Analytic Number Theory Class 4

- Today: 2017Q1, 2015Q3, 2014Q3(b,c)
- Did you receive solutions of the 2019 paper?
- Email wangr@maths.ox.ac.uk if there are any queries related to the notes.

2017, Q1
(a) $D_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$.
(b) [16] Let $f(n) = (-1)^{n+1}$.
Show that $D_f(s)$ defines a holomorphic function for $\Re s > 0$.
Give an expression for $D_f(s)$ in terms of the Riemann ζ-function, valid when $\Re s > 1$.
Hence, or otherwise, prove that $\zeta(s)$ extends to a meromorphic function on $\Re s > 0$ and that it has no real zeros on the segment $(0, 1)$.
(c) [8] Now let f be the multiplicative function for which $f(2) = -2$, $f(p) = 1$ when p is an odd prime and $f(p^j) = 0$ whenever p is a prime and $j \geq 2$. Assuming the Riemann hypothesis and any facts about ζ you need, show that $D_f(s)$ extends to a holomorphic function on $\Re s > \frac{1}{4}$.

Proof. By the mean value theorem.

$$D_f(s) = \sum_{n=1}^{\infty}(-1)^{n+1}n^{-s} = 1 - 2^{-s} + ... = -\sum_{k}\int_{2k-1}^{2k} sy^{-s-1}dy.$$

To bound the RHS:

$$\int_{2k-1}^{2k} sy^{-s-1}dy \leq s(2k-1)^{-\Re s-1}$$

which is summable uniformly in compact sets of $\Re s > 0$, and so the series $-\sum_k \int_{2k-1}^{2k} sy^{-s-1}dy$ is uniformly convergent. It follows from complex analysis that $D_f(s)$ defines a holomorphic function for $\Re s > 0$.

Note that for $\Re s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$$2^{-s}\zeta(s) = \sum_{n=1}^{\infty}(2n)^{-s}.$$

$$\zeta(s) - 2^{1-s}\zeta(s) = D_f(s).$$

The above identity yields

$$\zeta(s) = D_f(s)(1-2^{1-s})^{-1} \text{ for } \Re s > 1.$$

Since $D_f(s)$ and $1-2^{1-s}$ are holomorphic on $\Re s > 0$, the RHS is meromorphic on $\Re s > 0$. Hence, ζ extends to a meromorphic function on $\Re s > 0$. To show the final claim, note that $(1-2^{1-s})^{-1} \neq 0$ and $D_f(s) > 1 - 2^{-s} > 0$ on $(0, 1)$.

“Or otherwise”: alternatively, use Chapter 4 lecture notes.

(c) [8] Now let f be the multiplicative function for which $f(2) = -2$, $f(p) = 1$ when p is an odd prime and $f(p^j) = 0$ whenever p is a prime and $j \geq 2$. Assuming the Riemann hypothesis and any facts about ζ you need, show that $D_f(s)$ extends to a holomorphic function on $\Re s > \frac{1}{4}$

Proof. By multiplicativity,

$$D_f(s) = (1 - 2^{-s+1}) \prod_{p \geq 3} (1 + p^{-s}).$$

Since $1 - p^{-2s} = (1 + p^{-s})(1 - p^{-s})$, $\zeta(s) = \prod_p (1 + p^{-s} + p^{-2s} + ...) = \prod_p (1 - p^{-s})^{-1}$,

$$D_f(s) = \frac{1 - 2^{-s+1}}{1 + 2^{-s}} \prod_p (1 - p^{-2s}) \prod_p (1 - p^{-s})^{-1} = \frac{1 - 2^{-s+1}}{1 + 2^{-s}} \frac{\zeta(s)}{\zeta(2s)}.$$

- By RH, all zeros of $\zeta(2s)$ are on the line $s = 1/4$, so $\zeta(2s) \neq 0$ on $\Re s > 1/4$.
- $\zeta(s)(1 - 2^{-s+1})$ is entire since $\zeta(s)$ has a simple pole at $s = 1$ and $1 - 2^{-s+1} = 0$ at $s = 1$.
- $1 + 2^{-s}$ has no zeros on $\Re s > 1/4$.

