C3.1 Algebraic Topology

Oxford 2019

Prof. Alexander Ritter
ritter@maths.ox.ac.uk

Please be aware there are likely typos in these notes: comments/corrections are welcome!

Course Book

- Hatcher, Algebraic Topology — Chp. 2 & 3

  This is also freely available from the author's website. You are expected to read chapters 2 & 3.

Other references

- Ulrike Tillmann's C3.1 notes — see course page
- The discussion of orientations in my B3.2 Geometry of Surfaces notes may be helpful.

Other books

Massey, A basic course in Algebraic Topology

(Not "Algebraic Topology: An Introduction" ← does not treat homology)

MORE BASIC but full of ideas:

Fulton, Algebraic Topology : a first course

MORE ADVANCED:

May, A concise course in Algebraic Topology

Davis & Kirk, Lecture notes in Algebraic Topology

Bredon, Topology and Geometry

Bott & Tu, Differential forms in Algebraic Topology

Classics by **Spanier**, **Dold**, also see references in May's book

CONTENTS

0. OVERVIEW OF THE COURSE

Motivation, category theory, functors H_* and H^*: some computations

why functors are useful: Invariance of dimension, Brower fixed pt thm

1. ALGEBRAIC PRELIMINARIES : CHAIN COMPLEXES

Graded abelian groups

chain complexes, chain maps, subcomplex, quotient complex

chain map induces map on homology

exact sequence, snake lemma: SES induces LES on H_*, naturality of LES

5-Lemma, SES splits $\Leftrightarrow$ direct sum

2. Δ-COMPLEXES AND SIMPLICIAL HOMOLOGY

Δ^n, n-simplices, Δ-complex (structure), simplicial cx, triangulation

simplicial chain complex, $H_*^\Delta(S^n)$, $H_*^\Delta(T^2)$, remark about orientations

$H_*^\Delta(\sqcup \text{ conn.comp.}) \cong \oplus H_*^\Delta(\text{conn.comp.})$, $H_0^\Delta(X) \cong \mathbb{Z}^{\#\text{conn.comp.}}$

3. SINGULAR HOMOLOGY

Motivation, singular chain cx

naturality / functoriality, $H_*(\text{point})$

4. CHAIN HOMOTOPIES AND HOMOTOPY INVARIANCE

chain homotopy, prism operator

homotopic maps $f \simeq g$ (relative A), homotopy equivalent spaces $X \simeq Y$

contractible space, homotopic maps induce equal maps on homology

homotopy equivalence induces iso on H_*, $H_*(\mathbb{R}^n) = H_*(\mathbb{D}^n) = H_*(pt)$

pairs of spaces, relative homology $H_*(X,A)$, LES in H_* for pair

reduced homology $\widetilde{H}_*(X)$, LES for $\widetilde{H}_*$, $H_*(\mathbb{D}^n, S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})$

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5. EXCISION THEOREM AND QUOTIENTS

retractions, deformation retractions, excision thm, quotients,

good pairs $\Rightarrow H^*(X,A) \cong \widetilde{H}_*(X/A)$, generator of $H_n(S^n) \cong H_n(\mathbb{D}^n, S^{n-1}) \cong \mathbb{Z}$

another remark about orientations

6. MAYER-VIETORIS SEQUENCE

MV LES, $H_*(S^n)$

wedge sum $X \vee Y$, cone CX, suspension ΣX, connected sum $X \# Y$

7. DEGREE OF MAPS OF SPHERES

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local degree, proof of fundamental thm of algebra

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CW complexes, cellular complex, rank $H_n^{cw} \leq \#$n-cells

$H_*^{cw}(D^1 \times D^1)$, $H_*^{cw}(\mathbb{RP}^n)$, $H_*^{cw}(S^n)$, $H_*^{cw}(\Sigma_g)$

Δ-cx $\Rightarrow$ CW cx, $H_*^{cw}(X) \cong H_*^{\Delta}(X) \cong H_*(X)$, Axioms for homology

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10. CUP PRODUCT

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examples: $H^*(T^2)$, $H^*(\Sigma_2)$, remarks about intersection theory

11. KÜNNETH FORMULA AND PRODUCT SPACES

Tensor products of R-mods, tensor product of chain cxes,

algebraic Künneth thm, product spaces $X \times Y$, Euler characteristic χ

CW-cx for product space, Künneth thm, $H^*(S^n \times S^m)$, $H^*(T^n)$

12. UNIVERSAL COEFFICIENTS THEOREM

Universal coeff. thm, Background on Ext groups and free resolutions

(Co)homology with coefficients in a ring/field/module, $H^*(\mathbb{RP}^2; \mathbb{Z}/2)$

Univ. coeff. thm for PID R, Duality $H^*(X;\mathbb{F}) \simeq H_*(X;\mathbb{F})$ over fields

Structure thm for f.g. mods M over PID R, $\mathrm{Ext}^1_R(M;R)$, torsion shift H_* to H^{*+1}

13. MANIFOLDS: POINCARÉ-LEFSCHETZ DUALITY

Local orientation, orientation, fundamental class, degree, P. duality, L. duality,

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Alexander duality, Knot complements, Jordan curve thm

0. OVERVIEW OF THE COURSE

Motivation

Space X $\xrightarrow{\text{associate}}$ Algebraic object $A(X)$ like numbers, groups, rings, ...

Isomorphism of spaces $X \cong Y$ $\Longrightarrow$ Isomorphism $A(X) \cong A(Y)$

PAY-OFF Can tell spaces apart:

compute $A(X), A(Y)$ $\rightsquigarrow$ if $A(X) \not\cong A(Y)$ then $X \not\cong Y$

Examples

1\) Set $X \longrightarrow A(X) = \#X \in \mathbb{N}$

(bijection $X \to Y$) $\Longrightarrow$ same size

2\) Vector space $X \longrightarrow A(X) = \dim X \in \mathbb{N}$

(linear iso $X \to Y$) $\Longrightarrow$ same dim

3\) Topological Space X

$\longrightarrow$ $\#\pi_0(X) = \#$ **path components** $\} \in \mathbb{N}$

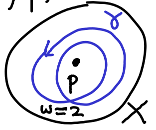
$\longrightarrow$ **#connected components** $\} \in \mathbb{N}$

$\longrightarrow$ $\chi(X)$ = **Euler characteristic** $\in \mathbb{Z}$

$\longrightarrow$ (for $X \subseteq \mathbb{R}^2$) Function $X \times \underbrace{\mathcal{L}X} \longrightarrow \mathbb{Z}$ ← loops $= C^0(S^1, X)$

$(p, \gamma) \longmapsto w(\gamma; p)$

Winding number of γ around p.



(Homeomorphism $X \to Y$) $\longrightarrow$ $A(X) = A(Y)$

CONVENTION for this course:

Space = topological space

Map of spaces = continuous map = cts map

For spaces, "$\cong$" means homeomorphism

"id" = identity map

All diagrams commute unless we say otherwise, e.g.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

means $\beta \circ \alpha = \delta \circ \gamma$

Category theory is the best language to phrase all this

Algebraic topology builds functors such as

$$\begin{tikzcd} & \left\{\begin{matrix}\text{Groups} \\ \text{and group homs}\end{matrix}\right\} \\ F : \left\{\begin{matrix}\text{Topological spaces} \\ \text{and continuous maps}\end{matrix}\right\} \arrow[ur] \arrow[r] \arrow[dr] & \left\{\begin{matrix}\text{Abelian groups} \\ \text{and group homs}\end{matrix}\right\} \\ & \left\{\text{Rings and ring homs}\right\} \end{tikzcd}$$

We will not use much category theory, just basic terminology:

Def A **category** C consists of the data:

$Ob(C)$ = a collection of **objects**

$Hom(A,B)$ = a set of **morphisms** between any $A, B \in Ob C$ ("**arrows**")

- with **composition** rule $Hom(B,C) \times Hom(A,B) \xrightarrow{\circ} Hom(A,C)$ which is associative.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

- with **identity** morphs $id_A \in Hom(A,A)$ s.t. $f \circ id_A = id_B \circ f = f$ $\forall (f : A \to B) \in Hom(A,B)$

Example
Sets = { sets with all maps between sets }
Top = { topological spaces with continuous maps }
Gps = { groups with group homs }

Def A (covariant) **functor** $F : C_1 \longrightarrow C_2$ is the data:

- an assignment $(A \in Ob\, C_1) \longmapsto (F(A) \in Ob\, C_2)$
- an assignment $\underset{Hom_{C_1}(A,B)}{\underset{\cap}{(A \xrightarrow{f} B)}} \longmapsto \underset{Hom_{C_2}(F(A),F(B))}{\underset{\cap}{(F(A) \xrightarrow{F(f)} F(B))}}$

Compatible with identities and compositions.

$$F(id_A) = id_{F(A)} \qquad F(g \circ f) = F(g) \circ F(f)$$

A **contravariant functor** is defined similarly except it reverses the direction of arrows: $(F(A) \xleftarrow{F(f)} F(B)) \in Hom(F(\underline{B}), F(\underline{A}))$
(so $F(g \circ f) = F(f) \circ F(g)$ reverses order of compositions)

Examples

1) $F : Top \longrightarrow Sets$, $A \longmapsto A$, $f \longmapsto f$ "forget the topology and continuity"

2) $F : Sets \longrightarrow Gps$, $A \longmapsto$ free abelian group generated by A

$$\mathbb{Z}\langle A \rangle = \left\{ \sum_{\text{finite}} n_i \cdot a_i \ : \ \begin{matrix} a_i \in A \\ n_i \in \mathbb{Z} \end{matrix} \right\}$$

$$(A \xrightarrow{f} B) \longmapsto \left(\begin{matrix} F(A) : \mathbb{Z}\langle A \rangle \longrightarrow \mathbb{Z}\langle B \rangle \\ \sum n_i \cdot a_i \longmapsto \sum n_i \cdot f(a_i) \end{matrix} \right)$$

When we say a construction is **natural** we mean functorial:

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[d, "f"'] \arrow[dd, bend right, "g \circ f"'] & A(X) \arrow[d, "A(f)"] \arrow[dd, bend left, "A(g \circ f) = A(g) \circ A(f)"] \\ Y \arrow[r, "A"] \arrow[d, "g"'] & A(Y) \arrow[d, "A(g)"] \\ Z \arrow[r, "A"] & A(Z) \end{tikzcd}$$

A : (a category of spaces) → (a cat. of algebraic objects)
The algebraic objects we assigned are assigned compatibly with maps of spaces, and the compatibility maps $A(f)$ are also compatible w.r.t. composition.
So we made compatible choices in constructing A.

Not to be confused with natural transformations of functors (later) which is about relating two such constructions A_1, A_2 in a compatible way

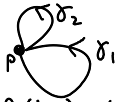
Example of a functor in algebraic topology

(see B3.5 Topology and Groups course)

$$\underset{\substack{\uparrow \\ \text{topological space}}}{\pi_1(X, p)} = \textbf{Fundamental group} = \left\{ \begin{matrix} \gamma \in \mathcal{L}X \\ \gamma(1) = p \end{matrix} \right\} \Big/ \begin{matrix}\text{continuous deformations} \\ \text{of loops based at } p\end{matrix}$$

$p \in X$

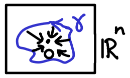
Group multiplication: concatenate loops $\gamma_1 * \gamma_2$



(each travelling twice as fast)

Examples

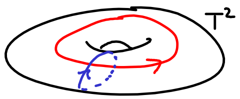
$\pi_1(\mathbb{R}^n) = 0$ ⟵ deform: $h : S^1 \times [0,1] \longrightarrow \mathbb{R}^n$, $h(t,s) = (1-s)\gamma(t)$



$\pi_1(S^1) \cong \mathbb{Z}$ ⟵ total # times wind around circle

$\pi_1(S^n) \cong 0$ $n \geq 2$ (not obvious)

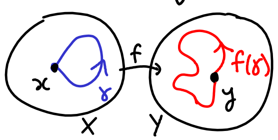
$\pi_1(\text{torus}) \cong \mathbb{Z}^2$



those loops generate π_1

FUNCTOR

$$\text{Based Top} = \left\{ \begin{matrix} \text{Topological spaces with choice of basepoint,} \\ \text{and continuous basepoint-preserving maps} \end{matrix} \right\} \xrightarrow{\pi_1} Gps$$



$$(X, p) \longmapsto \pi_1(X, p)$$

$$\left((X,x) \xrightarrow[f]{cts} (Y,y)\right) \longmapsto \left(\begin{matrix} \pi_1(X,x) \xrightarrow{gp. hom.} \pi_1(Y,y) \\ \gamma \longmapsto f \circ \gamma \end{matrix} \right)$$

Lemma Functors map isomorphisms to isomorphisms (iso. means ∃ inverse w.r.t. composition)

Pf $A \xrightarrow{f} B \xrightarrow{g} A$ ($\underbrace{}_{id}$) $\Rightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FA$ ($\underbrace{}_{F(id)=id}$), similarly for $B \xrightarrow{g} A \xrightarrow{f} B$ ($\underbrace{}_{id}$). □

Def **Natural transformation** $\alpha : F \longrightarrow G$ between functors $C_1 \overset{F}{\underset{G}{\rightrightarrows}} C_2$

is an association $(A \in \mathrm{Ob}\, C_1) \longmapsto (\alpha_A : F(A) \longrightarrow G(A))$ $\in \mathrm{Hom}_{C_2}(F(A), G(A))$

such that $(A \xrightarrow{f} B) \in \mathrm{Hom}_{C_1}(A,B)$ $\Rightarrow$

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"] & G(B) \end{tikzcd}$$

(commutes)

It is called a **natural isomomorphism** if each α_A is an isomorphism in C_2

Example of a natural transformation in algebraic topology

Let $H_1(X,p)$ = abelianisation of $\pi_1(X,p)$ (want to identify $ab=ba$ so quotient by $\langle aba^{-1}b^{-1}\rangle$) ← commutators

$\Rightarrow$ natural trans. $(\text{Based Top} \xrightarrow{\pi_1} \text{Gps}) \xrightarrow{\alpha} (\text{Based Top} \xrightarrow{H_1} \text{Gps})$

which associates $\underset{\in \text{BasedTop}}{(X,p)} \longmapsto (\alpha_{(X,p)} : \pi_1(X,p) \xrightarrow{\text{quotient}} H_1(X,p))$

Cultural Rmk **higher homotopy groups** $\pi_n(X,p) = \left\{ \underset{\text{basept} \mapsto p}{S^n \xrightarrow{\text{cts}} X} \right\} \Big/ \text{cts deform}^n$

FACT abelian for $n \geq 2$, but hard: e.g. $\pi_k(S^n)$ not all known.

