A <u>vector</u> is Defn (i) a list of numbers

Defn (ii) a geometric object w/ both magnitude and direction (an <u>arrow</u>)

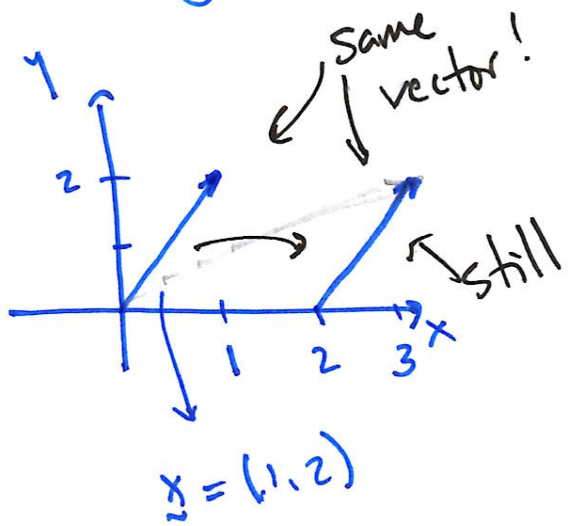
[vectors boldface in typed]

Defn (i) : vector $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ ← row vector - compare to column vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

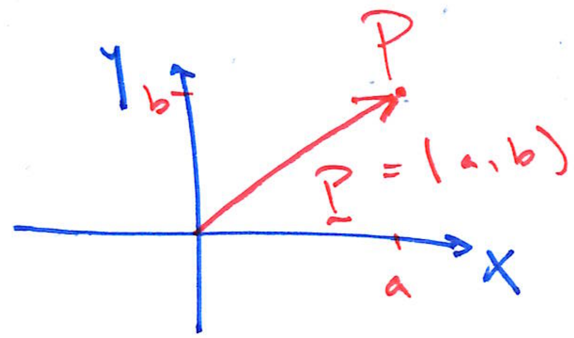
$x_i \in \mathbb{R}$ is ith coordinate of $\underset{\sim}{x}$

- the set of all $\underset{\sim}{x}$ is space $\mathbb{R}^n$

eg, $n=2$, $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$



still $\underset{\sim}{x} = (1,2)$. <u>translation</u> vector from (2,0) to (3,2)



- arrow pointing from origin to $x=a$, $y=b$ is <u>position</u> <u>vector</u> for point P

- <u>Coordinate system</u> $\underset{\sim}{0} = (0, 0, \dots, 0)$ is <u>origin</u>

zero vector

- all points $(0, 0, \dots x_i, 0, 0 \dots 0)$ for $x_i \in \mathbb{R}$ forms the "x_i axis"

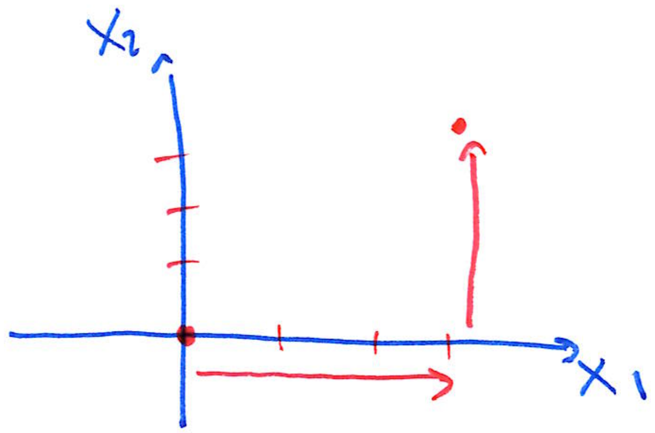
- The point $(x_1, x_2, \dots, x_n)$ ← arbitrary point in $\mathbb{R}^n$ is reached by:

moving x_1 units along the x_1 axis

" x_2 units parallel to x_2 axis

⋮

x_n " " " x_n axis



Adding / Subtracting vectors

• "Components view" $\underset{\sim}{u} = (u_1, u_2, \dots u_n)$, $\underset{\sim}{v} = (v_1, v_2, \dots v_n)$

$$\underset{\sim}{u} + \underset{\sim}{v} = (u_1+v_1, u_2+v_2, \dots, u_n+v_n)$$

$$\underset{\sim}{u} - \underset{\sim}{v} = (u_1-v_1, u_2-v_2, \dots, u_n-v_n)$$

[Note: if $v_i \in \mathbb{R}$, $\underset{\sim}{u} + v_i$ meaningless!