2015, Q3

(a) [5] Write down, without proof, the value of the integral

$$\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{x^s}{s(s+1)}\,\mathrm{d}s,$$

where c and x are positive real numbers. Show that

$$\sum_{n\le x}\frac{1+\Lambda(n)}{n}\left(1-\frac{n}{x}\right)=\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(\zeta(s+1)-\frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s(s+1)}\,\mathrm{d}s$$

for any real $c>0$. (You need not justify the interchange of summation and integration.)

(b) [14] Prove that

$$\sum_{n\le x}\frac{1+\Lambda(n)}{n}\left(1-\frac{n}{x}\right)-2\log x\to -2$$

as $x\to\infty$.

[*The following facts may be used without proof. Firstly, if* $s=\sigma+it$ *with* $1\le\sigma\le 2$ *and* $|t|\ge 2$, *then*

$$\zeta(s)=O(\log|t|),\quad \zeta'(s)=O(\log^2|t|)\ \ \textit{and}\ \ \frac{1}{\zeta(s)}=O((\log|t|)^7).$$

Secondly, for any given $T>0$ *there is a real number* $\alpha<1$ *depending on* T, *such that the rectangle*

$$R(\alpha):=\{s\in\mathbb{C}:\alpha\le\Re(s)\le 2,\ |\Im(s)|\le T\}$$

contains no zeros of $\zeta(s)$. *Thirdly, the function*

$$\left(\zeta(s+1)-\frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s(s+1)}$$

has a double pole at $s=0$ *with residue* $2(\log x-1)$.]

(c) [6] Deduce that

$$\sum_{n\le x}\frac{\Lambda(n)}{n}-\log x\to -\gamma$$

as $x\to\infty$, where γ is Euler's constant.

[*You may assume without proof that*

$$\sum_{n\le x}n^{-1}=\log x+\gamma+O(x^{-1})$$

for $x\ge 1$, *and that* $\psi(x)\sim x$.]

Note re. fact 2 zero-free region (for both this question and Q3 2014): there seem to be different versions either saying $\alpha>0$ or $\alpha<1$. The former assumption doesn't exclude the possibility that $\alpha>1$ which is used to bound the integral on the vertical path.

I think it's safe to use $\alpha<1$ by mentioning the classical zero-free region (or any other stronger results) without proofs. It might also be okay to assume so as $\zeta(s)$ has no zeros on $\Re s\ge 1$ is the well-known PNT.

(a) [5] Write down, without proof, the value of the integral

$$\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{x^s}{s(s+1)}\,\mathrm{d}s,$$

where c and x are positive real numbers. Show that

$$\sum_{n\le x}\frac{1+\Lambda(n)}{n}\left(1-\frac{n}{x}\right)=\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(\zeta(s+1)-\frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s(s+1)}\,\mathrm{d}s$$

for any real $c>0$. (You need not justify the interchange of summation and integration.)
Proof. We have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s(s+1)}\mathrm{d}s=\begin{cases}0 & \text{if } x\le 1;\\ 1-x^{-1} & \text{otherwise.}\end{cases}$$

Can be clarified using results from Chapter 8 notes.
Note that $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$ $-\zeta'(s)/\zeta(s)=\sum_{n=1}^{\infty}\Lambda(n)n^{-s}$ for $\Re s>1$. Use termwise integration and the identity stated at the beginning:

$$\begin{aligned}\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(\zeta(s+1)-\frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s(s+1)}\,\mathrm{d}s&=\sum_{n=1}^{\infty}\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{1+\Lambda(n)}{n}\frac{(x/n)^s}{s(s+1)}ds\\&=\sum_{n\le x}\frac{1+\Lambda(n)}{n}\left(1-\frac{n}{x}\right).\end{aligned}$$

(b) [14] Prove that

$$\sum_{n\le x} \frac{1+\Lambda(n)}{n}\left(1-\frac{n}{x}\right) - 2\log x \to -2$$

as $x \to \infty$.

