Vector Product

Let $\underset{\sim}{u}, \underset{\sim}{v}$ be vec's in $\mathbb{R}^3$, $\underset{\sim}{u} = (u_1, u_2, u_3)$

$\underset{\sim}{v} = (v_1, v_2, v_3)$

We define the vector product, or cross product as $\underset{\sim}{u} \wedge \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - v_2u_3,$

$u_3v_1 - u_1v_3,$

$u_1v_2 - u_2v_1)$

also written $\underset{\sim}{u} \times \underset{\sim}{v}$

Notes

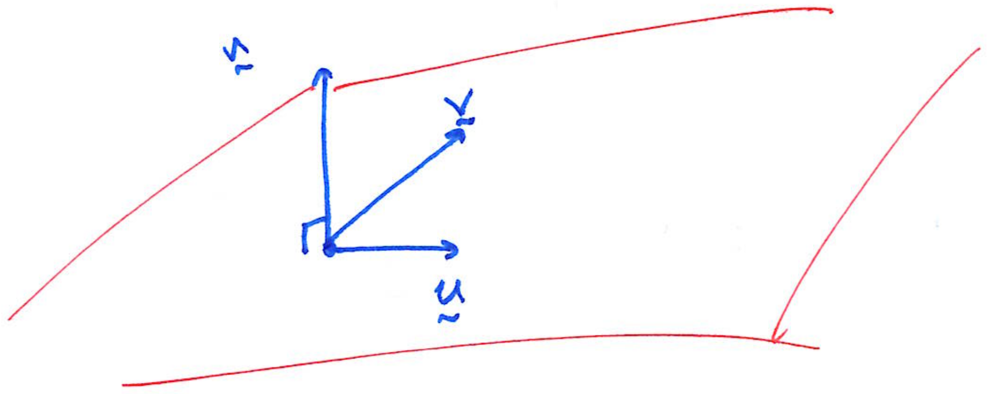
- $\underset{\sim}{u} \wedge \underset{\sim}{v}$ is a vector
- $\underset{\sim}{u} \wedge \underset{\sim}{v} = \underset{\sim}{0}$ iff $\underset{\sim}{u} = \lambda \underset{\sim}{v}$ for some λ (ie if $\underset{\sim}{u}, \underset{\sim}{v}$ are linearly dependent)
- $|\underset{\sim}{u} \wedge \underset{\sim}{v}|^2 = |\underset{\sim}{u}|^2 |\underset{\sim}{v}|^2 - (\underset{\sim}{u} \cdot \underset{\sim}{v})^2$ ← Problem sheets
- $(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) \wedge \underset{\sim}{w} = \alpha\, \underset{\sim}{u} \wedge \underset{\sim}{w} + \beta\, \underset{\sim}{v} \wedge \underset{\sim}{w}$

Geometric interpretation?

Observation 1 $(\underset{\sim}{u} \wedge \underset{\sim}{v}) \cdot \underset{\sim}{u} = 0 = (\underset{\sim}{u} \wedge \underset{\sim}{v}) \cdot \underset{\sim}{v}$

repeated row ⇒ determinant = 0

Proof $(\underset{\sim}{u} \wedge \underset{\sim}{v}) \cdot \underset{\sim}{u} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot (u_1\ u_2\ u_3) = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$



$\underset{\sim}{u} \wedge \underset{\sim}{v} = \lambda \underset{\sim}{n}$ w/ $\underset{\sim}{n}$ normal vector of plane spanned by $\underset{\sim}{u}, \underset{\sim}{v}$

Obs. 2

$|\underset{\sim}{u} \wedge \underset{\sim}{v}| = |\underset{\sim}{u}|\,|\underset{\sim}{v}| \sin\theta$, w/ θ is (smaller) angle b/t $\underset{\sim}{u}$, $\underset{\sim}{v}$

Pf $|\underset{\sim}{u} \wedge \underset{\sim}{v}|^2 = |\underset{\sim}{u}|^2 |\underset{\sim}{v}|^2 - \underbrace{(\underset{\sim}{u} \cdot \underset{\sim}{v})^2}_{|\underset{\sim}{u}|^2|\underset{\sim}{v}|^2\cos^2\theta} = |\underset{\sim}{u}|^2 |\underset{\sim}{v}|^2 (\underbrace{1 - \cos^2\theta}_{\sin^2\theta})$

* Thus $\underset{\sim}{u} \wedge \underset{\sim}{v}$ has max. magnitude when $\underset{\sim}{u}$ & $\underset{\sim}{v}$ are orthog.