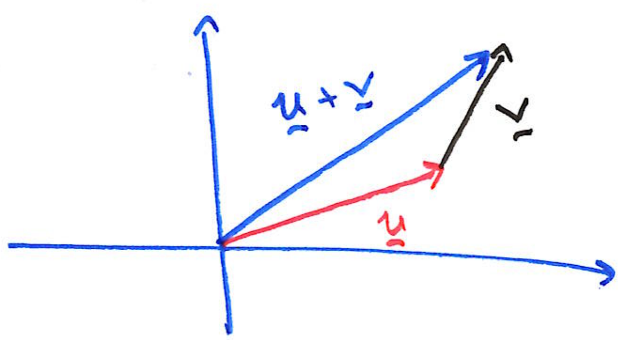
$\underset{\sim}{u} = (u_1, u_2 \dots u_n)$

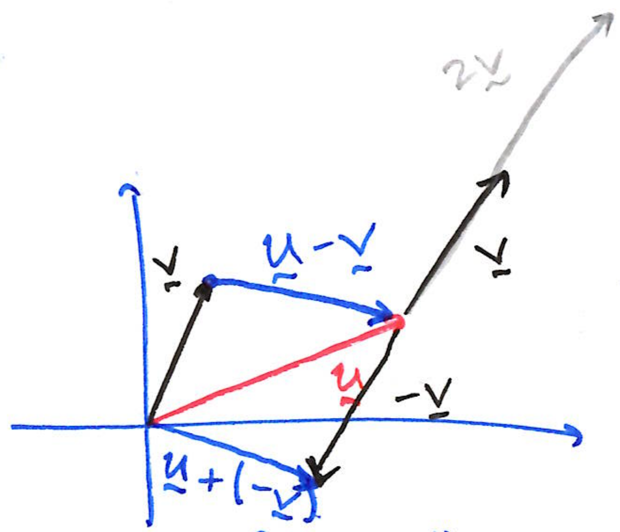
$\underset{\sim}{v} = (v_1, v_2, \dots v_m)$, $m \neq n$

$\underset{\sim}{u} + \underset{\sim}{v}$ meaningless!]

• Translation ("Movement") view : $\underset{\sim}{u} + \underset{\sim}{v}$ means move $u_i + v_i$ units in x_i direction

• "Arrow view"





Note: $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$, ie it is commutative

$\underset{\sim}{u} - \underset{\sim}{v} = -(\underset{\sim}{v} - \underset{\sim}{u})$

$\underset{\sim}{u} - \underset{\sim}{v}$ arrow from "end of v" to "end of u"

Scalar multiplication Let k be a real number and $\underset{\sim}{v} = (v_1, \dots, v_n)$ a vec.

The vector $k\underset{\sim}{v} = (kv_1, kv_2, \dots, kv_n)$ has the same ~~magnitude~~ direction as $\underset{\sim}{v}$, but magnitude scaled by k

Def'n: the vectors $\underset{\sim}{e}_1 = (1, 0, \ldots, 0)$, $\underset{\sim}{e}_2 = (0, 1, 0, \ldots, 0)$, $\vdots$, $\underset{\sim}{e}_n = (0, 0, \ldots, 1)$ form **standard basis**

$\uparrow$ nth

$n = 2, 3$: $\underset{\sim}{e}_1 \overset{\text{call}}{=} \underset{\sim}{i}$, $\underset{\sim}{e}_2 = \underset{\sim}{j}$, $\underset{\sim}{e}_3 = \underset{\sim}{k}$

Idea: can write any $\underset{\sim}{x} \in \mathbb{R}^n$ as linear combination of $\{\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_n\}$

Indeed, $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i \underset{\sim}{e}_i$ [eg $x_1 \underset{\sim}{e}_1 + x_2 \underset{\sim}{e}_2 = (x_1, x_2, 0, \ldots, 0)$]

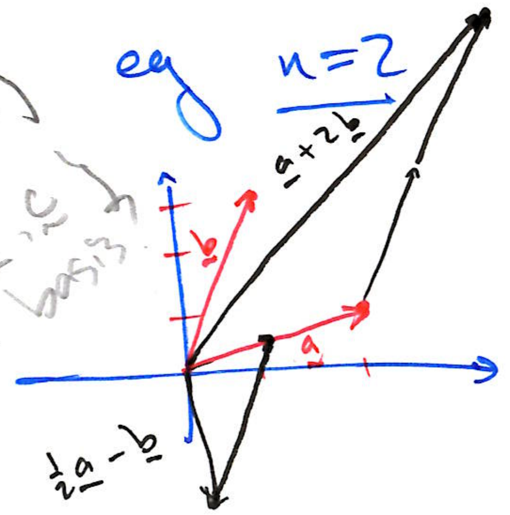
- Note: could define other bases

eg $\hat{\underset{\sim}{e}}_1 = (2, 0, 0, \ldots, 0)$, $\underset{\sim}{e}_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\underset{\sim}{e}_n = (0, 0, \ldots, 1)$

eg $n=2$

$\underset{\sim}{a} = (2, 1)$, $\underset{\sim}{b} = (1, 3)$ Claim: this is a basis for $\mathbb{R}^2$

Consider: $\underset{\sim}{c} = (4, 2)$ Does $\{\underset{\sim}{a}, \underset{\sim}{c}\}$ form a basis?