We will not study these in this course.
We will study simpler invariants called HOMOLOGY groups $H_n(X)$

FACT (Hurewicz) ∃ natural transformation $\pi_n \longrightarrow H_n$
which will make sense at the end of course:
$f : S^n \xrightarrow{\text{cts}} X$ gives rise to a class $f_*[S^n] \in H_n(X)$.

Exercise to practice these notions from category theory:
Summarise your undergraduate linear algebra as follows:

1) ∃ functor $F : \underbrace{\left\{ \begin{matrix} \mathbb{R}^n \text{ for } n \in \mathbb{N} \\ \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) = \{m \times n \text{ matrices}\} \end{matrix} \right\}}_{\text{Mat}} \to \underbrace{\left\{ \begin{matrix} \text{finite dimensional real vector spaces} \\ \text{and linear maps} \end{matrix} \right\}}_{\text{Vect}}$

2) A choice of basis for each vector space V determines a functor $G : \text{Vect} \to \text{Mat}$

3) Construct natural isomorphisms $G \circ F \xrightarrow{\alpha} \mathrm{Id}_{\text{Mat}}$, $F \circ G \xrightarrow{\beta} \mathrm{Id}_{\text{Vect}}$

When functors satisfying such natural isos exist, the categories are called **equivalent** (not isomorphic). So Mat, Vect are equivalent categories.

Aim of the course: build a functor

HOMOLOGY $H_* : \text{Top} \longrightarrow$ **Graded abelian groups**

$X \longmapsto H_*(X)$ ← grading $* \in \mathbb{Z}$

$(X \to Y) \longmapsto \left(H_*(X) \longrightarrow H_*(Y) \text{ grading preserving hom} \right)$

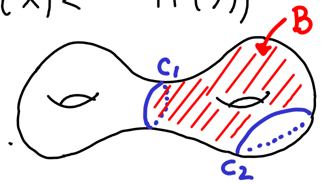
and a contravariant functor

COHOMOLOGY $H^* : \text{Top} \longrightarrow$ **Graded rings**

$X \longmapsto H^*(X)$

$(X \to Y) \longmapsto (H^*(X) \longleftarrow H^*(Y))$

Rough idea:
$H_* X$ is generated by "nice" subspaces $C \subseteq X$ which have no boundary: $\partial C = \emptyset$, modulo identify C_1, C_2 if $C_1 \cup C_2$ arises as a boundary ∂B.
Call such C_1, C_2 **homologous**.



Facts

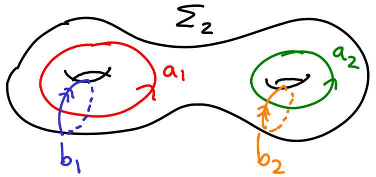
- $H_0(X) \cong \bigoplus_{\pi_0 X} \mathbb{Z}$ ← $\pi_0 X = \{\text{path-connected components}\}$ ← generated by a point in each path-comp.
- $X = \sqcup X_i$ path-components $\Rightarrow H_*(X) \cong \bigoplus H_*(X_i)$
- $\chi(X) = \sum_{d \geq 0} (-1)^d \operatorname{rank} H_d(X)$

↑ **Euler characteristic** ; rank ↑ max # $\mathbb{Z}$-linearly independent elements

Example: compact surfaces

$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}^{2g} & *=1 \\ \mathbb{Z} & *=2 \\ 0 & \text{else} \end{cases}$

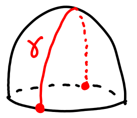
↑ orientable surface genus g
$\chi = 2-2g$



We will show that those 4 loops generate $H_1(\Sigma_2)$

$H_*(N_h) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/_2 \oplus \mathbb{Z}^{h-1} & *=1 \\ 0 & \text{else} \end{cases}$

↑ non-orientable surface S^2 with h Möbius bands attached
$\chi = 2-h$



$N_1 = \mathbb{RP}^2 = S^2/\pm \mathrm{Id}$

Notice γ is a loop.
It generates $H_1(N_1)$

Examples of homology calculations

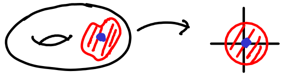
$$H_*(\mathbb{R}^n) \cong H_*(\mathbb{D}^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ 0 & \text{else} \end{cases}$$

n-dimensional ball $\mathbb{D}^n = \{x : \|x\| \leq 1\} \subseteq \mathbb{R}^n$

$$H_*(S^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$$

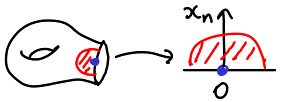
$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ n-dim sphere

Hausdorff top. space s.t. each pt has an open neighbourhood homeo to an open ball in $\mathbb{R}^n$ → **n-dimensional manifolds** X



$$H_*(X) = \begin{cases} 0 & \text{for } * < 0 \\ 0 & \text{for } * > n \\ \mathbb{Z} & \text{for } * = n \text{ for connected orientable compact manifold} \\ 0 & \text{for } * = n \text{ for } \begin{cases} \text{non-orientable} & '' \\ \text{non-compact} & '' \\ \text{connected manifolds with boundary} \neq \emptyset \end{cases} \end{cases}$$

boundary point has an open nbhd homeo to open nbhd of $0 \in$ half-space: $\{x \in \mathbb{R}^n : x_n \geq 0\}$

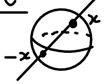


Rmk M compact connected n-mfd

$$\Rightarrow H_{n-1}(M) \cong \begin{cases} \mathbb{Z}^k & \text{some } k \geq 0 \text{ if orientable} \\ \mathbb{Z}^k \oplus \mathbb{Z}/2 & '' \quad '' \text{ non-orientable} \end{cases}$$

$$H_*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & \text{odd } * = 1, 3, 5, \ldots < n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

$\mathbb{RP}^n$: $S^n/\pm id$ real projective space



$\mathbb{RP}^n$ orientable ⇔ n odd (e.g. $\mathbb{RP}^1 \cong S^1$)

$$H_*(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \text{even } * = 0, 2, 4, \ldots, 2n \\ 0 & \text{else} \end{cases}$$

e.g. $\mathbb{CP}^1 \cong S^2$ stereographic projection

$\mathbb{CP}^n$: space of complex lines through $0 \in \mathbb{C}^{n+1}$

Complex projective space $\cong (\mathbb{C}^n \setminus 0)/\mathbb{C}^*\text{-rescaling}$
$= \{[z_0, z_1, \ldots, z_n] : z_j \in \mathbb{C} \text{ not all } 0\} / [z] = [\lambda z] \text{ for } \lambda \in \mathbb{C}^*$

Examples of cohomology calculations

$H^0(X) = \prod_{\pi_0 X} \mathbb{Z}$ ← if $\pi_0 X$ finite, then $\cong \bigoplus_{\pi_0 X} \mathbb{Z} \cong H_0 X$
but if infinite then not: here allow only finite sums

$H^*(X) \cong \prod H^*(X_i)$ ← X_i path-components of X

FACT If $H_n(X)$ finitely generated abelian gp, so
$H_n(X) \cong \mathbb{Z}^{r_n} \oplus T_n$ ← T_n = torsion elements = elements of finite order
Then $H^n(X) \cong \mathbb{Z}^{r_n} \oplus \underline{\underline{T_{n-1}}}$ as abelian groups

⇒ $H^*(\Sigma_g), H^*(\mathbb{R}^n), H^*(\mathbb{D}^n), H^*(S^n), H^*(\mathbb{CP}^n)$ same as for H_*, but:

$$H^*(N_h) = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}^{h-1} & *=1 \\ \mathbb{Z}/2 & *=2 \\ 0 & \text{else} \end{cases} \qquad H^*(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & *=0 \\ 0 & *=1 \\ \mathbb{Z}/2 & *=2 \\ 0 & \text{else} \end{cases} \ (h=1) \qquad H_*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/2 & * \text{ even} = 2, 4, \ldots \leq n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

and $H^n(\text{non-orientable compact n-mfd}) \cong \mathbb{Z}/2$.

⇒ The interesting feature is the ring structure:

$H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/x^{n+1}$ grading: $|x| = 2$ $\mathbb{Z}[x]$ = polynomials in x with $\mathbb{Z}$-coefficients

$H^*(S^n) \cong \mathbb{Z}[x]/x^2$ $|x| = n$

$H^*(T^n) \cong \Lambda[x_1, \ldots, x_n]$ $|x_i| = 1$

$T^n = S^1 \times \cdots \times S^1$ **n-torus**

$$\text{e.g. } n=2 \text{ (torus)} \begin{cases} \mathbb{Z} \cdot 1 & *=0 \\ \mathbb{Z}x_1 \oplus \mathbb{Z}x_2 & *=1 \\ \mathbb{Z}\, x_1 \wedge x_2 & *=2 \\ 0 & \text{else} \end{cases}$$

exterior algebra generated$/\mathbb{Z}$ by symbols $x_{i_1} \wedge \cdots \wedge x_{i_k}$ with $i_1 < \cdots < i_k$
product given by $\wedge$ using relations $x_i \wedge x_j = -x_j \wedge x_i$.

$H^*(\mathbb{RP}^{2n}) \cong \mathbb{Z}/2[x]/x^{n+1}$ $|x| = 2$

$H^*(\mathbb{RP}^{2n+1}) \cong \mathbb{Z}/2[x]/x^{n+1} \oplus \underbrace{\mathbb{Z}[-2n-1]}$ ← means: a copy of $\mathbb{Z}$ in degree $2n+1$

$H^*(\Sigma_g) \cong \Lambda[a_1, \ldots, a_g, b_1, \ldots, b_g] / \langle a_i b_j \text{ for } i \neq j,\ a_i b_i - a_j b_j,\ a_i a_j,\ b_i b_j \rangle$
$|a_i| = |b_i| = 1$ ← exterior alg. instead of poly.alg since $a_i b_i = -b_i a_i$

Why more information?

connected sum: remove a ball in each, glue along ∂ball

$S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ have same $H_* = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} \oplus \mathbb{Z} & *=2 \\ \mathbb{Z} & *=4 \end{cases}$

but the rings H^* are not iso, hence $S^2 \times S^2 \not\cong \mathbb{CP}^2 \# \mathbb{CP}^2$.

Example of why such functors are useful

Suppose $\exists\ F_* : \mathrm{Top} \longrightarrow \mathrm{Gps}$ functors s.t.

① $F_*(S^n) \neq 0 \Leftrightarrow * = n$ and ② $F_*(\mathbb{D}^n) = 0$ **all** $*$

<u>Rmk</u> We'll build such an F_*: **reduced homology $\widetilde{H}_*$**

s.t. $\widetilde{H}_* = H_*$ for $* \neq 0$, and $\widetilde{H}_0 \cong \mathbb{Z}^{\#(\text{path-components})-1}$

"homeomorphisms preserve dimension"

Non-trivial result because there are space-filling curves. e.g. Peano (1890) $\exists$ cts surjection $\underset{\text{interval}}{[0,1]} \to \underset{\text{square}}{[0,1]^2}$. The theorem implies this is <u>not</u> injective. (cts. bij: compact $\to$ Hausdorff $\Rightarrow$ homeo)

<u>Theorem</u> **Invariance of dimension**

$$\boxed{\begin{array}{lll} S^n \cong S^m & \Leftrightarrow & n=m \\ \mathbb{R}^n \cong \mathbb{R}^m & \Leftrightarrow & n=m \end{array}}$$

by ①

<u>Pf</u> Lemma $\Rightarrow F_n(S^n \xrightarrow{\cong} S^m)$ is iso $\underset{\neq 0}{F_n(S^n)} \xrightarrow{\cong} \underset{= 0 \text{ if } n\neq m}{F_n(S^m)}$ of gps. ✓

If $\mathbb{R}^n \underset{\varphi}{\xrightarrow{\cong}} \mathbb{R}^m$, then can extend φ to the one-point compactifications: $S^n \cong \mathbb{R}^n \cup \{\infty\} \xrightarrow{\cong} \mathbb{R}^m \cup \{\infty\} \cong S^m$, $\infty \mapsto \infty$. □

("Alexandroff extension")

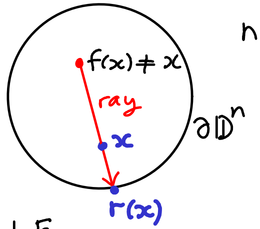
stereographic projection $(x_0, \dots, x_n) \mapsto \dfrac{(x_1, \dots, x_n)}{1-x_0}$

<u>Rmk</u> new open neighbourhoods at ∞ are $\{\infty\} \cup (\mathbb{R}^n \setminus C)$ where C is (closed &) compact. The extended map is cts since $\varphi^{-1}(C)$ is (closed &) compact since φ^{-1} is homeo.