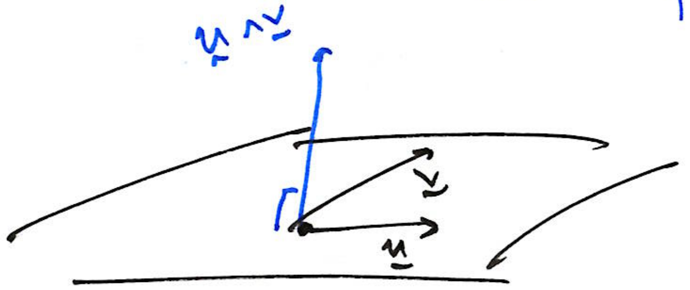
- Also, if $|\underset{\sim}{u}| = 1$, $|\underset{\sim}{v}| = 1$, and $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$, then $|\underset{\sim}{u} \wedge \underset{\sim}{v}| = 1$
  - in particular, note that $\underset{\sim}{i} \wedge \underset{\sim}{j} = \underset{\sim}{k}$, $\underset{\sim}{j} \wedge \underset{\sim}{k} = \underset{\sim}{i}$, $\underset{\sim}{k} \wedge \underset{\sim}{i} = \underset{\sim}{j}$

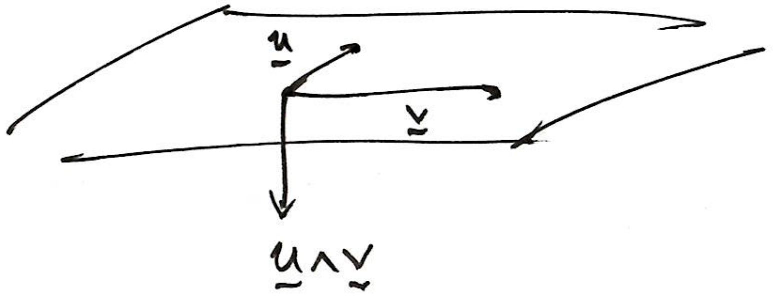
Obs. 3

Order matters! $\underset{\sim}{v} \wedge \underset{\sim}{u} = -(\underset{\sim}{u} \wedge \underset{\sim}{v})$

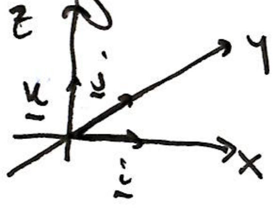
Which direction? Determined by "Right hand Rule"

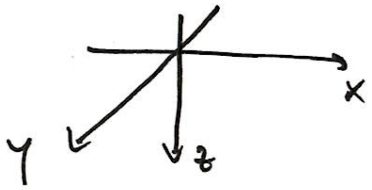
- orient ~~the~~ right hand st fingers point to $\underset{\sim}{u}$ when straight and point to $\underset{\sim}{v}$ when curled → then $\underset{\sim}{u} \wedge \underset{\sim}{v}$ is in direction of your (extended) thumb.





By convention, we use right handed coord. system:





Defn Let $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathbb{R}^3$. The *scalar triple product* is $\underset{\sim}{u} \cdot (\underset{\sim}{v} \wedge \underset{\sim}{w})$

- Often written $[\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}] = \underset{\sim}{u} \cdot (\underset{\sim}{v} \wedge \underset{\sim}{w})$

Notes 1. $[\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}] = \underset{\sim}{u} \cdot \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \leftarrow & \underset{\sim}{v} & \rightarrow \\ \leftarrow & \underset{\sim}{w} & \rightarrow \end{vmatrix} = \begin{vmatrix} \leftarrow & \underset{\sim}{u} & \rightarrow \\ \leftarrow & \underset{\sim}{v} & \rightarrow \\ \leftarrow & \underset{\sim}{w} & \rightarrow \end{vmatrix}$