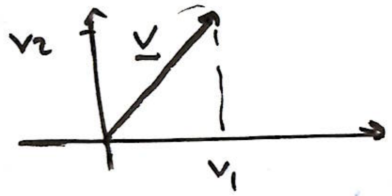


Exercise: write $\underset{\sim}{p} = (p_1, p_2) = c_1 \underset{\sim}{a} + c_2 \underset{\sim}{b} = (2c_1 + c_2, c_1 + 3c_2)$

$$\rightarrow \left.\begin{aligned} 2c_1 + c_2 &= p_1 \\ c_1 + 3c_2 &= p_2 \end{aligned}\right\} \xrightarrow{\text{solve}} \begin{aligned} c_1 &= \tfrac{1}{5}(3p_1 - p_2) \\ c_2 &= \tfrac{-1}{5}(p_1 - 2p_2) \end{aligned}$$

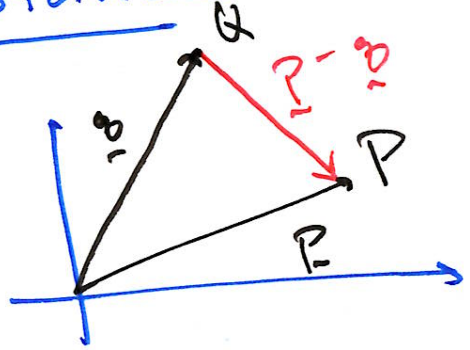
Def'n The <u>length</u> (or magnitude) of $\underset{\sim}{v} = (v_1, v_2, \dots, v_n)$

is $|\underset{\sim}{v}| = \sqrt{\sum_{i=1}^{n} v_i^2}$ ↖ n-D pythagorean Thm!



• $\underset{\sim}{v}$ is a <u>unit vector</u> if $|\underset{\sim}{v}| = 1$

Distance between pts P and Q w/ position vec's $\underset{\sim}{p}$ and $\underset{\sim}{q}$:



$$|\underset{\sim}{p} - \underset{\sim}{q}| = |\underset{\sim}{q} - \underset{\sim}{p}| = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$$

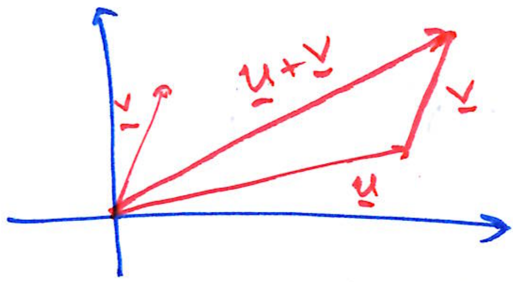
Notes: • $|\underset{\sim}{v}| = 0$ iff $\underset{\sim}{v} = \underset{\sim}{0}$

• $|\lambda \underset{\sim}{v}| = |\lambda|\,|\underset{\sim}{v}|$ for any $\lambda \in \mathbb{R}$

Triangle inequality let $\underset{\sim}{u}, \underset{\sim}{v} \neq \underset{\sim}{0}$ vec's in $\mathbb{R}^n$

Then $|\underset{\sim}{u} + \underset{\sim}{v}| \leq |\underset{\sim}{u}| + |\underset{\sim}{v}|$ w/ equality only if $\underset{\sim}{v} = \lambda \underset{\sim}{u}$ for some $\lambda > 0$

Proof $|\underset{\sim}{u} + \underset{\sim}{v}|^2 = |\underset{\sim}{u}|^2 + 2\sum_{i=1}^{n} u_i v_i + |\underset{\sim}{v}|^2 \underset{\text{want}}{\leq} (|\underset{\sim}{u}| + |\underset{\sim}{v}|)^2 = |\underset{\sim}{u}|^2 + 2|\underset{\sim}{u}|\,|\underset{\sim}{v}| + |\underset{\sim}{v}|^2$



True if $\sum_{i=1}^{n} u_i v_i \leq |\underset{\sim}{u}|\,|\underset{\sim}{v}|$ ★

Consider $|\mathbf{u} + t\mathbf{v}|^2 = |\mathbf{u}|^2 + 2t\sum_{i=1}^{n} u_i v_i + t^2|\mathbf{v}|^2 \geq 0$ and only $=0$

if $\mathbf{u} + t_0\mathbf{v} = \mathbf{0}$ ie $\mathbf{v} = \lambda\mathbf{u}$ $(\lambda = -t_0)$ in which case

$\mathbf{u} + \mathbf{v} = (1+\lambda)\mathbf{u}$, so $|\mathbf{u}+\mathbf{v}| = |1+\lambda|\,|\mathbf{u}| \leq |\mathbf{u}| + |\mathbf{v}|$

w/ equality if $\lambda > 0$

- otherwise $|\mathbf{u} + t\mathbf{v}|^2$ is quadratic fn of t with no real roots

$\Rightarrow$ negative discriminant $b^2 - 4ac \leq 0$

$\Rightarrow 4\left(\sum_{i=1}^{n} u_i v_i\right)^2 - 4|\mathbf{u}|^2|\mathbf{v}|^2 \leq 0$, and ☆ follows

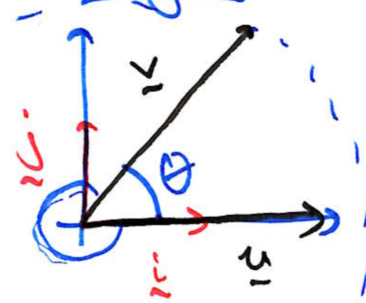
---

Defn $\mathbf{u}\cdot\mathbf{v} = \sum_{i=1}^{n} u_i v_i$ is <u>dot product</u> of $\mathbf{u}$ and $\mathbf{v}$ (also called <u>scalar</u> product, inner product)

Properties

- $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
- $(\lambda\mathbf{u})\cdot\mathbf{v} = \lambda(\mathbf{u}\cdot\mathbf{v})$
- $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$
- $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2 \Rightarrow \mathbf{u}\cdot\mathbf{u} = 0$ iff $\mathbf{u} = \mathbf{0}$
- $|\mathbf{u}\cdot\mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}|$ ← Cauchy Schwarz ineq.