<u>Theorem</u> **Brower fixed point thm** by ① & ②

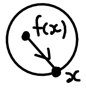
$f : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ continuous $\Rightarrow f$ has a fixed point ($f(p) = p$ some p)

<u>Proof</u> Suppose not. Let $r(x) = (\text{ray from } f(x) \text{ to } x) \cap \partial\mathbb{D}^n$



notice: • $r : \mathbb{D}^n \longrightarrow \partial\mathbb{D}^n = S^{n-1}$ continuous

• $r|_{\partial\mathbb{D}^n} = id_{S^{n-1}}$

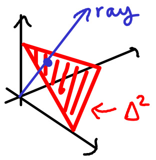


$$S^{n-1} = \partial\mathbb{D}^n \xrightarrow{\text{inclusion } i} \mathbb{D}^n \xrightarrow{r} S^{n-1}, \quad r \circ i = id$$

$\overset{\text{apply } F_{n-1}}{\Longrightarrow}$ $F_{n-1}(r) \circ F_{n-1}(i) = id \Longrightarrow F_{n-1}(i)$ injective $\underset{\neq 0}{F_{n-1}(S^{n-1})} \to \underset{= 0}{F_{n-1}(\mathbb{D}^n)}$ ↯ □

<u>Example</u> (Brower) $A = n \times n$ matrix, $A_{ij} > 0$ real $\Longrightarrow \exists$ evalue $\lambda > 0$ with real evector $(v_1, \dots, v_n)$ with $v_i \geq 0$

<u>Hint</u> $\mathbb{R}^3 \supseteq$



$X = \{\text{rays in "positive octant"}\}$ ← $x \in \mathbb{R}^n : x_i > 0\ \forall i$

notice $AX \subseteq X$

notice $X \cong \Delta^n = \{x \in \text{octant} : \sum x_i = 1\} \cong \mathbb{D}^n$

$\text{ray} \mapsto \text{ray} \cap \Delta^n$

1. ALGEBRAIC PRELIMINARIES : CHAIN COMPLEXES

Graded abelian groups

<u>Def</u> A **$\mathbb{Z}$-graded abelian group** C is an abelian group together with a direct sum decomposition

$$\boxed{C = \bigoplus_{n \in \mathbb{Z}} C_n} \leftarrow \text{abelian group}$$

<u>Convention</u>: always grade by $\mathbb{Z}$ unless say otherwise.

<u>Example</u> $C = \mathbb{Z}[x] = $ integer polynomials in x, $C_n = \mathbb{Z} \cdot x^n$ ← so grading by degree

A graded ab.gp. A is a **graded subgp** of C if • subgp • $A_n \subseteq C_n$.

A **homomorphism** $h : C \longrightarrow D$ of gr.ab.gps is hom of gps s.t.

$$h(C_n) \subseteq D_n$$

A **hom of degree k** is hom with

$$h(C_n) \subseteq D_{n+k}$$

Shift by k: $\mathbb{Z}$-gr.ab.gp. $C[k]$ with

$$\boxed{C[k]_n = C_{k+n}}$$

Notice: $C[k]_0 = C_k$ is now in degree zero, so shifted <u>down</u> by k

$\Rightarrow$ Can view gr. hom of deg k as a gr. hom $h : C \longrightarrow D[k]$

Abelian groups which are finitely generated

recall f.g. means $\exists$ surjection $\mathbb{Z}^m \to G$ for some m

<u>FACT</u> Finitely generated abelian groups are classified:

$$G \cong \underbrace{\mathbb{Z}^r}_{\text{free part}} \oplus \underbrace{\mathbb{Z}/_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}/_{p_k^{n_k}}}_{\text{torsion part}}$$

r called **rank G**

$n_i \in \mathbb{Z}$

p_i primes (possibly not distinct)

compare finite dimensional vectorspaces / field $\mathbb{F}$: $V \cong \mathbb{F}^r$ ← $r = \dim V$

Chain complexes

Def A chain complex $\overset{\text{differential or boundary homomorph}}{(C_*,\partial_*)}$ is a gr. ab. gp. C together with a hom ∂ of degree -1 such that $\partial\circ\partial=0$.

Thus:

$$\boxed{\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[rr, bend right, "\partial_n\circ\partial_{n+1}=0"'] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r] & \cdots \end{tikzcd}}$$

hence $\underset{\substack{\shortparallel\\ B_n}}{\operatorname{Im}\partial_{n+1}} \subseteq \underset{\substack{\shortparallel\\ Z_n}}{\operatorname{Ker}\partial_n}$

n-boundaries n-cycles

n-chains = elements of C_n

Now consider "cycles modulo boundaries":

Def The homology of (C_*,∂_*) is the gr. ab. gp.

$$\boxed{H_n(C_*,\partial_*) = \operatorname{Ker}\partial_n \big/ \operatorname{Im}\partial_{n+1}}$$

often abbreviate by $H_*(C)$

Two cycles are called homologous if they differ by a boundary

Def A chain map $h : (C_*,\partial_*) \longrightarrow (\tilde{C}_*,\tilde{\partial}_*)$ is a graded hom such that $\boxed{h\circ\partial_* = \tilde{\partial}_*\circ h}$

Example A chain subcomplex $C_*\subseteq\tilde{C}_*$ is a graded subgp with $\partial_* =$ restriction of $\tilde{\partial}_*$ to C_*.
So the inclusion $\text{incl} : (C_*,\partial_*)\longrightarrow(\tilde{C}_*,\tilde{\partial}_*)$ is a chain map.
Also get quotient complex $\tilde{C}_*/C_*$
with $\tilde{\partial}_*[\tilde{c}] = [\tilde{\partial}_*\tilde{c}]$ (well-defined: $\tilde{\partial}_*C_* = \partial_*C_* \subseteq C_*$)

Lemma A chain map induces a hom on homology

$$\begin{aligned} H_*(h) = h_* : H_*(C_*,\partial_*) &\longrightarrow H_*(\tilde{C}_*,\tilde{\partial}_*)\\ [x] &\longmapsto [h(x)] \end{aligned}$$

Proof $\operatorname{Ker}\partial_n \xrightarrow{\text{hom}} \operatorname{Ker}\tilde{\partial}_n$
$x \longmapsto h(x)$ since $\tilde{\partial}(h(x)) = h(\underbrace{\partial x}_{=0}) = 0$

Need $\operatorname{Im}\partial_n \longrightarrow \operatorname{Im}\tilde{\partial}_n$ to get well-defined hom

$H_n(C) = \operatorname{Ker}\partial_n/\operatorname{Im}\partial_{n+1} \longrightarrow \operatorname{Ker}\tilde{\partial}_n/\operatorname{Im}\tilde{\partial}_{n+1} = H_n(\tilde{C})$

Proof: $h(b) = h(\partial c) = \tilde{\partial}h(c) \in \operatorname{Im}(\tilde{\partial})$. □
$\uparrow b = \partial c\in\operatorname{Im}(\partial)$

The last step was a very simple example of a proof by "diagram chasing"

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "h_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "h_n"] & C_{n-1} \arrow[r] \arrow[d, "h_{n-1}"] & \cdots \\ \cdots \arrow[r] & \tilde{C}_{n+1} \arrow[r, "\tilde{\partial}_{n+1}"'] & \tilde{C}_n \arrow[r, "\tilde{\partial}_n"'] & \tilde{C}_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} c \arrow[r, "\partial"] \arrow[d, "h"'] & \partial c = b \arrow[d, "h"] \\ hc \arrow[r, "\tilde{\partial}"] & \tilde{\partial}(hc) = h\partial c = h(b) \end{tikzcd}$$

□

Def (C_*,∂_*) is exact (or acyclic) if $H_*(C)=0$
so $\operatorname{Im}\partial_{n+1} = \operatorname{Ker}\partial_n$

In general exact sequence means $\boxed{\text{"}\operatorname{Im}(\text{previous map}) = \operatorname{Ker}(\text{next map})\text{"}}$

A short exact sequence (SES) is an exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

Easy exercise

$$\left(0\to A \xrightarrow{i} B \xrightarrow{\pi} C\to 0\right)_{\text{exact}} \Leftrightarrow \begin{cases} i & \text{injective}\\ \pi & \text{surjective}\\ B/_{i(A)} \cong C \text{ via } [b]\longmapsto \pi(b)\end{cases}$$

Examples

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r, "\text{mod } 2"] & \mathbb{Z}/_2 \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\text{inclusion}"] & \mathbb{Z}\oplus\mathbb{Z}/_2 \arrow[r, "\text{project}"] & \mathbb{Z}/_2 \arrow[r] & 0 \end{tikzcd}$$

Note A, C do not determine B.

Snake Lemma A SES of chain complexes and chain maps yields a long exact sequence (LES) on homology:

$$\boxed{\cdots\to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{\pi_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*[-1]} \cdots}$$

degree -1 map $H_*(C)\to H_*(A)[-1]$ called connecting map

(So exact triangle:
$$\begin{tikzcd} H_*(A) \arrow[rr] & & H_*(B) \arrow[dl] \\ & H_*(C) \arrow[ul, "{[-1]}"] & \end{tikzcd}$$
)

Pf simplify notation by identifying A with $i(A)\subseteq B$: $\overset{\in A}{a} \equiv \overset{\in B}{i(a)}$, $A \xhookrightarrow{i} B$, $\partial a \equiv i\partial a = \partial i a$

$\Longrightarrow$ now $A_*\subseteq B_*$ inclusion of subcomplex:

$$0\longrightarrow (A_*,\partial_*) \xrightarrow{\text{incl}} (B_*,\partial_*) \xrightarrow{\pi} (C_*,\tilde{\partial}_*) \longrightarrow 0$$

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0 \\ & & \exists b \arrow[r, "\text{surj.}"] \arrow[d] & \underset{c}{\text{cycle}} = \pi(b) \arrow[d] & \\ & \partial b \arrow[r] & \partial b \arrow[r] & \tilde{\partial} c = 0 & \end{tikzcd}$$

$\nwarrow$ lifts to A by exactness

Define $\delta: H_*(C) \longrightarrow H_*(A)[-1]$
$c \longmapsto \partial b$ $\nwarrow$ where $b\in\pi^{-1}(c)$

(typically b is not in A, so ∂b need not be a bdry in A)

Well-defined?
- $\pi^{-1}(c)=\{b+a : a\in A\}$ and $\partial(b+a) = \partial b + \underbrace{\partial a}_{\text{boundary in } A}$ ✓
- cycle $\to$ cycle: $\partial(\partial b) = 0$ ✓
- boundary $\to$ boundary:

$$\begin{tikzcd} \exists \beta \arrow[r, "\text{surj.}"] \arrow[d] & x \in C_{n+1} \arrow[d] \\ \partial\beta \arrow[r] & \text{boundary } c = \tilde{\partial} x \arrow[d] \\ & 0 \end{tikzcd}$$

$\Rightarrow$ can pick $b = \partial\beta$

$\Rightarrow \partial b = \partial\partial\beta = 0$ ✓

Exactness at $H_n(C)$ (exercise: check exactness at H_*A, H_*B):

Need $\operatorname{Im}\pi_* = \operatorname{Ker}\delta$:

$\subseteq$: $\delta(\underset{\text{cycle}}{\pi_* b}) = \partial b = 0$ ✓

$\supseteq$:

$$\begin{tikzcd} \exists a \arrow[d] & \overset{\text{not necessarily cycle!}}{b} \arrow[r] \arrow[d] & \overset{\in \operatorname{Ker}\delta}{c} = \pi_* b \arrow[d] \\ \partial a \equiv \delta c = \partial b \arrow[r] & \partial b \arrow[r] & 0 \end{tikzcd}$$

assumption $\delta c = 0 \in H_* A$

$\pi_*(b-a) \overset{\pi_* A = 0}{=} c$

$\partial\underbrace{(b-a)}_{\text{thus cycle!}} = \partial b - \partial a = 0$

$\Rightarrow c = \pi_*(b-a) \in \operatorname{Im}\pi_*$ □

Rmk $0\to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ SES $\Rightarrow$ the connecting map of LES is

$$\boxed{\begin{aligned} \delta: H_*(C) &\longrightarrow H_*(A)[-1] \\ c &\longmapsto i^{-1}(\partial b) \end{aligned}}$$

$\forall\, b\in B$ with $\pi(b) = c$.

