Re s = 1$.

   (c) [13 marks] Suppose we know that $\zeta(s)$ does not vanish on or to the right of the line $\Re s = \alpha$, for some $\alpha \in (\frac{1}{2}, 1]$. Show that

$$\sum_n \Lambda(n)W\left(\frac{n}{X}\right) = X\int W + o(X^\alpha)$$

   as $X \to \infty$.

(a) [2 marks] State the partial fraction expansion for $\zeta'/\zeta$.

(b) [10 marks] Show that $\zeta'(s)/\zeta(s) = O(|s|^3)$ uniformly for all $s \in \mathbb{C}$ with $|s| \geqslant 2$ and at distance at least 1 from any zero of $\zeta$.

a) For us: Partial fraction expansion: For $Re(s) > -\frac{1}{4}$:

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + \sum_{|\rho - s| \leqslant \frac{1}{100}} \frac{1}{s-\rho} + O(\log(2+|t|)).$$

b) For $Re(s) > -\frac{1}{4}$ this is trivial (no terms in the sum) by the above.

(This was bookwork for old syllabus).

(c) [3 marks] Show that this bound may be improved to $O(1)$ uniformly for $\Re s \geqslant 2$.

(d) [10 marks] Show that $\zeta''(s)/\zeta(s) = O(1)$ uniformly for $\Re s \geqslant 2$.

c) $\mathrm{Re}(s) > 1$ we have

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

$$\therefore \left|\frac{\zeta'}{\zeta}(s)\right| \leqslant \sum_{n=1}^{\infty} \frac{\log n}{n^{\mathrm{Re}(s)}}$$

so for $\mathrm{Re}(s) \geqslant 2$ $\quad \left|\frac{\zeta'}{\zeta}(s)\right| \leqslant \sum_{n=1}^{\infty} \frac{\log n}{n^2} \leqslant \int_1^{\infty} \frac{\log t}{t^2} dt \ll 1.$

d) For $\mathrm{Re}(s) > 1$ we have

$$\zeta''(s) = \sum_{n=1}^{\infty} \frac{(\log n)^2}{n^s}.$$

(Can be proved as in lectures doing $\zeta'(s)$).

$\therefore$ For $\mathrm{Re}(s) \geqslant 2$ $\quad |\zeta''(s)| \leqslant \sum_{n=1}^{\infty} \frac{(\log n)^2}{n^2} \leqslant \int_1^{\infty} \frac{(\log t)^2}{t^2} dt \ll 1.$

$$|\zeta(s)| = \left|\prod_p \left(1 - \frac{1}{p^s}\right)^{-1}\right|$$

$$\geqslant \prod_p \left(1 + \frac{1}{p^2}\right)^{-1} = \exp\left(-\sum_p \log\left(1 + \frac{1}{p^2}\right)\right)$$

but $\log\left(1+\frac{1}{p^2}\right) \ll \frac{1}{p^2}$ and $\sum \frac{1}{p^2} \ll 1.$

so $\exp\left(-\sum_p \log\left(1+\frac{1}{p^2}\right)\right) \gg 1.$

3. *You may assume that $\zeta$ extends to a meromorphic function, holomorphic except for a simple pole at 1, and not vanishing at 0. You may assume the Euler product for $\zeta$ and that $\sum |\rho|^{-2}$ is finite, where the sum is over all zeros $\rho$, trivial and non-trivial, of $\zeta$. You may use bounds on the Mellin transform of smooth functions without proof, provided they are stated correctly.*

   (a) [2 marks] State the explicit formula, giving an expression for $\sum_n \Lambda(n) W(\frac{n}{X})$ where $W$ is a smooth, compactly support function supported on $[c, \infty)$.

   (b) [10 marks] Show that $\zeta(s)$ does not vanish on the line $\Re s = 1$.

a) Our explicit formula is: for $2 \leq T \leq x$

$$\sum_{n \leq x} \Lambda(n) = x - \sum_{|\rho| \leq T} \frac{x^\rho}{\rho} + O\left( \frac{x (\log x)^3}{T} \right)$$

The version used for this version of the course was

$$\sum \Lambda(n) W\left(\frac{n}{x}\right) = x \int_0^\infty W(t)\,dt - \sum_\rho \tilde{W}(\rho) x^\rho \quad \text{for } X \geq \frac{1}{c}$$

where $\tilde{W}(s) := \int_0^\infty W(t) t^{s-1}\,dt$, and $W$ is as given by the question.

b) This is bookwork. (In our version of the course this would be quite a long bookwork question, so would likely give you the inequality

$$4 \operatorname{Re}\left( \frac{\zeta'}{\zeta}(\sigma + it) \right) \leq -\operatorname{Re}\left( \frac{\zeta'}{\zeta}(\sigma + 2it) \right) - 3 \frac{\zeta'}{\zeta}(\sigma),$$

and ask you to go from there)

(c) [13 marks] Suppose we know that $\zeta(s)$ does not vanish on or to the right of the line $\Re s = \alpha$, for some $\alpha \in (\frac{1}{2}, 1]$. Show that

$$\sum_n \Lambda(n) W\left(\frac{n}{X}\right) = X \int W + o(X^\alpha)$$

as $X \to \infty$.

This is similar to a question from the sheets.

$$\sum_n \Lambda(n) W\left(\frac{n}{X}\right) = X \int W - \sum_\rho \tilde{W}(\rho) X^\rho \quad \text{(explicit formula)}$$

$$|\tilde{W}(s)| \ll \frac{1}{|s|^2}$$ (Fact about Mellin transforms of smooth functions - by integration by parts $\tilde{W}(s) = \int_0^\infty \frac{W''(t) t^{s+1} dt}{s(s+1)}$ )

Split the sum into $|\rho| < T$ and $|\rho| > T$, for some $T$ which w

For $|\rho| < T$ the fact $Re(\rho) < \alpha \ \forall \rho$ means that there is an $\varepsilon_T > 0$ depending only on $T$

s.t. $|X^\rho| \leq X^{\alpha - \varepsilon_T}$

$\therefore$ These contribute $\leq X^{\alpha - \varepsilon_T} \sum_{|\rho| \leq T} |\tilde{W}(\rho)|$

$\ll X^{\alpha - \varepsilon}$ (since $\sum_\rho \frac{1}{|\rho|^2}$ converges and $|\tilde{W}(s)| \leq \frac{1}{|s|^2}$).

For $|\rho| > T$ we see that they contribute

$$\leq X^\alpha \sum_{|\rho| > T} |\tilde{W}(\rho)| \ll X^\alpha \sum_{|\rho| > T} \frac{1}{|\rho|^2}$$

$= o_{T \to \infty}(X^\alpha)$ as $\sum_\rho \frac{1}{|\rho|^2}$ converges so

$\therefore \sum_{|\rho| > T} \frac{1}{|\rho|^2} \to 0$ as $T \to \infty$.

$$\therefore \quad \sum \Lambda(n) W\left(\frac{n}{X}\right) = X \int W + O\left(X^{\alpha - \varepsilon_T}\right) + o_{T \to \infty}(X^{\alpha}).$$

Given any $\delta > 0$, can choose $T$ large enough s.t. $o_{T \to \infty}(X^{\alpha}) \leq \delta X^{\alpha}$

and can then choose $X$ large enough s.t. $X^{\alpha - \varepsilon_T} \leq \delta X^{\alpha}$

$\therefore$ for any $\delta > 0$ error term is $\leq 2\delta X^{\alpha}$ for $X$ large enough

$\therefore \quad = o(X^{\alpha})$.

(This was quite similar to something that they covered in lectures in 2017, so don't worry if this seems difficult/new).