2. $[\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}] = [\underset{\sim}{w}, \underset{\sim}{u}, \underset{\sim}{v}] = [\underset{\sim}{v}, \underset{\sim}{w}, \underset{\sim}{u}] = -[\underset{\sim}{u}, \underset{\sim}{w}, \underset{\sim}{v}] = -[\underset{\sim}{v}, \underset{\sim}{u}, \underset{\sim}{w}] = -[\underset{\sim}{w}, \underset{\sim}{v}, \underset{\sim}{u}]$

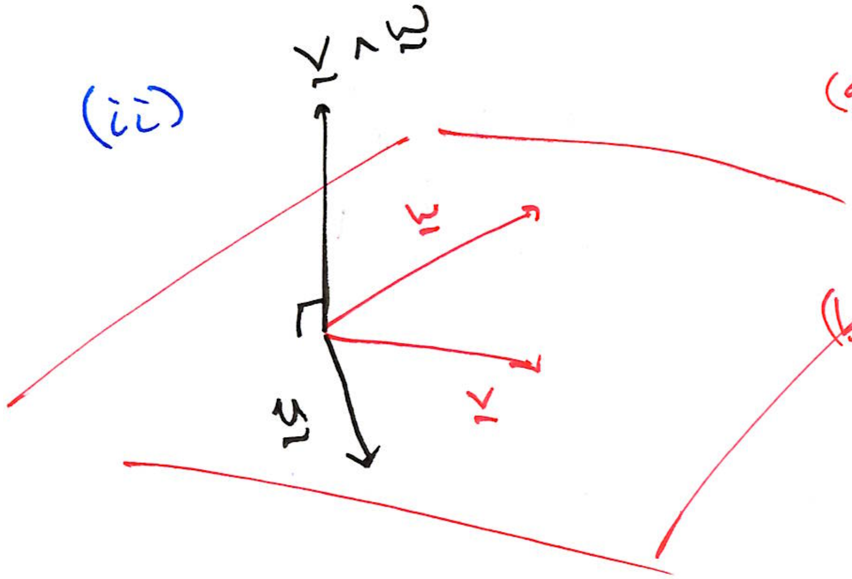
3. $[\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}] = 0$ iff $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ are *linearly dependent*

↳ $\exists\ c_1, c_2, c_3$ not all 0 st $c_1 \underset{\sim}{u} + c_2 \underset{\sim}{v} + c_3 \underset{\sim}{w} = \underset{\sim}{0}$

Proofs of 3: (i) if A is a square matrix,

$\det A = 0$ iff A is singular iff rows of A are lin. dependent

(ii)

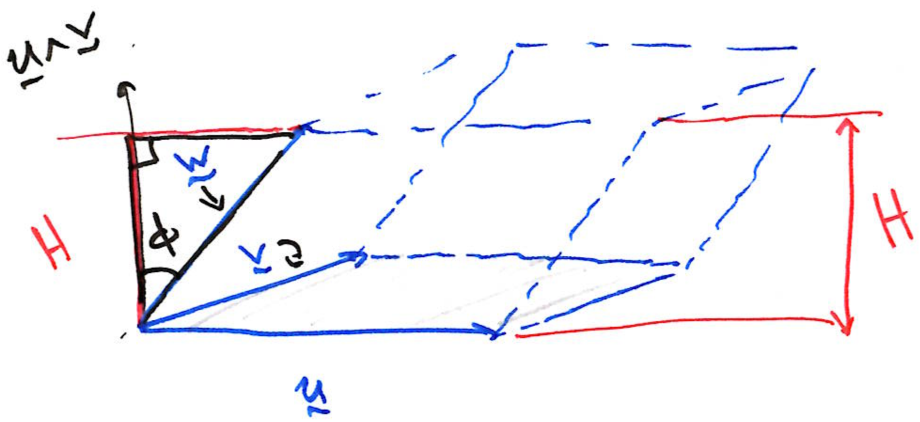


(a) $\underset{\sim}{v} \wedge \underset{\sim}{w} = \underset{\sim}{0}$ iff $\underset{\sim}{v} = \lambda \underset{\sim}{w}$ iff $\underset{\sim}{v}, \underset{\sim}{w}$ are lin. dep