Lemma The construction of δ is natural (i.e. functorial)

Pf

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "\pi"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & \tilde{A} \arrow[r, "\tilde{i}"] & \tilde{B} \arrow[r, "\tilde{\pi}"] & \tilde{C} \arrow[r] & 0 \end{tikzcd}$$

all chain maps $\rightarrow$

$$\begin{tikzcd} \delta c = a \arrow[r] \arrow[d, "f"] & \partial b \arrow[d, "g"] & b \arrow[r] \arrow[d, "g"] & c \arrow[d, "h"] \\ fa \arrow[r] & g\partial b = \tilde{\partial} g b = \delta h c & gb \arrow[r] & hc \end{tikzcd}$$

$$\begin{aligned} \Rightarrow \delta h c &= \tilde{i}^{-1}\tilde{\partial} g b \\ &= \tilde{i}^{-1} g \partial b \\ &= f a \\ &= f\delta c \end{aligned}$$ □

Exercise Deduce the LES is natural, so

$$\begin{tikzcd} \cdots \arrow[r] & H_* A \arrow[r, "i_*"] \arrow[d, "f_*"] & H_* B \arrow[r, "\pi_*"] \arrow[d, "g_*"] & H_* C \arrow[r, "\delta"] \arrow[d, "h_*"] & H_{*-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\ \cdots \arrow[r] & H_*\tilde{A} \arrow[r] & H_*\tilde{B} \arrow[r, "\tilde{\pi}_*"] & H_*\tilde{C} \arrow[r, "\delta"] & H_{*-1}(\tilde{C}) \arrow[r] & \cdots \end{tikzcd}$$

5-Lemma

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha", "\cong"'] & B \arrow[r] \arrow[d, "\beta", "\cong"'] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta", "\cong"'] & E \arrow[d, "\varepsilon", "\cong"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

exact rows $\Rightarrow$ γ also iso.

Pf exercise (diagram chase) □

Splitting Lemma

Cor $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ SES of abelian gps

If $\begin{tikzcd} B \arrow[r, "\beta"] & C \arrow[l, bend left, "\exists \gamma"] \end{tikzcd}$ s.t. $\beta \circ \gamma = id_C$ then the SES **splits**: $B \cong A \oplus C$

(converse is obvious)

Pf

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A\oplus C \arrow[r] \arrow[d, "\alpha+\gamma"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

□

Exercise If $\begin{tikzcd} A \arrow[r, "\alpha"] & B \arrow[l, bend left, "\exists \mu"] \end{tikzcd}$ s.t. $\mu \circ \alpha = id_A$ then it splits: $B \xrightarrow[\mu \oplus \beta]{\cong} A \oplus C$

Exercise If C is a free abelian group $(C \cong \bigoplus_{i \in I} \mathbb{Z})$ then the SES splits.

Rmk A free $\not\Rightarrow$ splits, e.g. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \longrightarrow 0$

Cultural Rmk Splitting Lemma generalises the **rank-nullity theorem** from linear algebra: $V \xrightarrow{\beta} W$ linear map of vector spaces $\Rightarrow$ $\text{Im}\beta \oplus \text{Ker}\beta \cong V$

Pf $0 \to \text{Ker}\,\beta \xrightarrow{\text{incl}} V \xrightarrow{\beta} \text{Im}\,\beta \to 0$ is SES, and splits since $\text{Im}\beta$ free.

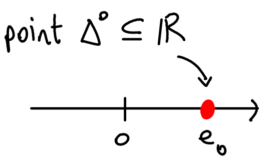
2. Δ-COMPLEXES AND SIMPLICIAL HOMOLOGY

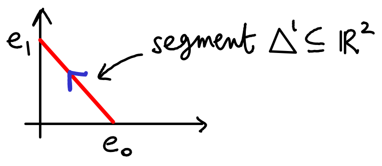
Standard n-simplex

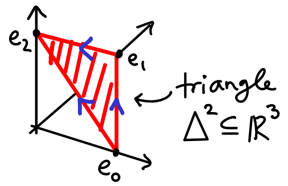
$$\Delta^n = \left\{ \underset{\textstyle \sum t_i e_i}{\overset{}{(t_0, \dots, t_n)}} \in \mathbb{R}^{n+1} : \begin{array}{l} t_i \geq 0 \\ \sum t_i = 1 \end{array} \right\}$$

$(t_0,\dots,t_n) = \sum t_i e_i$ ← standard basis of $\mathbb{R}^{n+1}$ $e_0, \dots, e_n$ $(e_0 = (1,0,\dots,0), \dots$

Examples

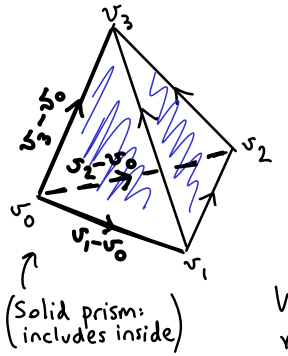






Def For $\{v_0, \dots, v_n\} \subseteq \mathbb{R}^{n+k}$ s.t. ← any $k \geq 0$

$v_1 - v_0, \dots, v_n - v_0$ $\mathbb{R}$-linearly independent



(Solid prism: includes inside)

$[v_0, \dots, v_n]$ = **n-Simplex** spanned by $v_0, \dots, v_n$

= convex hull of $v_0, \dots, v_n$

$= \{ \sum t_i v_i : t_i \geq 0 \text{ and } \sum t_i = 1 \}$

= Image of linear homeo $\sigma : \Delta^n \longrightarrow \mathbb{R}^{n+k}$

$\sigma(e_i) = v_i$

canonical homeomorphism

Will often blur the distinction between map σ and its image,

$\sigma \equiv [\sigma e_0, \dots, \sigma e_n]$

but the ordering of the v_j will be important (so the map σ is more precise)

We encode this extradata by orienting the edges $\overset{v_i}{\bullet} \longrightarrow \overset{v_j}{\bullet}$ if $i<j$

Def **d-dimensional faces** $[v_{i_0}, \dots, v_{i_d}]$ for $i_0 < \dots < i_d$

Example 0-dim faces are the vertices $v_0, \dots, v_n$

facets = $(n-1)$-dimensional faces

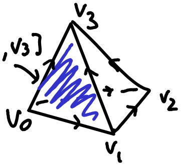
$= [v_0, \dots, \hat{v}_k, \dots, v_n]$ where we omit v_k

$= \{ \sum t_i v_i \in [v_0, \dots, v_n] : t_k = 0 \}$

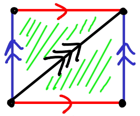
$= \text{Image } \sigma|_{\Delta_k^{n-1}} : \Delta_k^{n-1} \longrightarrow \mathbb{R}^{n+k}$

$\Delta_k^{n-1} = \{ t \in \Delta^n : t_k = 0 \}$

$[v_0, v_1, \hat{v}_2, v_3] = [v_0, v_1, v_3]$



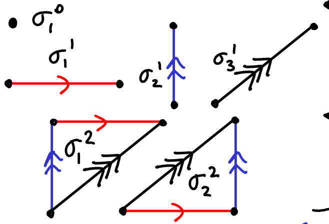
Example Can build a torus out of simplices:

$T^2 =$ 

1 0-simplex $\bullet\ \sigma_1^0$

3 1-simplices

2 2-simplices

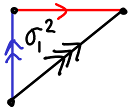
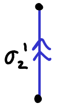


each facet is associated to another simplex, and we identify them linearly

(The simplices here are abstract simplices: don't confuse them with their images in T^2)

$T^2 =$ quotient space $\bigsqcup \sigma_i^n$ / canonical homeos associated to the facets

(don't confuse the abstract simplices with their images in T^2=quotient space)

for example identify facet  of  with  via linear homeo (orientation-preserving)

Def **Δ-complex** is determined by data

- indexing set I_n, for each $n \in \mathbb{N}$
- choice of n-simplex σ_α^n (not necessarily standard) for each $\alpha \in I_n$
- gluing data: for each $\alpha \in I_n$, $0 \le i \le n$, associate some $\beta(\alpha,i) \in I_{n-1}$
- consitency condition (see later)

The Δ-complex is the quotient space

$$\boxed{X = \bigsqcup_{\substack{\alpha\in I_n\\ n\in\mathbb{N}}} \sigma_\alpha^n \Big/ \begin{array}{l} i\text{-th facet of } \sigma_\alpha^n \text{ is identified with } \sigma^{n-1}_{\beta(\alpha,i)} \\ \text{via the order-preserving canonical linear homeo}\end{array}}$$

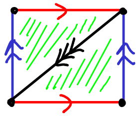
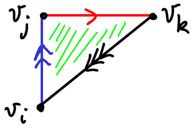
(quotient topology: $U \subseteq X$ is open $\Leftrightarrow$ U intersects σ_α^n in an open set, $\forall \alpha, n$)

A **Δ-complex structure** on a top.space Y is a homeo from a Δ-cx $X \simeq Y$.

Explicit description of the facet identification

$$\begin{tikzcd}
\{\sum s_i w_i\} = [w_0,\dots,w_{n-1}] \arrow[r] & {[v_0,\dots,v_n]} = \{\sum t_i v_i\} & \\
 & & \{s_0v_0+\dots+s_{i-1}v_{i-1}+s_iv_{i+1}+\dots+s_{n-1}v_n\} = [v_0,\dots,\hat{v}_i,\dots,v_n] \arrow[ul, phantom, "\cup"] \\
\Delta^{n-1} \arrow[u, "\sigma^{n-1}_{\beta(\alpha,i)}"] \arrow[r] & \Delta_i^{n-1} \subseteq \Delta^n \arrow[u, "\sigma_\alpha^n"'] \arrow[ur, "\sigma_\alpha^n|_{\Delta_i^{n-1}}"'] &
\end{tikzcd}$$

$$(s_0,\dots,s_{n-1}) \mapsto (s_0,\dots,s_{i-1},0,s_i,\dots,s_{n-1})$$

Non-example  because: 

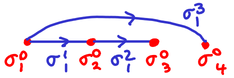
This decomposition for T^2 is not a Δ-complex.

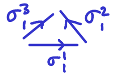
vertices are not totally ordered: $i<j<k<i$ ↯

Consistency condition

We want to additionally ensure that each point of X lies in the interior of exactly one σ_α^n, because we want to avoid unexpected identifications.

Example: $X =$ 



then glue $\sigma_1^2 =$  via 

notice how σ_3^0, σ_4^0 get identified in the quotient, but we only notice this after gluing σ_1^2

(If you try to run the definition of simplicial homology – defined later – you notice that the differential cannot satisfy $\partial_1 \circ \partial_2 = 0$)

Equivalently: the facet gluing maps are compatible under double restriction: $\forall i<j$

$$\begin{tikzcd}
{[v_0,\dots,v_n]} \arrow[r, squiggly, "\text{facet}"] \arrow[dr, squiggly, "\text{facet}"'] & {[v_0,\dots,\hat{v}_i,\dots,v_n]} \arrow[r, phantom, "\overset{\text{identify}}{\cong}"] & {[w_0,\dots,w_{n-1}]} \arrow[r, squiggly, "\text{facet}"] & {[w_0,\dots,\hat{w}_{j-1},\dots,w_{n-1}]} \arrow[r, phantom, "\overset{\text{identify}}{\cong}"] & {[x_0,\dots,x_{n-2}]} \\
 & {[v_0,\dots,\hat{v}_j,\dots,v_n]} \arrow[r, phantom, "\overset{\text{identify}}{\cong}"] & {[z_0,\dots,z_{n-1}]} \arrow[r, squiggly, "\text{facet}"] & {[z_0,\dots,\hat{z}_i,\dots,z_{n-1}]} \arrow[ur, phantom, "\underset{\text{identify}}{\cong}"] &
\end{tikzcd}$$

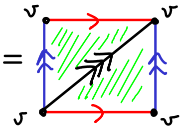
this ensures that $[v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]$ is identified with the same $[x_0,\dots,x_{n-2}]$ whether we first restrict to $t_i=0$ (omit v_i) or first restrict to $t_j=0$ (omit v_j).

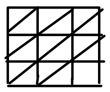
Another equivalent condition: can define the **k-th skeleton** of Δ-cx X, X^k = quotient space you get by gluing all simplices of dimensions $\le k$. Consistency is the condition that the boundary of each σ_α^n should map continuously into X^{n-1}

(in the above Example consider the vertex → $\triangle = \partial\sigma_1^2$)

Rmk (see Part A Topology) A **simplicial complex** is a Δ-complex in which each d-dim face is uniquely determined by d distinct vertices. (more precisely, the "topological realisation" of a simpl. complex)

A homeo from such a complex to X is a **triangulation** of X.

Non-example $T^2 =$  both 2-simplices have vertices v,v,v

whereas $T^2 =$  is a triangulation.