Angles



$\mathbf{u} = \overset{u_1}{|\mathbf{u}|}\,\mathbf{i} + \overset{u_2}{0}\,\mathbf{j}$

$\mathbf{v} = \underbrace{|\mathbf{v}|\cos\theta}_{v_1}\,\mathbf{i} + \underbrace{|\mathbf{v}|\sin\theta}_{v_2}\,\mathbf{j}$

$\Rightarrow \boxed{\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta}$

The angle b/t $\mathbf{u}, \mathbf{v}$ is $\theta = \cos^{-1}\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right)$

<u>Notes</u> • clear in $\mathbb{R}^2$, $\mathbb{R}^3$ but true in $\mathbb{R}^n$ (how we <u>define</u> angles)

• $\dfrac{|\underline{u}\cdot\underline{v}|}{|\underline{u}|\,|\underline{v}|} \leq 1 \Rightarrow$ can always choose $\theta \in [0, \pi]$ (smaller of angles)

• $\underline{u}\cdot\underline{v} = 0$ iff $\underline{u}$, $\underline{v}$ are or<u>thogonal</u> (perpendicular)

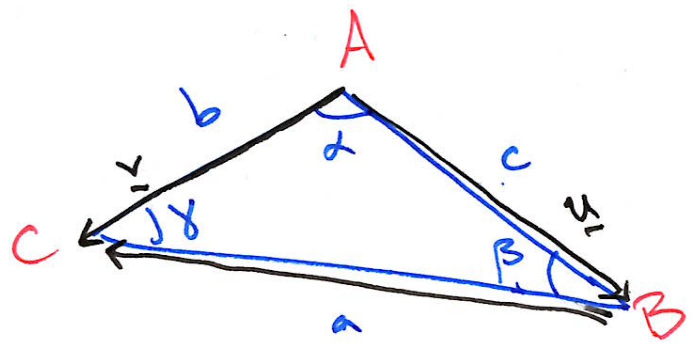
• Dot product is operator that measures how <u>aligned</u> 2 vec's are.

Triangles and Circles

Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos\alpha$

Proof Define $\underset{\sim}{u} = \overrightarrow{AB}$, $\underset{\sim}{v} = \overrightarrow{AC}$

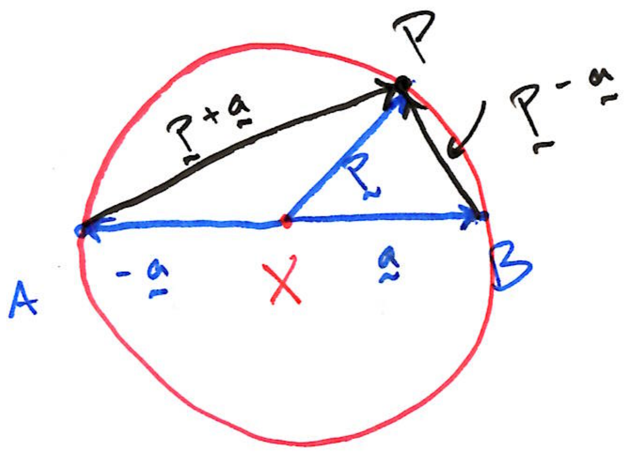
$(\Rightarrow \overrightarrow{BC} = \underset{\sim}{v} - \underset{\sim}{u})$



$$a^2 = |\underset{\sim}{v} - \underset{\sim}{u}|^2 \overset{\text{expand}}{=} |\underset{\sim}{v}|^2 + |\underset{\sim}{u}|^2 - 2\underbrace{\underset{\sim}{u}\cdot\underset{\sim}{v}}_{= |\underset{\sim}{u}||\underset{\sim}{v}|\cos\alpha} = b^2 + c^2 - 2bc\cos\alpha$$

- Given 2 pts A, B, where can place P st $\measuredangle APB$ is right angle?