[*The following facts may be used without proof. Firstly, if* $s = \sigma + it$ *with* $1 \le \sigma \le 2$ *and* $|t| \ge 2$, *then*

$$\zeta(s) = O(\log|t|), \quad \zeta'(s) = O(\log^2|t|) \quad \text{and} \quad \frac{1}{\zeta(s)} = O((\log|t|)^7).$$

Secondly, for any given $T > 0$ *there is a real number* $\alpha < 1$ *depending on* T, *such that the rectangle*

$$R(\alpha) := \{s \in \mathbb{C} : \alpha \le \Re(s) \le 2,\ |\Im(s)| \le T\}$$

contains no zeros of $\zeta(s)$. *Thirdly, the function*

$$\left(\zeta(s+1) - \frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s(s+1)}$$

has a double pole at $s = 0$ *with residue* $2(\log x - 1)$.]

Proof. We choose $c = (\log x)^{-1}$. By the previous part, we have the identity

$$\sum_{n\le x} \frac{1+\Lambda(n)}{n}\left(1-\frac{n}{x}\right) = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(\zeta(s+1) - \frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s(s+1)}\,\mathrm{d}s.$$

First, since $\zeta(s) = O(\log|t|)$ and $\zeta'(s)/\zeta(s) = O(\log^9|t|)$ for $\Re s = c$ and $|t| > 2$ ($1 < c+1 < 2$, fact 1), it follows that for any $|t| > 2$

$$\left(\zeta(s+1) - \frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s(s+1)} \ll \log^9|t|\frac{x^{\log x}}{|t|^2} \ll |t|^{-3/2},$$

and so for any $T > 2$

$$\int_{c+iT}^{c+i\infty}\left(\zeta(s+1) - \frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s(s+1)}\,\mathrm{d}s = O(T^{-1/2}).$$

Thus, for any $\epsilon > 0$ there exists some T such that the above expression is less than ϵ. Fix this choice of T.

To bound the integral between $c - iT$ and $c + iT$, we apply Cauchy's residue theorem as follows. Let $\alpha = \alpha_T$ be the real number provided by fact 2. We move the contour of integration so as to go from $c - iT$ to $\alpha - 1 - iT$, to $\alpha - 1 + iT$, and then to $c + iT$. By the second fact, $\zeta(s+1)$ doesn't have any zeros in this region, so the only pole is at $s = 0$, whose residue $2(\log x - 1)$ by fact 3. To bound the integrals along the contour, by fact 2 the function $\left(\zeta(s+1) - \frac{\zeta'(s+1)}{\zeta(s+1)}\right)$ is continuous on the contour, and so by fact 1 it is uniformly bounded by some constant $M = M(T)$. Hence,

$$\int_{\alpha-1+iT}^{c+iT}\left(\zeta(s+1) - \frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s(s+1)}\,\mathrm{d}s \ll M\int_{\alpha-1}^{c} x^y dy \ll M\frac{x^c - x^{\alpha-1}}{\log x} < \frac{eM}{\log x}$$

and one can estimate the integral from $c - iT$ to $\alpha - 1 - iT$ similarly. The vertical integral is bounded by

$$\int_{\alpha-1-iT}^{\alpha-1+iT}\left(\zeta(s+1) - \frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s(s+1)}\,\mathrm{d}s \ll 2TMx^{\alpha-1}.$$

Thus for sufficiently large x (depending on T), the integrals on the contour is bounded by ϵ.

(c) Deduce that

$$\sum_{n\le x}\frac{\Lambda(n)}{n}-\log x\to-\gamma$$

as $x\to\infty$, where γ is Euler's constant.
[*You may assume without proof that*

$$\sum_{n\le x}n^{-1}=\log x+\gamma+O(x^{-1})$$

for $x\ge 1$, *and that* $\psi(x)\sim x$.]