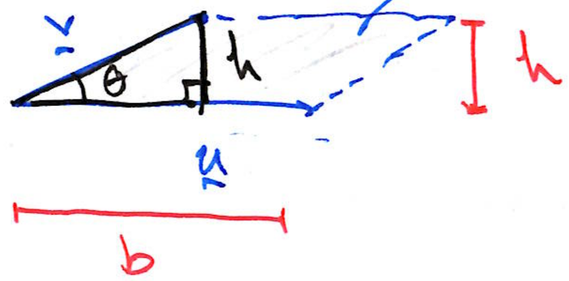
then $\underset{\sim}{u} \cdot (\underset{\sim}{v} \wedge \underset{\sim}{w}) = 0$

(b) $\underset{\sim}{v} \wedge \underset{\sim}{w} \neq \underset{\sim}{0}$ $\quad \underset{\sim}{u} \cdot (\underset{\sim}{v} \wedge \underset{\sim}{w}) = 0$ iff $\underset{\sim}{u}$ lives in plane spanned by $\underset{\sim}{v}$ & $\underset{\sim}{w}$ iff $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ lin dep.

- if construct a parallelepiped of $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$



claim: volume is $\left| [\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}] \right|$



area of parallelogram is $b \cdot h$

$b = |\underset{\sim}{u}|$, $h = |\underset{\sim}{v}| \sin\theta$

$\Rightarrow$ area $= |\underset{\sim}{u}| \, |\underset{\sim}{v}| \sin\theta = |\underset{\sim}{u} \wedge \underset{\sim}{v}|$

Volume of parallelepiped is $\underbrace{\text{area of base}}_{|\underset{\sim}{u} \wedge \underset{\sim}{v}|} \times \underset{\displaystyle H}{\underset{\shortparallel}{\text{height}}}$, $H = |\underset{\sim}{w}| \, |\cos\phi|$

- ϕ is angle between $\underset{\sim}{w}$ and $\underset{\sim}{u} \wedge \underset{\sim}{v}$

$$\Rightarrow \text{volume} = |\underset{\sim}{u} \wedge \underset{\sim}{v}| \, |\underset{\sim}{w}| \, |\cos\phi| = |(\underset{\sim}{u} \wedge \underset{\sim}{v}) \cdot \underset{\sim}{w}| = \left| [\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}] \right|$$

- Consider volume is zero iff "box is squashed", ie flat, ie $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ all live in same plane ie lin. dep!

Def^n Let $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ in $\mathbb{R}^3$, the vector triple product is $\underset{\sim}{u} \wedge (\underset{\sim}{v} \wedge \underset{\sim}{w})$. - this satisfies

$$\boxed{\underset{\sim}{u} \wedge (\underset{\sim}{v} \wedge \underset{\sim}{w}) = (\underset{\sim}{u} \cdot \underset{\sim}{w})\underset{\sim}{v} - (\underset{\sim}{u} \cdot \underset{\sim}{v})\underset{\sim}{w}}$$

Pf Let's choose $\underset{\sim}{i}$ to align w/ $\underset{\sim}{u}$. So $\underset{\sim}{u} = u\underset{\sim}{i}$, then

$$(\underset{\sim}{u} \cdot \underset{\sim}{w})\underset{\sim}{v} - (\underset{\sim}{u} \cdot \underset{\sim}{v})\underset{\sim}{w} = u w_1 \underset{\sim}{v} - u v_1 \underset{\sim}{w} = u(0, w_1 v_2 - v_1 w_2, w_1 v_3 - v_1 w_3)$$

$$\updownarrow \ \textcircled{=}$$

$$\underset{\sim}{i} \wedge \underset{\sim}{x} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 0 & 0 \\ x_1 & x_2 & x_3 \end{vmatrix} = (0, -x_3, x_2) \Rightarrow u\underset{\sim}{i} \wedge (\underset{\sim}{v} \wedge \underset{\sim}{w}) = u(0, w_1 v_2 - v_1 w_2, w_1 v_3 - v_1 w_3)$$

Def^n The scalar quadruple product : $(\underset{\sim}{a} \wedge \underset{\sim}{b}) \cdot (\underset{\sim}{c} \wedge \underset{\sim}{d})$

Claim $= (\underset{\sim}{a} \cdot \overset{c}{\cancel{\underset{\sim}{d}}})(\underset{\sim}{b} \cdot \underset{\sim}{d}) - (\underset{\sim}{a} \cdot \underset{\sim}{d})(\underset{\sim}{b} \cdot \underset{\sim}{c})$