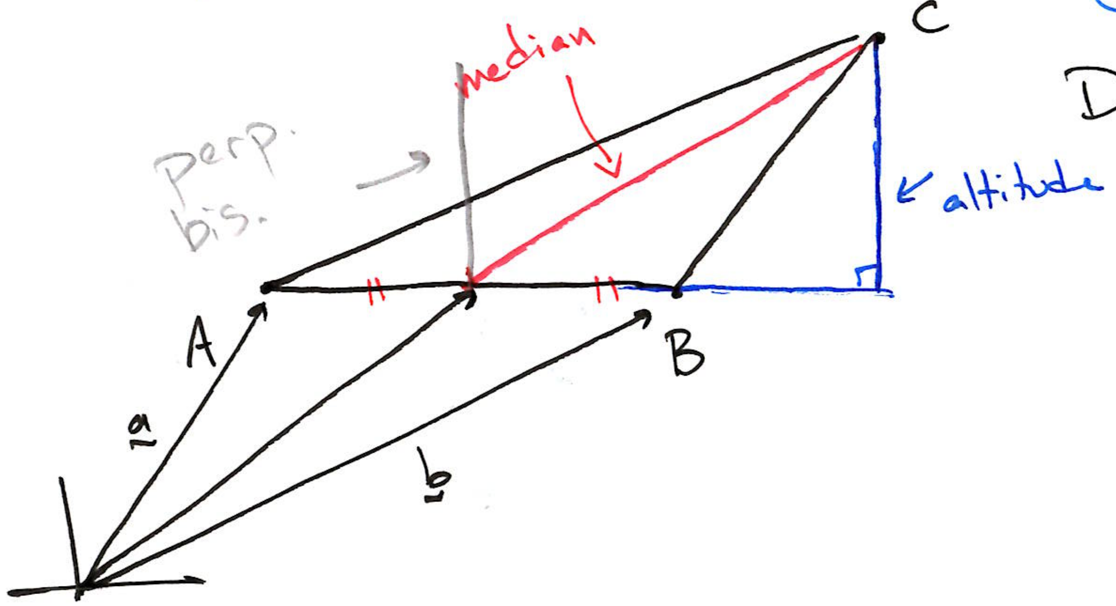
Thales Thm: Form a circle w/ $\overline{AB}$ a diameter. Then $\measuredangle APB$ is a right angle iff P is also on the circle.



Pf $(\underset{\sim}{p}+\underset{\sim}{a})\cdot(\underset{\sim}{p}-\underset{\sim}{a}) = |\underset{\sim}{p}|^2 - |\underset{\sim}{a}|^2 = 0$ iff $|\underset{\sim}{p}| = |\underset{\sim}{a}|$ ie

if P is on the circle. ✓

· Let ABC be a triangle w/ pos. vec's $\mathbf{a}, \mathbf{b}, \mathbf{c}$ from an origin



Define: - a median connects a vertex to opposing midpoint

- altitude is distance from vertex to opposing side (possibly extended)

- perpendicular bisector divides an edge w/ perp. line

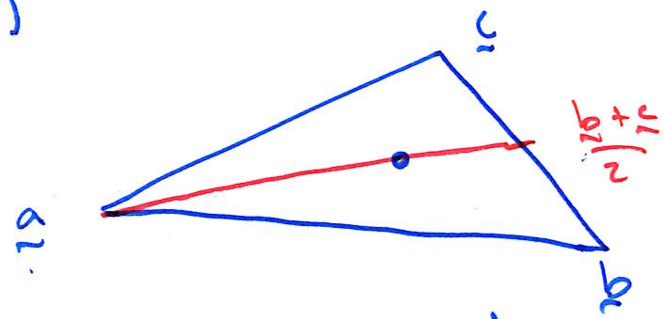
Claim: 1. The medians intersect at a single point (the centroid)

2. Altitudes .. .. .. (orthocentre)

3. Perp. bis. .. .. (circumcentre)

4. The centroid, orthocentre, and circumcentre are collinear, ie all lie on a single line (Euler line)

Challenge: prove w/ vector identities/properties



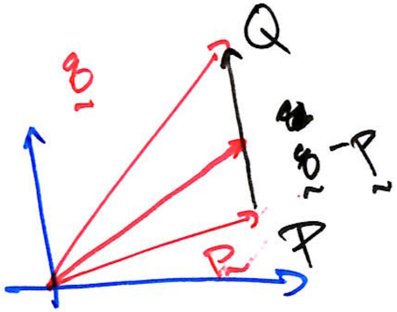
Pf of 1 The centroid is arithmetic mean: $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$

The midpts of the edges given by $\dfrac{\mathbf{a}+\mathbf{b}}{2}, \dfrac{\mathbf{a}+\mathbf{c}}{2}, \dfrac{\mathbf{b}+\mathbf{c}}{2}$.