Proof. Substitute all the identities. Recall that by (b)

$$\sum_{n\le x}\frac{1+\Lambda(n)}{n}\left(1-\frac{n}{x}\right)-2\log x\to-2$$

as $x\to\infty$.

$$\sum_{n\le x}\frac{1+\Lambda(n)}{n}\left(1-\frac{n}{x}\right)=\sum_{n\le x}\frac{\Lambda(n)}{n}+\sum_{n\le x}n^{-1}-1-\sum_{n\le x}\Lambda(n)x^{-1}.$$

By the limits provided,

$$\sum_{n\le x}\frac{\Lambda(n)}{n}+\log x+\gamma-2-2\log x\to-2$$

as $x\to\infty$.

2014, Q3
Let $f(n)$ be a non-negative arithmetic function and write

$$\eta(x) = \sum_{n \le x} f(n) \quad \text{and} \quad \eta_1(x) = \int_0^x \eta(t) \mathrm{d}t.$$

You may assume in the rest of the question that if $\eta_1(x) \sim x^2/2$ *as* $x \to \infty$ *then* $\eta(x) \sim x$.
(a) Not covered in the current course.
(b) [12] Define the arithmetic function $\lambda(n)$ by setting $\lambda(1) = 1$ and

$$\lambda(p_1^{e_1} \dots p_k^{e_k}) = (-1)^{e_1 + \dots + e_k},$$

for any distinct primes $p_1, \dots, p_k$.

Using the facts below, and starting from the formula

$$\sum_{n \le x} \lambda(n)(x - n) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{\zeta(2s)}{\zeta(s)} x^{s+1} \frac{\mathrm{d}s}{s(s+1)} \quad (c > 1),$$

which you may also use without proof, show that

$$\sum_{n \le x} \lambda(n)(x - n) = o(x^2).$$

[*The following facts may be used without proof. Firstly, if* $1 \le \Re(s) \le 2$ *and* $|\Im(s)| \ge 2$, *then*

$$\frac{\zeta(2s)}{\zeta(s)} = O((\log |t|)^7),$$

where $t = \Im(s)$. *Secondly, for any given* $T > 0$ *there is a real number* $\alpha < 1$ *depending on* T, *such that the rectangle*

$$R(\alpha) := \{s \in \mathbb{C} : \alpha \le \Re(s) \le 2, \ |\Im(s)| \le T\}$$

contains no zeros of $\zeta(s)$. *Thirdly,* $\zeta(2s)/\zeta(s)$ *has a removable singularity at* $s = 1$.]
(c) [6] By applying the Tauberian result (stated at the beginning) to $f(n) = 1 + \lambda(n)$ show that

$$\sum_{n \le x} \lambda(n) = o(x),$$

and deduce that there is a number x_0 such that $\lambda(n)$ changes sign at least once in the interval $[\frac{99}{100}x, x]$, whenever $x \ge x_0$.

(b) [12] Define the arithmetic function $\lambda(n)$ by setting $\lambda(1) = 1$ and

$$\lambda(p_1^{e_1} \dots p_k^{e_k}) = (-1)^{e_1+\dots+e_k},$$

for any distinct primes $p_1, \dots, p_k$.

Using the facts below, and starting from the formula

$$\sum_{n \le x} \lambda(n)(x-n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(2s)}{\zeta(s)} x^{s+1} \frac{\mathrm{d}s}{s(s+1)} \quad (c > 1),$$

which you may also use without proof, show that

$$\sum_{n \le x} \lambda(n)(x-n) = o(x^2).$$

[*The following facts may be used without proof. Firstly, if* $1 \le \Re(s) \le 2$ *and* $|\Im(s)| \ge 2$, *then*

$$\frac{\zeta(2s)}{\zeta(s)} = O((\log|t|)^7),$$

where $t = \Im(s)$. *Secondly, for any given* $T > 0$ *there is a real number* $\alpha < 1$ *depending on* T, *such that the rectangle*

$$R(\alpha) := \{s \in \mathbb{C} : \alpha \le \Re(s) \le 2,\ |\Im(s)| \le T\}$$

contains no zeros of $\zeta(s)$. *Thirdly,* $\zeta(2s)/\zeta(s)$ *has a removable singularity at* $s = 1$.]