Simplicial chain complex

Def For a Δ-complex X, let X_n = set of n-simplices of X

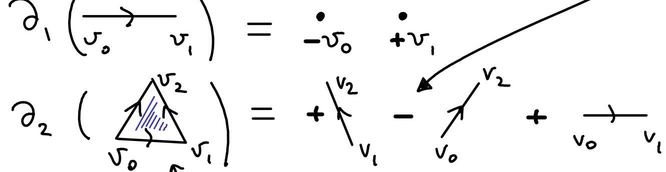
$$C_n^\Delta(X) = \text{free abelian group generated by the set } X_n$$
$$= \Big\{ \sum_{\alpha\in I_n} c_\alpha \cdot \sigma_\alpha^n : c_\alpha \in \mathbb{Z} \text{ and only finitely many } c_\alpha \neq 0 \Big\}$$

differential: $\boxed{\partial_n \sigma_\alpha = \sum (-1)^i \cdot \sigma_{\beta(\alpha,i)}}$

so: $\boxed{\partial_n [v_0,\dots,v_n] = \sum (-1)^i \cdot [v_0,\dots,\hat{v}_i,\dots,v_n]}$ $\Big\}$ and extend linearly

will show $\partial\circ\partial=0$, so get **simplicial homology**: $\boxed{H_*^\Delta(X) = H_*(C_*^\Delta, \partial_*)}$

Examples



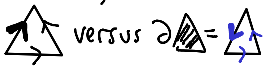
(this)

$\partial_2\circ\partial_1$ (this) $= +(v_2-v_1) - (v_2-v_0) + (v_1-v_0) = 0$

$\partial\circ\partial = 0$ fails for

(not Δ-complex), try!

Later: The $(-1)^i$ signs keep track of whether the orientation agrees/disagrees with geometric boundary orientation, so



Lemma $\boxed{\partial\circ\partial = 0}$

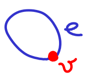
Pf $\partial_{n-1}(\partial_n [v_0,\dots,v_n]) = \sum (-1)^i \partial_{n-1} [v_0,\dots,\hat{v}_i,\dots,v_n]$

$$= \sum_{j<i} (-1)^i (-1)^j [v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]$$ antisymmetric if swap i,j

$$+ \sum_{j>i} (-1)^i \underline{\underline{(-1)^{j-1}}} [v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]$$

$= 0$ □

Example $S^1 =$



Δ-cx: X_0: 1 0-simplex • $e_1^0 = e_{\beta(1,0)} = e_{\beta(1,1)}$

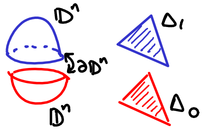
X_1: 1 1-simplex ⟶ e_1^1

$$0 \to \underset{\mathbb{Z}e}{\overset{}{C_1^\Delta}} \to \underset{\mathbb{Z}v}{C_0^\Delta} \to 0$$

$e \longmapsto v - v = 0$

$$\Rightarrow H_*^\Delta(S^1) = \begin{cases} \mathbb{Z} & *=0,1 \\ 0 & \text{else} \end{cases}$$

Example Δ-cx structure on S^n:



$S^n = \Delta^n \sqcup \Delta^n / \text{glue along } \partial\Delta^n$

call this Δ_1 this Δ_0

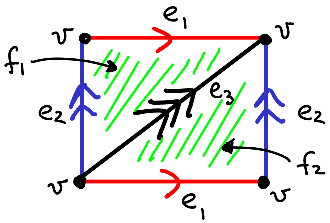
One can deduce: ← but messy!

$$H_* S^n = \begin{cases} \mathbb{Z}\langle \text{point}\rangle \cong \mathbb{Z} & *=0 \\ \mathbb{Z}\langle \Delta_1 - \Delta_0\rangle \cong \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$$

pick any vertex

Example

$X = T^2 =$



$$0 \to \underset{\mathbb{Z}f_1+\mathbb{Z}f_2}{\overset{}{C_2^\Delta}} \longrightarrow \underset{\mathbb{Z}e_1+\mathbb{Z}e_2+\mathbb{Z}e_3}{C_1^\Delta} \longrightarrow \underset{\mathbb{Z}v}{C_0^\Delta} \longrightarrow 0$$

$f_1 \longmapsto e_1 - e_3 + e_2$

$f_2 \longmapsto e_2 - e_3 + e_1$

$e_1, e_2, e_3 \longmapsto v - v = 0$

$$H_*^\Delta(T^2) = \begin{cases} \mathbb{Z}v & *=0 \\ (\mathbb{Z}e_1+\mathbb{Z}e_2+\mathbb{Z}e_3)/\mathbb{Z}(e_1-e_3+e_2) & *=1 \\ \mathbb{Z}\cdot(f_1-f_2) & *=2 \\ 0 & \text{else} \end{cases}$$ ← freely generated by e_1, e_2

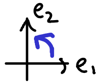
$$\cong \begin{cases} \mathbb{Z} & *=0,2 \\ \mathbb{Z}^2 & *=1 \\ 0 & \text{else} \end{cases}$$

(Smith normal form of ∂_2: $\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{row op.}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{col. op.}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ so after $\mathbb{Z}$-isos of C_2, C_1 we get $\mathbb{Z}^2 \to \mathbb{Z}^3$, $(a,b) \to (a,0,0)$)

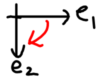
Remark about orientations (see also my B3.2 Geometry of Surfaces notes)

For vector space an **orientation** is a choice of basis modulo linear endomorphisms of det > 0

Example $\mathbb{R}^2$



right-hand orientation (positive) $\xrightarrow[\det<0]{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$



left-hand orientation (negative)

Fact $GL(n,\mathbb{R})$ has 2 path-components $\big\langle \begin{smallmatrix} A: \det A > 0 \\ A: \det A < 0 \end{smallmatrix}$ so can always continuously deform a basis to another within same orientation

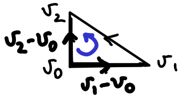
Canonical orientation on $\mathbb{R}^n$: $e_1,\dots,e_n$ standard basis ← **"positive orientation"**

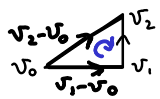
Example $[v_0,\dots,v_n]$ simplex $\Rightarrow$ $v_1-v_0,\dots,v_n-v_0$ is a basis of vector subspace

$$V = \{\sum a_i v_i : \sum a_i = 0\} \subseteq \mathbb{R}^{n+k}$$

hence a choice of orientation of V, and each transposition of vertices $v_0,\dots,v_n$ switches the orientation class.

If $v_0,\dots,v_n \in \underline{\underline{\mathbb{R}^n}}$ then $V=\mathbb{R}^n$ so simplex's orientation can be compared with $\mathbb{R}^n$-orient^n.

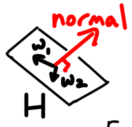
Example In $\mathbb{R}^2$: 

positively oriented 

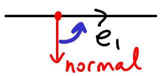
negatively oriented

- No canonical choice of orientation for abstract vector space. Need choose basis $v_1,\dots,v_n$ then declare another basis positively oriented if the change of basis matrix has det >0.

- For hyperplane $H \subseteq \mathbb{R}^n$ with choice of normal can declare orientation of basis $w_1,\dots,w_{n-1}$ of H positive if normal, $w_1,\dots,w_{n-1}$ is positive $\mathbb{R}^n$-basis



convention "outward normal first"

Example 

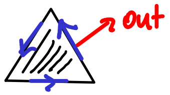
$H \subseteq \mathbb{R}^2 \Rightarrow e_1$ positive basis for H

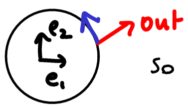
$(\text{normal}, e_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\det = +1 > 0$

Example $\Delta^n \subseteq \mathbb{R}^{n+1}$ with normal $(1,1,\dots,1)$ is positively oriented.

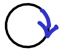
UPSHOT For an n-simplex $[v_0,\dots,v_n]$ in $\underline{\underline{\mathbb{R}^n}}$, each facet lies in a hyperplane and have canonical choice of normal: outward normal. Hence facets are canonically oriented.

Example

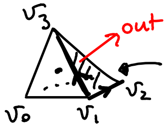
$\mathbb{R}^2 \supseteq$ 

in smooth world: $\mathbb{D}^2$ 

so $\partial \mathbb{D}^2 = S^1$  anticlockwise

Any reflection of $\mathbb{R}^n$ will swap orientation: after $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ get  clockwise

Example

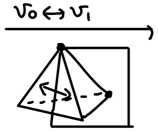


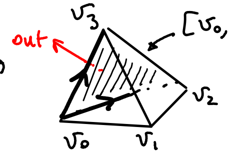
$[\hat{v}_0, v_1, v_2, v_3]$

out, v_2-v_1, v_3-v_1

Positive $\mathbb{R}^3$-basis

reflect $v_0 \leftrightarrow v_1$





$[v_0, \hat{v}_1, v_2, v_3]$

out, v_2-v_0, v_3-v_0

negative $\mathbb{R}^3$-basis

UPSHOT $(-1)^i$ in $(-1)^i [v_0,\dots,\hat{v}_i,\dots,v_k]$ in definition of simplicial ∂ is there to ensure that orientations are consistent (crucial for $\partial\partial=0$)

Lemma $H^{\Delta}_*(X) \cong \oplus H^{\Delta}_*(X_i)$ where X_i are the path-components of X.

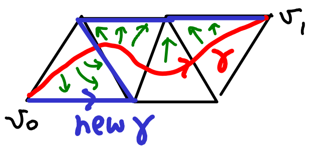
Pf $\bigoplus C^{\Delta}_*(X_i) \longrightarrow C^{\Delta}_*(X)$, $\oplus c_i \longmapsto \sum c_i$ is chain isomorphism since any simplex $\sigma: \Delta^k \to X$ has path-connected image, so $\subseteq X_i$ some i. ☐

since Δ^k path-conn.

Theorem X has Δ-cx structure $\Rightarrow H^{\Delta}_0(X) \cong \bigoplus\limits_{\substack{\text{path-conn.}\\\text{components}}} \mathbb{Z}$

Pf By Lemma, WLOG X path-connected

- vertex v $\Rightarrow$ $v \in C_0(X) \xrightarrow{\partial} C_{-1}(X) \equiv 0$ $\Rightarrow [v] \in H_0(X)$
- vertices $v_0, v_1 \in X$ $\Rightarrow$ $\exists$ path γ from v_0 to v_1



$\Rightarrow$ can homotope path so that go along edges (continuously deform)

$\Rightarrow$ γ is sum of 1-chains s.t. $\partial\gamma = v_1 - v_0$

$\Rightarrow$ $[v] \in H_0(X)$ independent of choice of v

$\Rightarrow$ $H_0(X) = \langle [v] \rangle$

- $H_0(X) = \langle [v] \rangle \leftarrow \mathbb{Z}$ is injective?

$nv \leftarrow\!\shortmid n$ Suppose $nv = \partial c$ some $c \in C_1(X)$

consider the augmentation hom

$$C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

$$\underbrace{\sum n_i \sigma_i}_{\text{0-simplices}} \longmapsto \sum n_i$$

notice composite is 0 since $\partial \left(\overset{\text{1-simplex}}{\sigma_0 \to \sigma_1} \right) = \sigma_1 - \sigma_0 \xrightarrow{\varepsilon} 1-1=0$

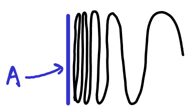
$\Rightarrow n = \varepsilon(nv) = \varepsilon\partial c = 0$. ☐

Rmk X top. space $\Rightarrow$ path conn. component $\subseteq$ connected component since path-conn. $\Rightarrow$ connected. For Δ-cx, these are same (since connected + locally path-conn. $\Rightarrow$ path-connected).

However for top. space they need not be the same, e.g. Topologist's sine curve

$\{(x, \sin\frac{1}{x}) : x \in (0,1]\} \cup \underbrace{0 \times [0,1]}_{A} \subseteq \mathbb{R}^2$

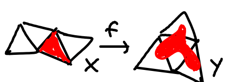
2 path-conn. components



- connected
- not path-connected
- not locally path-connected

3. SINGULAR HOMOLOGY

Motivation Not obvious that H_*^Δ is functorial: $\Delta^k \xrightarrow{\sigma} X \xrightarrow{f} Y$ ↖continuous map

then $f\circ\sigma$ typically not a simplex: $\triangle \xrightarrow{\sigma}$ 

Solution 1: only allow **simplicial maps** $f: X \to Y$ (so $f\circ\sigma$ simplex $\forall\sigma$)

Solution 2: show that any cts map $f: X\to Y$ can be approximated arbitrarily well by a simplicial map after having performed barycentric subdivision on X, Y enough times. Also any two such approximations induce the same map $H_*^\Delta(X)\to H_*^\Delta(Y)$

Solution 3: develop new homology $H_*(X)$ which allows any cts map $\Delta^n \to X$ and prove $H_*^\Delta(X) \cong H_*(X)$ for Δ-complexes X.

WILL DO THIS.