Now, writing centroid as $\dfrac{\mathbf{a}}{3} + \dfrac{2}{3}\left(\dfrac{\mathbf{b}+\mathbf{c}}{2}\right)$ implies it lies on line connecting $\mathbf{a}$ to $\dfrac{\mathbf{b}+\mathbf{c}}{2}$ $\left(\mu = \frac{1}{3}\right)$

Similarly, can write as $\dfrac{\mathbf{b}}{3} + \dfrac{2}{3}\left(\dfrac{\mathbf{a}+\mathbf{c}}{2}\right)$ or $\dfrac{\mathbf{c}}{3} + \dfrac{2}{3}\left(\dfrac{\mathbf{a}+\mathbf{b}}{2}\right)$

$\Rightarrow$ centroid lies on all 3 medians

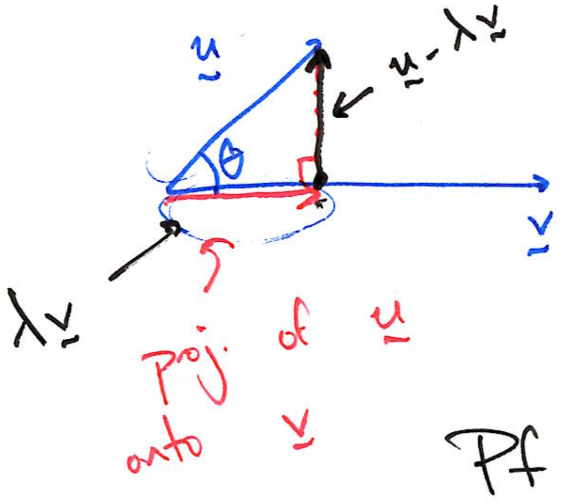


Aside: in general, the pts between P and Q are given by $\underbrace{\mu \underset{\sim}{q} + (1-\mu)\underset{\sim}{p}}_{\parallel}$ for $\mu \in [0,1]$

$$\underset{\sim}{p} + \mu(\underset{\sim}{q} - \underset{\sim}{p})$$

(the midpoint is $\dfrac{\underset{\sim}{p} + \underset{\sim}{q}}{2}$)

Vector Projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$ is the orthogonal projection of $\underset{\sim}{u}$ onto a line parallel to $\underset{\sim}{v}$ – "shadow of $\underset{\sim}{u}$ on $\underset{\sim}{v}$"



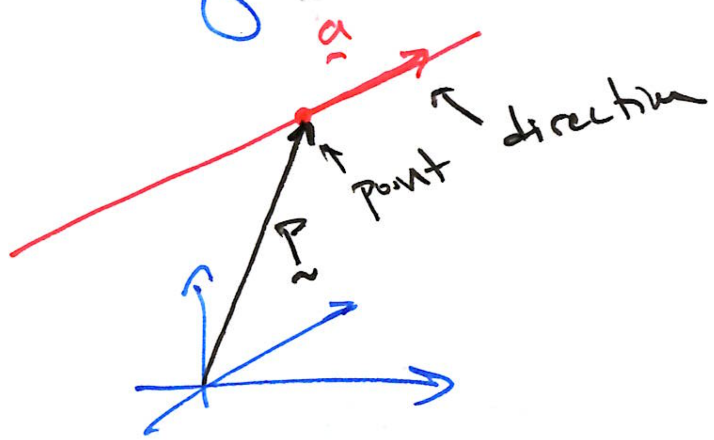
Ex 21 Show $\exists$ unique $\lambda \in \mathbb{R}$ st $\underset{\sim}{u} - \lambda \underset{\sim}{v}$ is perp. to $\underset{\sim}{v}$

Pf $\underset{\sim}{u} - \lambda \underset{\sim}{v}$ perp. to $\underset{\sim}{v}$ iff $(\underset{\sim}{u} - \lambda \underset{\sim}{v}) \cdot \underset{\sim}{v} = 0 = \underset{\sim}{u} \cdot \underset{\sim}{v} - \lambda \underset{\sim}{v} \cdot \underset{\sim}{v}$

iff $\boxed{\lambda = \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2}}$ (for $\underset{\sim}{v} \neq \underset{\sim}{0}$) $\lambda = \dfrac{|\underset{\sim}{u}|\,|\underset{\sim}{v}| \cos\theta}{|\underset{\sim}{v}|^2} \Rightarrow \lambda |\underset{\sim}{v}| = |\underset{\sim}{u}| \cos\theta$

Equations of Lines and Planes Goal: describe parametrically ito vectors

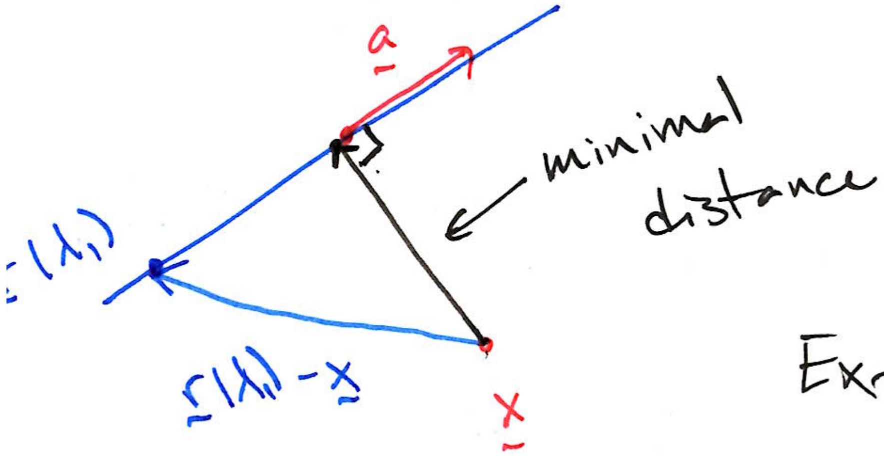
"Ingredients" of line: a point (on line) and direction



all points on line: $\boxed{\underset{\sim}{r}(\lambda) = \underset{\sim}{p} + \lambda \underset{\sim}{a}}$, $-\infty < \lambda < \infty$

parametric form w/ parameter λ

<u>Ex 22</u> Given a point $\underset{\sim}{x}$ not on line, show that $|\underset{\sim}{r}(\lambda) - \underset{\sim}{x}|$ is minimal when $(\underset{\sim}{x} - \underset{\sim}{r}(\lambda)) \cdot \underset{\sim}{a} = 0$