Pf. $(\underset{\sim}{a} \wedge \underset{\sim}{b}) \cdot (\underset{\sim}{c} \wedge \underset{\sim}{d}) = [\underset{\sim}{a} \wedge \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}] = [\underset{\sim}{d}, \underset{\sim}{a} \wedge \underset{\sim}{b}, \underset{\sim}{c}] = \underset{\sim}{d} \cdot ((\underset{\sim}{a} \wedge \underset{\sim}{b}) \wedge \underset{\sim}{c})$

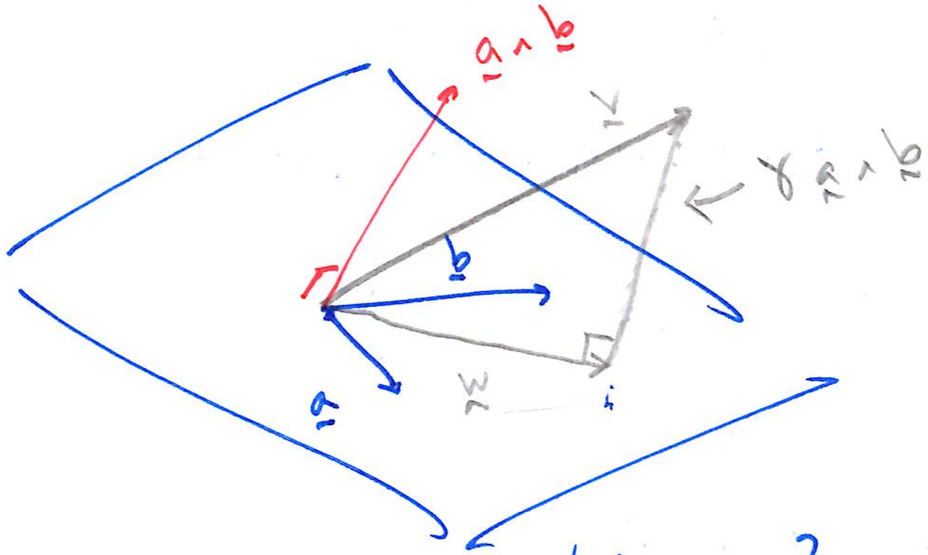
$= \underset{\sim}{d} \cdot \left((\underset{\sim}{c} \cdot \underset{\sim}{a})\underset{\sim}{b} - (\underset{\sim}{c} \cdot \underset{\sim}{b})\underset{\sim}{a} \right) = (\underset{\sim}{a} \cdot \underset{\sim}{c})(\underset{\sim}{b} \cdot \underset{\sim}{d}) - (\underset{\sim}{c} \cdot \underset{\sim}{b})(\underset{\sim}{a} \cdot \underset{\sim}{d})$

Ex. If $a, b \in \mathbb{R}^3$ are lin. independent, $\Big|\, \begin{array}{l} a \neq \gamma b \text{ for any } \gamma) \\ a \wedge b \neq 0 \} \end{array}$

then a, b, $a \wedge b$ form a basis for $\mathbb{R}^3$

- Formally, for any $v \in \mathbb{R}^3$, $\exists$ unique $\{\alpha, \beta, \gamma\}$ (scalars) st $v = \alpha a + \beta b + \gamma\, a \wedge b$

Pf



define $w = v - \gamma\, a \wedge b$ where $\gamma = \dfrac{v \cdot (a \wedge b)}{|a \wedge b|^2}$

$\uparrow$

$\cdot (a \wedge b)$, using $w \cdot (a \wedge b) = 0$

& $w = \alpha a + \beta b$ for some α, β

Unique? suppose $\alpha_1 a + \beta_1 b + \gamma_1\, a \wedge b = v = \alpha_2 a + \beta_2 b + \gamma_2\, a \wedge b$

Yes, ~~⇒~~ if $\underbrace{\alpha a + \beta b + \gamma\, a \wedge b = 0}$ ~~then~~ $\Rightarrow$ $\alpha = \beta = \gamma = 0$

↳ Dot w/ $a \wedge b \Rightarrow \gamma = 0$. $\Rightarrow \alpha = \beta = 0$ since a, b are lin. indep.