Def **Singular n-simplex** is any $\boxed{\text{continuous map } \sigma: \Delta^n \longrightarrow X}$ ← X is any top. space

Singular n-chains $C_n(X)$ = free abelian group generated by σ's

$$= \left\{ \sum_{\substack{\text{singular}\\ n\text{-simplices } \sigma}} c_\sigma \cdot \sigma \ : \ \begin{matrix} c_\sigma \in \mathbb{Z} \\ \text{only finitely many } c_\sigma \neq 0 \end{matrix} \right\}$$

$$\boxed{\partial_n \sigma = \sum (-1)^i \cdot \sigma|_{\Delta_i^{n-1}}} \quad \text{(and extend linearly)}$$

← i-th facet

Rmk Here $\Delta_i^{n-1} = [e_0,..,\hat{e}_i,..,e_n]$ is identified canonically with Δ^{n-1} (send $e_k \to e_k$ $k<i$; $e_k \to e_{k-1}$ for $k>i$)

Will show $\partial\circ\partial = 0$, so get **singular homology**: $\boxed{H_*(X) = H_*(C_*, \partial_*)}$

For Δ-complex X have inclusion of subcomplex $C_*^\Delta \to C_*$

$\Rightarrow$ induces $H_*^\Delta(X) \longrightarrow H_*(X)$ Fact: isomorphism (proof later, see cellular $H_*^{CW} \cong H_*$)

Corollary $H_*^\Delta(X)$ is independent of choice of Δ-cx structure on X

Lemma $\partial\circ\partial = 0$

Proof $\partial_{n+1}(\partial_n \sigma) = \partial_{n+1}\left(\sum (-1)^i \sigma|_{\Delta_i^{n-1}}\right)$ ← $[e_0,..,\hat{e}_i,..,e_n]$

$$= \sum_{j<i} (-1)^i (-1)^j \sigma|_{[e_0,..,\hat{e}_j,..,\hat{e}_i,..,e_n]}$$

$$+ \sum_{j>i} (-1)^i \underline{\underline{(-1)^{j-1}}} \sigma|_{[e_0,..,\hat{e}_i,..,\hat{e}_j,..,e_n]}$$

$= 0$ □

Example $X = \text{point} \Rightarrow C_n(X) = \mathbb{Z}\cdot(\sigma_n : \Delta^n \to X$ the constant map$)$

$$\partial\sigma_n = \sum(-1)^i \sigma_n|_{\Delta_i^{n-1}} = \underbrace{\sum(-1)^i}_{\left\{\begin{smallmatrix}1 & \text{if } n \text{ even}\\ 0 & \text{"} \text{ odd}\end{smallmatrix}\right.} \sigma_{n-1} \quad \left.\begin{matrix} \Rightarrow \cdots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \underset{\overset{\|\wr}{C_1}}{\mathbb{Z}} \xrightarrow{0} \underset{\overset{\|\wr}{C_0}}{\mathbb{Z}} \to 0 \\ \Rightarrow H_*(pt) \cong \begin{cases}\mathbb{Z} & * = 0\\ 0 & \text{else}\end{cases}\end{matrix}\right.$$

Lemma $\boxed{H_*(X) \cong \bigoplus H_*(X_i)}$ where X_i are path-components of X

Pf Image of cts map $\Delta^n \to X$ is path conn. so lies in some X_i. □

Cor $\boxed{H_0(X) = \bigoplus_{X_i} \mathbb{Z}}$

Pf By Lemma, WLOG X path-connected. $\Delta^0 = pt \to X$ is cycle since $C_{-1}(X) = \emptyset$ ←generators of $C_0(X)$

Given 2 points $x, y \in X$, a path $\Delta^1 = [0,1] \xrightarrow{\gamma} X$, $\gamma(0) = x$, $\gamma(1) = y$ is also a 1-chain!

So $x - y = \partial\gamma$, so x, y are homologous. Finally if $n\cdot[x] = 0 \in H_0(X)$ then $nx = \partial c$ some $c \in C_1(X)$ = generated by paths. Now run the augmentation hom. trick like we did for H_0^Δ: $n = \varepsilon(nx) = \varepsilon\partial c = 0$ as $\varepsilon\circ\partial = 0$. □

Naturality (i.e. functoriality)

Lemma $f: X \to Y$ continuous

$\Longrightarrow H_*(f) = f_*: H_*(X) \to H_*(Y)$ induced by chain map

$f_*: C_*(X) \longrightarrow C_*(Y)$

induced map

$\boxed{f_*(\sigma) = f\circ\sigma}$ and extend linearly

$$\begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[dr, "f_*\sigma"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Pf $\partial_n(f_*\sigma) = \sum(-1)^n f\circ\sigma|_{\Delta_i^{n-1}} = f_*\left(\sum(-1)^n \sigma|_{\Delta_i^{n-1}}\right) = f_*(\partial_n\sigma)$ □

Properties 1) $X \xrightarrow{f} Y \xrightarrow{g} Z \quad\Longrightarrow\quad (g\circ f)_* = g_*\circ f_*$

2) $id_* = id$

Pf 1) $(g\circ f)_*\sigma = g\circ f\circ\sigma = g_*(f\circ\sigma) = g_*(f_*\sigma)$ ✓

2) $id_*\sigma = id\circ\sigma = \sigma$ ✓ □

Cor $H_*: \left\{\begin{matrix}\text{topological spaces}\\ \& \\ \text{cts maps}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{graded abelian groups}\\ \& \\ \text{graded homs}\end{matrix}\right\}$ is a functor

Cor $X \cong Y \Rightarrow H_*(X) \cong H_*(Y)$

4. CHAIN HOMOTOPIES AND HOMOTOPY INVARIANCE

Algebra: chain homotopies

$f_*, g_* : (C_*, \partial_*) \longrightarrow (\tilde{C}_*, \tilde{\partial}_*)$ chain maps

Def f_*, g_* are **chain homotopic** if $\exists$ (degree $+1$) hom $h : C_* \longrightarrow \tilde{C}_*[1]$ s.t.

$$\boxed{\tilde{\partial} \circ h + h \circ \partial = f - g}$$

h is called a **chain homotopy**

Consequence $f_* = g_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$ on homology

Pf

$$\begin{tikzcd} C_{n+1} \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[dl, "h_{n-1}"'] \\ \tilde{C}_{n+1} \arrow[r, "\tilde{\partial}_{n+1}"'] & \tilde{C}_n \arrow[r] & \tilde{C}_{n-1} \end{tikzcd}$$

c cycle $\in C_n$

$$\Rightarrow f_n(c) - g_n(c) = \underbrace{\tilde{\partial}_{n+1} \circ h_n(c)}_{\text{boundary}} + h_{n-1} \circ \underbrace{\partial_n(c)}_{=0}$$

$$\Longrightarrow f_n(c) = g_n(c) \text{ in } H_*(\tilde{C})$$ □

Theorem $i_0 : X \longrightarrow X \times I$, $i_0(x) = (x, 0)$ where $I = [0,1]$

$i_1 : X \longrightarrow X \times I$, $i_1(x) = (x, 1)$

$\Longrightarrow i_{0*}, i_{1*} : C_*(X) \to C_*(X \times I)$ are chain hpic.

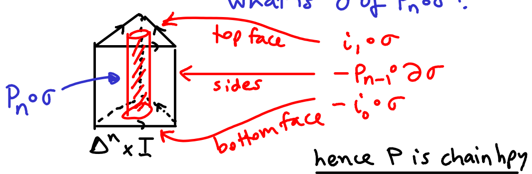
Key idea Need "prism operator" which cuts $\Delta^n \times I$ into a sum Γ_n of $(n+1)$-simplices in $\Delta^n \times I$:

$(\sigma : \Delta^n \to X) \mapsto \sigma \times id : \Delta^n \times I \to X \times I$

$\Gamma_n =$ combo of maps $\uparrow$ Δ^{n+1}

prism operator P_n $\to$ $(\sigma \times id) \circ \Gamma_n : \Delta^{n+1} \to X \times I$

What is ∂ of $P_n \circ \sigma$?



hence P is chain hpy

Pf ← Non-examinable

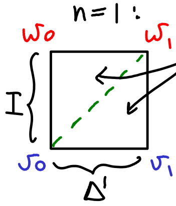
bottom facet $\Delta^n \times 0 = [v_0, \ldots, v_n]$ ← $v_i = e_i \times 0$

top facet $\Delta^n \times 1 = [w_0, \ldots, w_n]$ ← $w_i = e_i \times 1$

$\} \subseteq \Delta^n \times [0,1] \subseteq \mathbb{R}^{n+1}$

Examples

$n = 1$:

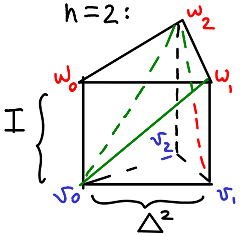


two 2-simplices

$[v_0, v_1, w_1]$

$[v_0, w_0, w_1]$

$n = 2$:



three 3-simplices:

$[v_0, v_1, v_2, w_2]$

$[v_0, v_1, w_1, w_2]$

$[v_0, w_0, w_1, w_2]$

Let $s_i = [v_0, \ldots, v_i, w_i, \ldots, w_n]$

Claim The s_i cover $\Delta \times [0,1]$ and give Δ-cx structure on $\Delta^n \times I$

Pf $\sum_{k \leq i} t_k v_k + \sum_{k \geq i} s_k w_k = (t_0, \ldots, t_{i-1}, \underline{t_i + s_i}, s_{i+1}, \ldots, s_n, s_i + \cdots + s_n)$

So given $(x_0, \ldots, x_n, a) \in \Delta^n \times I$, equate and solve:

$t_0 = x_0, \ldots, t_{i-1} = x_{i-1}$, $s_{i+1} = x_{i+1}, \ldots, s_n = x_n$, and $\begin{cases} s_i = a - x_{i+1} - \cdots - x_n \\ t_i = x_i - s_i \end{cases}$

Note $x_k \geq 0$, $\Sigma x_k = 1$, $a \in [0,1]$ hence $\Sigma t_k + \Sigma s_k = 1$ ✓ $\begin{cases} t_k \geq 0 \text{ for } k < i \\ s_k \geq 0 \text{ for } k > i \end{cases}$ ✓

but $\left.\begin{matrix} s_i \geq 0 \\ t_i \geq 0 \end{matrix}\right\} \Leftrightarrow \begin{cases} a \geq x_{i+1} + \cdots + x_n \\ x_i + x_{i+1} + \cdots + x_n \geq a \end{cases}$. Thus a solution exists if we pick $i = \min\{k : a \geq x_{i+1} + \cdots + x_n\}$

There are multiple solutions if $x_{i+1} = x_{i+2} = \cdots = x_j = 0$, but that is as expected: those points of $\Delta^n \times I$ belong to the faces of $s_i, s_{i+1}, \ldots, s_j$. □

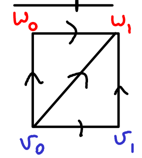
Def

$\Gamma_n = \sum (-1)^i s_i \in C_{n+1}(\Delta^n \times [0,1])$ ← geometrically this "represents" $\Delta^n \times I$ as a simplicial chain

$$\Rightarrow \partial \Gamma_n = \sum_i \sum_{j \leq i} (-1)^i (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_i, w_i, \ldots, w_n] + \sum_i \sum_{j \geq i} (-1)^i (-1)^{j+1} [v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j, \ldots, w_n]$$

} geometrically this "represents" $\partial(\Delta^n \times I) = (\partial \Delta^n \times I) \sqcup (\Delta^n \times \partial I)$

Example



$\Gamma_1 = [v_0, w_0, w_1] - [v_0, v_1, w_1]$ "is the square"

$\partial \Gamma_1 = [w_0, w_1] - \cancel{[v_0, w_1]} + [v_0, w_0] - [v_1, w_1] + \cancel{[v_0, w_1]} - [v_0, v_1]$

"is ∂ of square"

"inside facets" cancel

Prism operator $P : C_n(X) \longrightarrow C_{n+1}(X \times [0,1])$

$$\boxed{P(\sigma) = (\sigma \times \mathrm{id})_*(\Gamma_n)}$$

$\sigma : \Delta^n \to X$

$\sigma \times \mathrm{id} : \Delta^n \times [0,1] \to X \times [0,1]$

$(\sigma \times \mathrm{id})(x,t) = (\sigma(x), t)$

this abbreviated notation means the map

$(t_0,\dots,t_n) \mapsto (t_0\sigma e_0 + \dots + \widehat{t_j \sigma e_j} + t_j \sigma e_{j+1} + \dots + t_{i-1}\sigma e_i + t_i \sigma e_i + \dots + t_n \sigma e_n, t_i + \dots + t_n) \in X \times I$, $\overset{\uparrow}{\Delta^n}$

$$\partial P(\sigma) = \partial(\sigma \times \mathrm{id})_*(\Gamma_n)$$
$$= (\sigma \times \mathrm{id})_*(\partial \Gamma_n)$$
$$= \sum_i \sum_{j \leq i} (-1)^i (-1)^j [i_0\sigma e_0, \dots, \widehat{i_0 \sigma e_j}, \dots, i_0 \sigma e_i, i_1 \sigma e_i, \dots, i_1 \sigma e_n]$$
$$+ \sum_i \sum_{j \geq i} (-1)^i (-1)^{j+1} [i_0\sigma e_0, \dots, i_0 \sigma e_i, i_1 \sigma e_i, \dots, \widehat{i_1 \sigma e_j}, \dots, i_1 \sigma e_n]$$
$$= \underset{\substack{\uparrow \\ i=j=0 \\ 1^{st}\text{ sum}}}{i_{1*}\sigma} - \underset{\substack{\uparrow \\ i=j=n \\ 2^{nd}\text{ sum}}}{i_{0*}\sigma} - \underbrace{P\partial\sigma}_{((\partial\sigma) \times \mathrm{id})_* \Gamma_{n-1}}$$

⍟
$$(\sigma \times \mathrm{id})(v_i)$$
$$= (\sigma \times \mathrm{id})(e_i, 0)$$
$$= (\sigma(e_i), 0)$$
$$= i_0(\sigma)(e_i)$$
$$(\sigma \times \mathrm{id})(w_i)$$
$$= (\sigma(e_i), 1)$$
$$= i_1(\sigma)(e_i)$$

$\Gamma_{n-1} = \sum_i (-1)^i [v_0, \dots, v_i, w_i, \dots, w_{n-1}]$

now use ⍟ and

$\partial \sigma = \sum (-1)^k [\sigma e_0, \dots, \widehat{\sigma e_k}, \dots, \sigma e_n]$. □

Homotopy invariance

$f_0, f_1 : X \longrightarrow Y$

Def $f_0 \simeq f_1$ **homotopic** if $\exists$ continuous map $F : X \times [0,1] \longrightarrow Y$ called **homotopy**

s.t. $f_0 = F \circ i_0$
$f_1 = F \circ i_1$.