<u>Soln</u> Define $d(\lambda) = |\underset{\sim}{r}(\lambda) - \underset{\sim}{x}|^2$

we want λ st $d'(\lambda) = 0$

Expand $|\underset{\sim}{r} - \underset{\sim}{x}|^2 = |\underset{\sim}{p} + \lambda \underset{\sim}{a} - \underset{\sim}{x}|^2 = |\underset{\sim}{p}|^2 + 2\lambda(\underset{\sim}{p}\cdot\underset{\sim}{a} - \underset{\sim}{x}\cdot\underset{\sim}{a}) + \lambda^2|\underset{\sim}{a}|^2 + |\underset{\sim}{x}|^2$

$$\Rightarrow d'(\lambda) = 2(\underset{\sim}{p}\cdot\underset{\sim}{a} - \underset{\sim}{x}\cdot\underset{\sim}{a} + \lambda\,\underset{\sim}{a}\cdot\underset{\sim}{a}) = 2(\underbrace{\underset{\sim}{p} + \lambda\underset{\sim}{a} - \underset{\sim}{x}}_{= \underset{\sim}{r}(\lambda) - \underset{\sim}{x}})\cdot\underset{\sim}{a} = 0$$

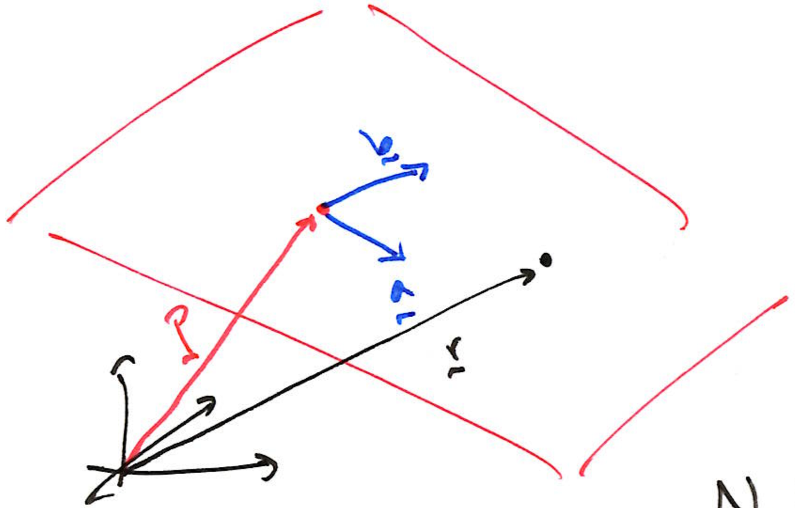
<u>Note</u> in $\mathbb{R}^3$, we could write $\underset{\sim}{p} = (p, q, r)$, $\underset{\sim}{a} = (a, b, c)$, $\underset{\sim}{r} = (x, y, z)$

Then $\underset{\sim}{r} = \underset{\sim}{p} + \lambda\underset{\sim}{a} \rightarrow \begin{cases} x = p + \lambda a \\ y = q + \lambda b \\ z = r + \lambda c \end{cases}$ or, equivalently, if solve for λ:

$$\frac{x-p}{a} = \frac{y-q}{b} = \frac{z-r}{c}$$

Eq'n of a plane : ingredients : a point in plane, and 2 unique directions a, b

Unique means: $a \neq \gamma b$ for any γ - same as a and b are linearly indep. - not scalar multiples

