C3.3 Differentiable Manifolds

Problem Sheet 3

Michaelmas Term 2019–2020

1. Let (x_0, x_1, x_2, x_3) be coordinates on $\mathcal{S}^3 \subseteq \mathbb{R}^4$. Let

$$X = -x_1\partial_0 + x_0\partial_1 - x_3\partial_2 + x_2\partial_3$$

restricted to $\mathcal{S}^3$ and

$$\omega = -x_2 \mathrm{d}x_0 + x_3 \mathrm{d}x_1 + x_0 \mathrm{d}x_2 - x_1 \mathrm{d}x_3.$$

   (a) Compute the flow of X and hence $\mathcal{L}_X\omega$ using the definition of Lie derivative.

   (b) Compute $\mathrm{d}\omega$ and $\mathrm{d}(i_X\omega)$ and hence compute $\mathcal{L}_X\omega$ using Cartan's formula.

2. A Riemann surface is a 2-dimensional manifold with an atlas $\{(U_i, \varphi_i) : i \in I\}$ whose transition maps $\varphi_j \circ \varphi_i^{-1}$ for $i, j \in I$ are maps from an open set $\varphi_i(U_i \cap U_j)$ of $\mathbb{C} = \mathbb{R}^2$ to another open set $\varphi_j(U_i \cap U_j)$ which are holomorphic and invertible. Show that a Riemann surface is orientable.

3. Show that a product of orientable manifolds is orientable.

4. Let M be a manifold and let G act freely and properly discontinuously by diffeomorphisms f_g for $g \in G$ on M. Let $\pi : M \to M/G$ be the projection map.

   (a) Suppose that M/G is orientable, so that there is a volume form Ω on M/G. Show that $\Upsilon = \pi^*\Omega$ is a volume form on M such that $f_g^*\Upsilon = \Upsilon$ for all $g \in G$.

   (b) Suppose that Υ is a volume form on M such that $f_g^*\Upsilon = \Upsilon$ for all $g \in G$. Show that there is a volume form Ω on M/G such that $\pi^*\Omega = \Upsilon$, and hence that M/G is orientable.

   (c) Is $\mathcal{S}^2 \times \mathbb{RP}^2$ orientable? What about $\mathbb{RP}^2 \times \mathbb{RP}^2$?

5. Define $f : (0,1) \times (0, 2\pi) \to B^2$, where B^2 is the unit ball centred at 0 in $\mathbb{R}^2$, by

$$f(r, \theta) = (r\cos\theta, r\sin\theta)$$

and let (y_1, y_2) be coordinates on B^2. Let $B_s \subseteq B^2$ denote the open ball centred at 0 of radius s, for $s \in (0,1)$, with its standard orientation. Let $k \in \{1, -1\}$.

   (a) Compute

$$f^*\left(4(1 - y_1^2 - y_2^2)^{2k} \mathrm{d}y_1 \wedge \mathrm{d}y_2\right).$$

   (b) Hence, or otherwise, calculate

$$\int_{B_s} 4(1 - y_1^2 - y_2^2)^{2k} \mathrm{d}y_1 \wedge \mathrm{d}y_2$$

   in each of the cases $k = 1$ and $k = -1$. What happens as $s \to 1$ in each case?

6. Use Stokes Theorem for manifolds with boundary to prove the following results.

(a) Let $\gamma : \mathcal{S}^1 \to \mathbb{R}^2$ be an embedding and let D be the region in $\mathbb{R}^2$ bounded by $C = \gamma(\mathcal{S}^1)$. Let $u_1, u_2 : \mathbb{R}^2 \to \mathbb{R}$ be smooth functions. Then

$$\int_C u_1 \mathrm{d}x_1 + u_2 \mathrm{d}x_2 = \int_D \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \mathrm{d}x_1 \mathrm{d}x_2.$$

(b) Let V be an open subset of $\mathbb{R}^3$ with compact closure and smooth boundary $S = \partial V$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth. Then

$$\int_V \mathrm{div} F \, \mathrm{d}V = \int_S F \cdot \mathrm{d}S.$$

(c) Let Σ be a compact oriented surface in $\mathbb{R}^3$ with smooth boundary $\Gamma = \partial\Sigma$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth. Then

$$\int_\Sigma \mathrm{curl} F \cdot \mathrm{d}\Sigma = \int_\Gamma F \cdot \mathrm{d}\Gamma.$$