Let $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^3$. Consider all points $\underset{\sim}{r}$ st $\boxed{\underset{\sim}{r} \wedge \underset{\sim}{a} = \underset{\sim}{b}}$

be linearly indep.

- $\{\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{a} \wedge \underset{\sim}{b}\}$ forms basis $\Rightarrow$ can write $\underset{\sim}{r} = \alpha \underset{\sim}{a} + \beta \underset{\sim}{b} + \gamma\, \underset{\sim}{a} \wedge \underset{\sim}{b}$

So $\underset{\sim}{r} \wedge \underset{\sim}{a} = \alpha \overset{\underset{\sim}{0}}{\cancel{\underset{\sim}{a} \wedge \underset{\sim}{a}}} + \beta\, \underset{\sim}{b} \wedge \underset{\sim}{a} + \gamma \left((\underset{\sim}{a} \cdot \underset{\sim}{b}) \underset{\sim}{a} - (\underset{\sim}{a} \cdot \underset{\sim}{a}) \underset{\sim}{b} \right) = \underset{\sim}{b}$

Since $\{\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{a} \wedge \underset{\sim}{b}\}$ is a basis, any pt has unique representation

$\Rightarrow \beta = 0$. Case 1: if $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, $\gamma = \frac{1}{|\underset{\sim}{a}|^2}$

Case 2: if $\underset{\sim}{a} \cdot \underset{\sim}{b} \neq 0$ No soln's! ✓ γ can't equal 0 and $\frac{1}{|\underset{\sim}{a}|^2}$!

α is free! $\Rightarrow \boxed{\underset{\sim}{r} = \frac{\underset{\sim}{a} \wedge \underset{\sim}{b}}{|\underset{\sim}{a}|^2} + \alpha \underset{\sim}{a}}$ is soln for $\boxed{\text{any } \alpha}$

$\therefore$ $\underset{\sim}{r} \wedge \underset{\sim}{a} = \underset{\sim}{b}$, $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, is a <u>line</u> thru pt $\frac{\underset{\sim}{a} \wedge \underset{\sim}{b}}{|\underset{\sim}{a}|^2}$, in direction $\underset{\sim}{a}$

Compare: $\underset{\sim}{r} \cdot \underset{\sim}{n} = c$ plane w/ normal $\underset{\sim}{n}$

$\underset{\sim}{r} \wedge \underset{\sim}{a} = \underset{\sim}{b}$ line w/ dir. $\underset{\sim}{a}$

Ex Find intersection of $\underset{\sim}{r} \cdot \underset{\sim}{n} = c$, $\underset{\sim}{r} \wedge \underset{\sim}{a} = \underset{\sim}{b}$ $\quad (\underset{\sim}{a} \cdot \underset{\sim}{b} = 0)$

(expect 0, 1, or ∞ soln's)

- pts on line satisfy $\underset{\sim}{r} = \dfrac{\underset{\sim}{a} \wedge \underset{\sim}{b}}{|\underset{\sim}{a}|^2} + \alpha \underset{\sim}{a}$

plug into $\underset{\sim}{r} \cdot \underset{\sim}{n} = c$: $\underbrace{\boxed{\dfrac{(\underset{\sim}{a} \wedge \underset{\sim}{b})}{|\underset{\sim}{a}|^2}}}_{= \text{call } \underset{\sim}{p}} \cdot \underset{\sim}{n} + \alpha \, \underset{\sim}{a} \cdot \underset{\sim}{n} = c$ ↙ eq'n for α

Possibilities

- $\underset{\sim}{a} \cdot \underset{\sim}{n} \neq 0 \longrightarrow$ one unique sol'n for α, $\alpha = \dfrac{c - \underset{\sim}{p} \cdot \underset{\sim}{n}}{\underset{\sim}{a} \cdot \underset{\sim}{n}}$
- $\underset{\sim}{a} \cdot \underset{\sim}{n} = 0$ AND $\underset{\sim}{p} \cdot \underset{\sim}{n} = c \Rightarrow$ true for any α, so ∞'ly many sol'ns
- $\underset{\sim}{a} \cdot \underset{\sim}{n} = 0$ AND $\underset{\sim}{p} \cdot \underset{\sim}{n} \neq c \Rightarrow$ false for all α ⇒ No sol'ns!

line is in plane →

line is // to plane →