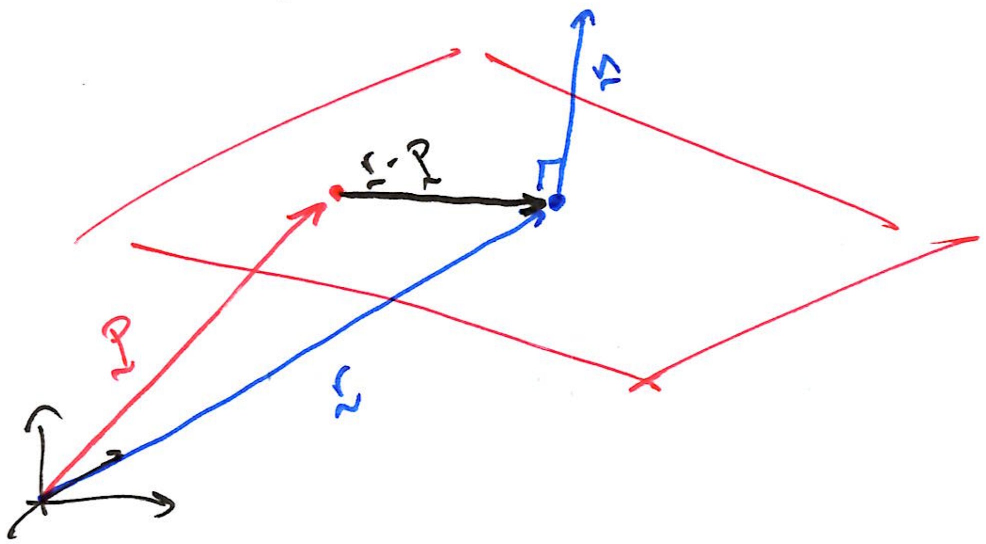


All points in plane given by

$$\boxed{r(\lambda, \mu) = p + \lambda a + \mu b} \quad \text{for } -\infty < \lambda, \mu < \infty$$

Note: this construction holds in any dimension ≥ 2

• in $\mathbb{R}^3$, an alternative description: ingredients are a point and a single vector perpen. to plane



For any r in plane, $(r - p)\cdot n = 0$

ie $\boxed{r \cdot n = p \cdot n = c}$ or call

← n is normal vector to plane

Conversely, the equation $\underset{\sim}{r} \cdot \underset{\sim}{n} = c$ for given $\underset{\sim}{n} \neq \underset{\sim}{0}$ and $c \in \mathbb{R}$ describes a plane with normal $\underset{\sim}{n}$

Let $\underset{\sim}{r} = (x, y, z)$, ~~[illegible]~~ $\underset{\sim}{p} = (p_1, p_2, p_3)$, $\underset{\sim}{a} = (a_1, a_2, a_3)$, $\underset{\sim}{b} = (b_1, b_2, b_3)$

and consider plane $\underset{\sim}{r} = \underset{\sim}{p} + \lambda \underset{\sim}{a} + \mu \underset{\sim}{b}$. In comp. form:

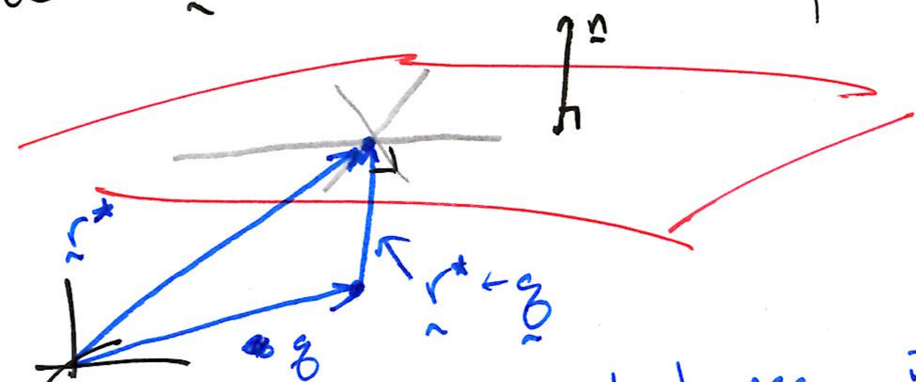
$$x = p_1 + \lambda a_1 + \mu b_1$$
$$y = p_2 + \lambda a_2 + \mu b_2$$
$$z = p_3 + \lambda a_3 + \mu b_3$$

- eliminate λ & μ: ... we get

$$x \cdot (b_3 a_2 - b_2 a_3) + y(b_1 a_3 - b_3 a_1) + z(b_2 a_1 - b_1 a_2)$$
$$= p_1 (\quad) + p_2 (\quad) + p_3 (\quad)$$

Has form $\underset{\sim}{r} \cdot \underset{\sim}{n} = c$ w/ $\underset{\sim}{n} = (b_3 a_2 - b_2 a_3,\ b_1 a_3 - b_3 a_1,\ b_2 a_1 - b_1 a_2)$, $c = \underset{\sim}{p} \cdot \underset{\sim}{n}$

can verify: $\underset{\sim}{a} \cdot \underset{\sim}{n} = \underset{\sim}{b} \cdot \underset{\sim}{n} = 0$, ie $\underset{\sim}{n}$ is normal to plane



__Ex 29__ Find distance b/t a pt $\underset{\sim}{q}$ and plane $\underset{\sim}{r} \cdot \underset{\sim}{n} = c$

Claim: At closest pt, $\underline{\underset{\sim}{r}^* - \underset{\sim}{q} = \lambda \underset{\sim}{n}}$

Also, $\underset{\sim}{r}^* \cdot \underset{\sim}{n} = c$ $\quad$ ↳ dot w/ $\underset{\sim}{n}$: $c - \underset{\sim}{q} \cdot \underset{\sim}{n} = \lambda |\underset{\sim}{n}|^2$ $\quad \therefore$ The min. distance is $|\lambda \underset{\sim}{n}| = |\lambda|\,|\underset{\sim}{n}| = \dfrac{|c - \underset{\sim}{q} \cdot \underset{\sim}{n}|}{|\underset{\sim}{n}|}$