Proof. Proceed similarly as 2015 Q2(b). Choose $c = 1 + (\log x)^{-1}$.

Bound the integral for $s = c + it$ and $|t| > T$: fact 1 states $\zeta(2s)/\zeta(s) = O(\log^7|t|)$ for $\Re s = c$ and $|t| > 2$ $(1 < c + 1 < 2$, fact 1), it follows that for any $|t| > 2$

$$\frac{\zeta(2s)}{\zeta(s)} x^{s+1} \frac{\mathrm{d}s}{s(s+1)} \ll \log^7|t| \frac{x^{1+\log x+1}}{|t|^2} \ll x|t|^{-3/2},$$

and so for any $T > 2$

$$\int_{c+iT}^{c+i\infty} \frac{\zeta(2s)}{\zeta(s)} x^{s+1} \frac{\mathrm{d}s}{s(s+1)} \,\mathrm{d}s = O(x^2 T^{-1/2}).$$

Thus, for any $\epsilon > 0$ there exists some T such that the above expression is less than ϵ. Fix this choice of T.

To bound the integral between $c - iT$ and $c + iT$, we apply Cauchy's residue theorem as follows. Let $\alpha = \alpha_T$ be the real number provided by fact 2. We move the contour of integration so as to go from $c - iT$ to $\alpha - iT$, to $\alpha + iT$, and then to $c + iT$. By the second fact, $\zeta(s)$ doesn't have any zeros in this region, so the only singularity is at $s = 1$, which is told to be removable by fact 3.

To bound the integrals along the contour, by fact 2 the function $\zeta(2s)/\zeta(s)$ is continuous on the contour, and so by fact 1 it is uniformly bounded by some constant $M = M(T)$. Hence,

$$\int_{\alpha+iT}^{c+iT} \frac{\zeta(2s)}{\zeta(s)} x^{s+1} \frac{\mathrm{d}s}{s(s+1)} \,\mathrm{d}s \ll M \int_{\alpha}^{c} x^{y+1} dy \ll M \frac{x^{c+1} - x^{\alpha+1}}{\log x} < \frac{eMx^2}{\log x}$$

and one can estimate the integral from $c - iT$ to $\alpha - iT$ similarly. The vertical integral is bounded by

$$\int_{\alpha-iT}^{\alpha+iT} \frac{\zeta(2s)}{\zeta(s)} x^{s+1} \frac{\mathrm{d}s}{s(s+1)} \,\mathrm{d}s \ll 2TMx^{\alpha+1}.$$

Thus for sufficiently large x (depending on T), the integrals on the contour is bounded by ϵ.

(c) [6] By applying the Tauberian result (stated at the beginning) to $f(n) = 1 + \lambda(n)$ show that

$$\sum_{n \le x} \lambda(n) = o(x),$$

and deduce that there is a number x_0 such that $\lambda(n)$ changes sign at least once in the interval $[\frac{99}{100}x, x]$, whenever $x \ge x_0$.

The Tauberian result: let f be a non-negative arithmetic function, $\eta(x) = \sum_{n \le x} f(n)$ and $\eta_1(x) = \int_0^x \eta(t)\mathrm{d}t$. If $\eta_1(x) \sim x^2/2$ as $x \to \infty$ then $\eta(x) \sim x$.

Proof. f is nonnegative. Then $\eta_1(x) = \sum_{n \le x}(x-n)(1+\lambda(n)) = \sum_{n \le x}(x-n) + o(x^2)$ by (b). Thus the Tauberian bound says

$$\eta(x) = \sum_{n \le x}(1 + \lambda(n)) = x(1 + o(1))$$

and the first part of this question follows.
By subtraction

$$\sum_{n=99x/100}^{x} \lambda(n) = \sum_{n=1}^{x} \lambda(n) - \sum_{n=1}^{99x/100} \lambda(n) = o(x).$$

In particular, for sufficiently large x

$$\sum_{n=99x/100}^{x} \lambda(n) < x/200.$$

If there isn't any sign change, then $|\sum_{n=99x/100}^{x} \lambda(n)| > x/200$ which is a contradiction.

Good luck!