Idea Think of this as a continuous family of maps $f_t = F(\cdot, t) : X \to Y$ from f_0 to f_1.

Exercise $\simeq$ is an equivalence relation.

Homotopic **relative to $A \subseteq X$** if $F(a,t) = f_0(a) = f_1(a)$ all $a \in A$, all t.

write "$f \simeq g$ **rel A**"

Def $X \simeq Y$ **homotopy equivalent** spaces if $\exists$ maps

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$ with $g \circ f \simeq \mathrm{id}$, $f \circ g \simeq \mathrm{id}$

Rmk homeo $\Rightarrow$ hpy equivalent

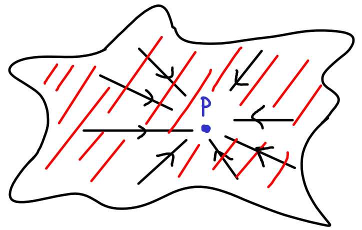
Def X **contractible** if $X \simeq pt$

equivalently $(X \xrightarrow{id} X) \simeq (X \xrightarrow{const} \text{point} \in X)$

Example • $\mathbb{R}^n \simeq pt$

$F(x,t) = tx$ then $f_0 \equiv 0$, $f_1 = id$.

• (star-shaped subsets of $\mathbb{R}^n$) $\simeq pt$



contains line segments to a specific point p

WLOG $p=0$ & use same F
↑
translate

(examples: $\mathbb{D}^n$, convex sets, ...)

Theorem $f_0 \simeq f_1 \Rightarrow f_{0*} = f_{1*} : H_* X \to H_* Y$

Pf
$$f_{1*} - f_{0*} = F_* i_{1*} - F_* i_{0*}$$
$$= F_*(i_{1*} - i_{0*})$$
$$\overset{\text{previous Thm}}{=} F_*(\partial P + P\partial)$$
$$\overset{F_*\text{ chain map}}{=} \partial \circ (F_* \circ P) + (F_* \circ P) \circ \partial$$
$\Rightarrow F_* \circ P$ is chain hpy from f_{0*} to f_{1*} □

(where F = homotopy, i_0, i_1 as in previous Thm)

Cor $X \simeq Y \Rightarrow H_* X \cong H_* Y$

Pf $f_* g_* = id_*$, $g_* f_* = id_*$ □

Example X contractible $\Rightarrow H_* X \cong H_*(pt) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Cultural Rmk (Whitehead's theorem) For nice topological spaces ← (CW complexes – see later in course)
if X, Y are simply connected and $\exists f : X \to Y$ inducing isomorphisms on H_* then $X \simeq Y$ are homotopy equivalent.

Relative homology

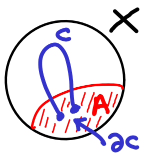
Def (X,A) **pair of spaces** if $A \subseteq X$ topological subspace

$\Rightarrow$ $i = \mathrm{incl}: A \hookrightarrow X$ induces a subcx $i_*: C_*A \to C_*X$

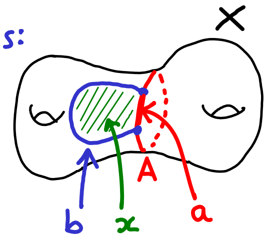
$\Rightarrow$ C_*X/C_*A quotient chain cx (recall $\partial[x] = [\partial x]$)

$$\boxed{H_*(X,A) = H_*(C_*X/C_*A)}$$

Idea: **relative cycles**: $c \in C_*X$ s.t. $\partial c \in C_*A$



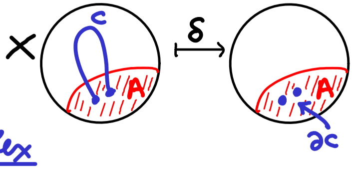
relative boundaries: $b \in C_*X$ s.t. $\exists\, x \in C_{*+1}X$ $\partial x = b + a$, $a \in C_*A$



$\Rightarrow$ $0 \to C_*A \to C_*X \to C_*X/C_*A \to 0$ SES

Cor $\boxed{\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} \cdots}$

LES for the pair



$\delta c = \partial c$, $\partial c \in C_*A$

Need not be ∂a some $a \in C_*A$

Reduced homology

$\tilde{H}_*X = H_*$ of **augmented chain complex**

$$\cdots \to C_n(X) \to \cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

augmentation $\varepsilon(\Sigma n_i \cdot p_i) = \Sigma n_i$, $n_i \in \mathbb{Z}$, points $p_i \in X$

can view $C_{-1}(X) = \mathbb{Z} \cdot (\text{map } \emptyset \to X)$ where allow the empty simplex $\emptyset$

For $X \neq \emptyset$, $\tilde{H}_*X = \mathrm{Ker}\; H_*X \to H_*(pt)$ induced by $X \to pt$

Example $\tilde{H}_*(\mathbb{R}^n) \cong \tilde{H}_*(\mathbb{D}^n) \cong \tilde{H}_*(pt) = 0$

Check
- $H_*X = \tilde{H}_*X$ $* \neq 0$, and $H_0X \cong \tilde{H}_0X \oplus \mathbb{Z}$ for $X \neq \emptyset$
- $f: X \to Y$ $\Rightarrow$ $f_*: \tilde{H}_*X \to \tilde{H}_*Y$

Lemma (X,A) pair $\Rightarrow \exists$ LES

if $A = \emptyset$ we end with $\tilde{H}_{-1}A = \mathbb{Z}$

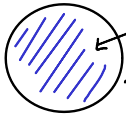
$$\boxed{\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \to H_n(X,A) \to \tilde{H}_{n-1}(A) \xrightarrow{i_*} \cdots \to H_0(X,A) \to 0}$$

Pf use augmented ch. cx. and $\mathbb{Z}/\mathbb{Z} = 0$ □

Cor $H_*(X,pt) \cong \tilde{H}_*(X)$

Pf $\tilde{H}_*(pt) = 0$. □

Example LES: $\tilde{H}_*(S^{n-1}) \to \tilde{H}_*(\mathbb{D}^n) = 0$



$\mathbb{D}^n \subseteq \mathbb{R}^n$, $\partial\mathbb{D}^n = S^{n-1}$

$$\begin{tikzcd} & \tilde{H}_*(S^{n-1}) \arrow[r] & \tilde{H}_*(\mathbb{D}^n)=0 \arrow[dl] \\ & H_*(\mathbb{D}^n, S^{n-1}) \arrow[u, "{[-1]}"] & \end{tikzcd}$$

$\Rightarrow$ $\boxed{H_*(\mathbb{D}^n, S^{n-1}) \cong \tilde{H}_{*-1}(S^{n-1})}$

Naturality of the LES for pairs

Def A **map of pairs of spaces** $(X,A) \xrightarrow{f} (Y,B)$ means $f: X \to Y$ and $f(A) \subseteq B$.

Lemma

$$\begin{tikzcd} \cdots \arrow[r] & H_*A \arrow[r] \arrow[d, "f_*"] & H_*X \arrow[r] \arrow[d, "f_*"] & H_*(X,A) \arrow[r] \arrow[d] & H_{*-1}A \arrow[r] \arrow[d, "f_*"] & \cdots \\ \cdots \arrow[r] & H_*B \arrow[r] & H_*Y \arrow[r] & H_*(Y,B) \arrow[r] & H_{*-1}B \arrow[r] & \cdots \end{tikzcd}$$

and similarly for $\tilde{H}_*$.

Pf

$$\begin{tikzcd} 0 \arrow[r] & C_*A \arrow[r] \arrow[d, "f_*"] & C_*X \arrow[r] \arrow[d, "f_*"] & C_*X/C_*A \arrow[r] \arrow[d, "f_*"] & 0 \\ 0 \arrow[r] & C_*B \arrow[r] & C_*Y \arrow[r] & C_*Y/C_*B \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow$ claim follows by naturality of LES induced by SESs of chain cxes. □

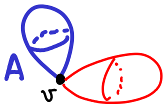
5. EXCISION THEOREM AND QUOTIENTS

(X,A) pair

Def $r: X \to X$ **retraction** onto A if $\begin{cases} r(X) = A \\ r|_A = id_A \end{cases}$

(equivalently $r^2 = r$ then define $A = im(r)$)

Example



$X = \underbrace{S^2}_{A} \vee S^2$ = two spheres glued at one point v (**wedge sum**)

$r: X \to A$ map second sphere to v

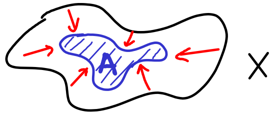
Example In Pf of Brower fixed pt thm we built a retraction r by contradiction

Cor r retraction $\Rightarrow$ $r_*: H_*X \to H_*A$ surjective

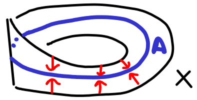
$\mathrm{incl}_*: H_*A \to H_*X$ injective

Pf $\begin{tikzcd} A \arrow[r, "incl"] \arrow[rr, bend right, "id_A"'] & X \arrow[r, "r"] & A \end{tikzcd}$ now use H_* functorial □

Def $r: X \to X$ **deformation retraction** onto A if $\begin{cases} r \text{ retraction} \\ r \simeq \mathrm{id}_X \text{ rel } A \end{cases}$



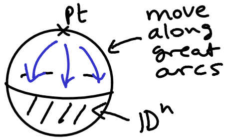
Example X = Möbius strip
A = equator



Lemma r def. retr. $\Rightarrow$ · $A \xrightarrow[\simeq]{\text{incl}} X$ is a homotopy equivalence.
· incl_* and r_* are isos on H_*, so $H_*A \cong H_*X$

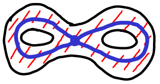
Pf $A \underset{r}{\overset{\text{incl}}{\rightleftarrows}} X$ $\quad \text{incl} \circ r = r \simeq \mathrm{id}_X$, $\quad r \circ \text{incl} = r|_A = \mathrm{id}_A$ □

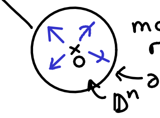
Example $S^n \setminus pt$ def. retracts to $\mathbb{D}^n \cong$ lower hemisphere:



$\Rightarrow S^n \setminus pt \simeq \mathbb{D}^n$

$\Rightarrow S^n \setminus \{2 \text{ points}\} \simeq \mathbb{D}^n \setminus pt \cong \mathbb{D}^n \setminus 0 \simeq S^{n-1}$

$\Rightarrow S^n \setminus \{3 \text{ points}\} \underset{\text{def.ret.}}{\simeq}$  $\underset{\text{ret.}}{\simeq}$  $S^{n-1} \vee S^{n-1}$

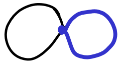
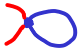
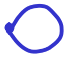


Excision theorem

$E \subseteq A \subseteq X$ subspaces with $\overline{E} \subseteq A^\circ$ $\Longrightarrow$ $(X \setminus E, A \setminus E) \longrightarrow (X, A)$ induces iso

$$\boxed{H_*(X \setminus E, A \setminus E) \xrightarrow{\cong} H_*(X, A)}$$

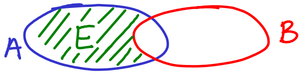
Proof Later.

Example $X = S^1 \vee S^1 =$  $\supseteq A =$  $\supseteq E =$  $\cong S^1$

$\Rightarrow H_*(X, A) \underset{\text{exc. thm.}}{\cong} H_*($  $) \underset{\text{hpy invce}}{\cong} H_*(\mathbb{D}^1, \underbrace{\partial \mathbb{D}^1}_{S^0}) \cong \widetilde{H}_0(\underset{\text{2 points}}{S^0}) \cong \mathbb{Z}$

Rephrasing of Excision Thm

$X = A^\circ \cup B^\circ$ $\Longrightarrow$ $\boxed{H_*(X, A) \cong H_*(B, A \cap B)}$ induced by inclusion $(X, A) \leftarrow (B, A \cap B)$
$(A, B \subseteq X$ subspaces$)$



Pf Take $E = X \setminus B$ so $X \setminus E = B$ and $A \cap B = A \setminus E$. □

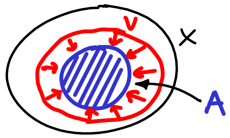
Idea why excision holds: $C_*(A) + C_*(B) \longrightarrow C_*(X)$ is a homotopy equivalence **and** $C_*(A) \cap C_*(B) = C_*(A \cap B)$. Idea: can subdivide chains in X many times, and small enough chains lie either in A or in B (or in both).

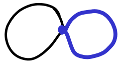
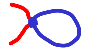
Good pairs and quotients

(X, A) pair

· **Quotient** $X/A = X/\sim$ ← equiv. relation $x \sim y \Leftrightarrow x = y$ or $x, y \in A$

· (X, A) **good pair** if $\begin{cases} A \neq \emptyset \\ A \text{ closed} \\ A \text{ deformation retract of nbhd } V \text{ of } A \end{cases}$

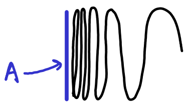


Example $X = S^1 \vee S^1 =$  $\supseteq V =$  $\supseteq A =$  $\cong S^1$

$X/A \cong$  ← all points of A are identified with the node

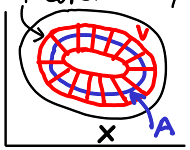
Non-example **Topologist's sine curve**

$\{(x, \sin\frac{1}{x}) : x \in (0, 1]\} \cup \underbrace{0 \times [0, 1]}_{A} \subseteq \mathbb{R}^2$



connected
not path-connected
not locally connected
not locally path-connected

Cultural Rmk
Smooth submanifold $\subseteq$ smooth manifold is a good pair (tubular neighbourhood theorem)



Cor (X, A) good $\Rightarrow (X, A) \longrightarrow (X/A, pt)$ induces iso

$$\boxed{H_*(X, A) \longrightarrow H_*(X/A, pt) = \widetilde{H}_*(X/A)}$$

Pf good $\Rightarrow \exists$ nbhd V of A, and $A \xrightarrow[\text{incl}]{\simeq} V$.

$$\begin{tikzcd} H_n(X,A) \arrow[r, "\cong"] \arrow[d, "\text{quot.}"] & H_n(X,V) \arrow[d, "\text{quot.}"] & H_n(X\setminus A, V\setminus A) \arrow[l, "\cong"'] \arrow[d, "="', "\mathrm{id}_* = \text{identity}"] \\ H_n(X/A, A/A) \arrow[r, "\cong"] & H_n(X/A, V/A) & H_n(X/A \setminus p, V/A \setminus p) \arrow[l, "\cong"'] \end{tikzcd}$$

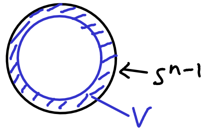
LES for pairs & 5-Lemma since $A \simeq V$, $A/A \simeq V/A$

excision

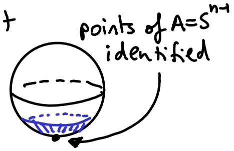
A/A: call this point p

Hence all arrows are isos. □

Example $\mathbb{D}^n \supseteq S^{n-1}$ good:



$\xrightarrow{\text{quotient}}$



$\Longrightarrow H_*(\mathbb{D}^n, S^{n-1}) \underset{\text{Cor}}{\cong} \widetilde{H}_*(\mathbb{D}^n / S^{n-1}) \cong \widetilde{H}_*(S^n)$ ← $\mathbb{D}^n / S^{n-1} \cong S^n$

Recall we proved $\widetilde{H}_*(\mathbb{D}^n,S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})$ (from LES & $\widetilde{H}_*(\mathbb{D}^n)=0$)

$\Rightarrow$ inductively, using Example

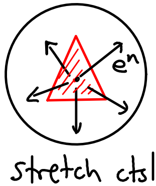
$$\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1}) \cong \dots \cong \widetilde{H}_{k-n}(\underset{\text{2 points}}{S^0}) = \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{else} \end{cases}$$

$H_0(2\text{pts}) = \mathbb{Z}\oplus\mathbb{Z}$

Generator of $H_n(S^n) \cong \widetilde{H}_n(\mathbb{D}^n/S^{n-1}) \cong H_n(\mathbb{D}^n,S^{n-1}) \cong \mathbb{Z}$

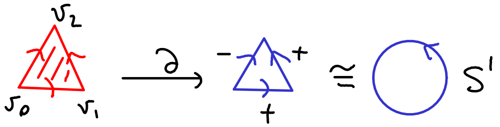
Observe $\exists$ homeo $e^n : \Delta^n \cong \mathbb{D}^n$ (homework) inducing Δ-cx structure on S^{n-1}:

$$\boxed{\partial\Delta^n \cong \partial\mathbb{D}^n = S^{n-1}}$$



Stretch ctsly outwards from barycentre(Δ^n)

Example $\mathbb{D}^2 \cong$



Upshot $(n\geq 2)$

$$\boxed{\begin{aligned} H_n(\mathbb{D}^n, S^{n-1}) &= \mathbb{Z}\cdot e^n \\ H_{n-1}(S^{n-1}) &= \mathbb{Z}\cdot\partial e^n \\ \widetilde{H}_n(\mathbb{D}^n/S^{n-1}) &= \mathbb{Z}\cdot[e^n] \end{aligned}}$$

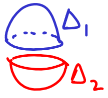
LES

← for $n-1\geq 1$, so $n\geq 2$

← by Cor

$[e^n]$ really lives in $H_n(\mathbb{D}^n, S^{n-1}) \cong H_n(\mathbb{D}^n/_{S^{n-1}}, S^{n-1}/_{S^{n-1}})$

Exercise Recall another Δ-cx structure on S^n:



$$\boxed{S^n = \Delta^n \sqcup \Delta^n / \text{glue along } \partial\Delta^n}$$

call this Δ_1 — this Δ_0

then $\boxed{H_n S^n = \mathbb{Z}\cdot(\Delta_1 - \Delta_0)}$ and $H_n(S^n,\Delta_0) \overset{\text{exc.}}{\cong} H_n(\Delta_1, \partial\Delta_1) \cong H_n(\mathbb{D}^n, \partial\mathbb{D}^n)$

$\Delta_1 - \Delta_0 \mapsto \Delta_1$

Another remark about orientations

Fact $\{\text{homeos } \Delta^n \to \mathbb{D}^n\}$ has 2 path-components

Above we chose a path-component by constructing e^n.

If r is any reflection in $\mathbb{R}^{n+1}$ then $e^n \circ r$ is in the other path-component

e.g. swap 2 coordinates in Δ^n

$$H_n(S^n) \cong H_n(\mathbb{D}^n, S^{n-1}) \xrightarrow{\cong} \mathbb{Z}$$
$$e^n \longmapsto +1$$
$$e^n\circ r \longmapsto -1$$

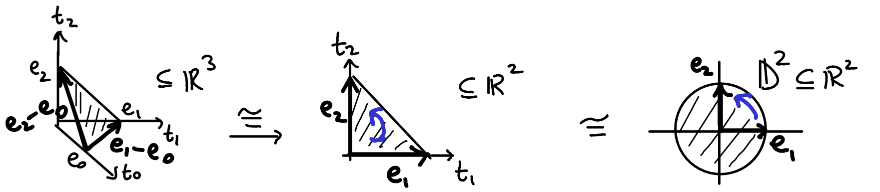
We will see later in the course that this corresponds to a choice of orientation of $\mathbb{D}^n$ and S^n.

Our choice is consistent with the inclusion $\mathbb{D}^n \subseteq \mathbb{R}^n$ (with the positive (canonical) orientation of $\mathbb{R}^n$) and the inclusion

$$(\Delta^n \subseteq \mathbb{R}^{n+1}) \xrightarrow{\cong} \{(t_1,\dots,t_n)\in\mathbb{R}^n : t_i \geq 0, \sum_{i=1}^n t_i \leq 1\}$$
$$(t_0,\dots,t_n) \longmapsto (t_1,\dots,t_n)$$

$t_i \geq 0, \Sigma t_i = 1$

Example $\Delta^2 = [e_0,e_1,e_2]$



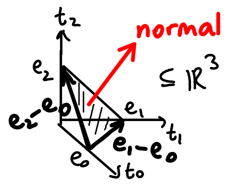
e_1, e_2 positive $\mathbb{R}^2$-basis

standard orientation

Our choice is also consistent with the "normal first" convention for orienting hyperplanes with a given choice of normal:

$\Delta^n \subseteq \text{hyperplane } \{(t_0,\dots,t_n) : \Sigma t_j = 1\} \subseteq \mathbb{R}^{n+1}$ normal $(1,1,\dots,1)$ (so pointing to ∞ in positive quadrant)

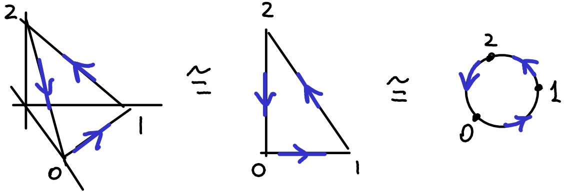
Example $\Delta^2 = [e_0,e_1,e_2]$



normal, e_1-e_0, e_2-e_0 positive $\mathbb{R}^3$-basis

Consistent also with the geometric boundary orientation (outward normal first convention)

$\partial_{\text{geometric}}\Delta^2 =$



$S^1 = \partial\mathbb{D}^2 \subseteq \mathbb{R}^2$ standard orientation

Compare $\partial\Delta = +[\hat{e}_0, e_1, e_2] - [e_0, \hat{e}_1, e_2] + [e_0, e_1, \hat{e}_2]$

This $-[e_0,e_2]$ is not equal to singular chain $[e_2,e_0]$ since they are different maps and we take free abelian group generated by maps.

But $[e_0,e_2] + [e_2,e_0]$ is homologous to 0 (Homework).

Locality (or "small simplices theorem")

$\mathcal{U} = \{\text{subspaces } U_i \subseteq X\}$ whose interiors cover X:

$$X = \bigcup U_i^{\circ}$$

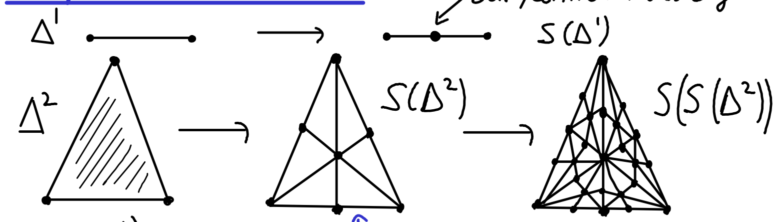
Def $C^{\mathcal U}_*(X) \subseteq C_*(X)$ subcx generated by n-simplices σ with $\sigma(\Delta^n) \subseteq U_i$ some i

Theorem $\boxed{H_*(C^{\mathcal U}_*(X)) \cong H_*(C_*(X)) = H_* X}$

Sketch Pf ① **Barycentric subdivision**

↑ Non-examinable

barycentre of $[v_0, \dots, v_n]$ is $\frac{1}{n+1}(v_0 + \dots + v_n)$



subdivide the boundary (inductively by dimension) then draw the new faces obtained by convex combinations involving the new vertices and the barycentre

$\Rightarrow$ chain map $S: C_*(X) \longrightarrow C_*(X)$

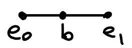
$\sigma \longmapsto$ "$\sigma \circ S$"

and $S(C^{\mathcal U}_*) \subseteq C^{\mathcal U}_*$

Construction of "$\sigma \circ S$" is inductive:

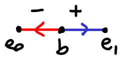
On linear simplices (then for maps σ you restrict $\sigma|_{\dots}$)

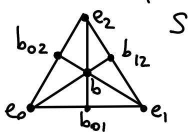
- $S[e_0] = [e_0]$



$S[e_0, e_1] = [b, e_1] - [b, e_0]$ $\quad$ (= "$[b, S\partial[e_0,e_1]]$")

geometrically





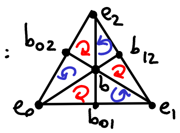
$S[e_0,e_1,e_2] =$ "$[b, S\partial[e_0,e_1,e_2]]$"

$=$ "$[b, S[e_1,e_2]] - [b, S[e_0,e_2]] + [b, S[e_0,e_1]]$"

$= ([b, b_{12}, e_2] - [b, b_{12}, e_1]) - ([b, b_{02}, e_2] - [b, b_{02}, e_0])$

$+ ([b, b_{01}, e_1] - [b_1, b_{01}, e_0])$

geometrically:



So for $\sigma: \Delta^2 \to X$ you take $S(\sigma) = \sigma|_{[b, b_{12}, e_2]} - \sigma|_{[b, b_{12}, e_1]} - \dots$

② S chain hpic to id:

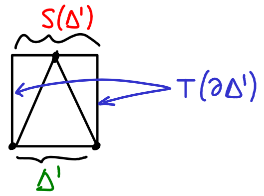
$T: C_n(X) \longrightarrow C_{n+1}(X)$

$T(\sigma): \Delta^n \times I \xrightarrow{\text{project}} \Delta^n \xrightarrow{\sigma} X$

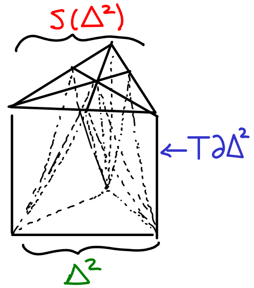
exercise: $\partial T + T\partial = S - id$

$\Rightarrow \boxed{S_*: H_*(X) \xrightarrow{id} H_*(X)}$

Idea: $\Delta^1 \times I =$



$\Delta^2 \times I =$



③ $\forall$ n-simplex $\sigma: \Delta^n \to X$, apply $S(.)$ enough times until σ(each n-simplex of subdivision)$\subseteq U_i$ some i

③ $\Rightarrow$ $\forall$ cycle c, $\exists n$ s.t. $S^n(c) \in C^{\mathcal U}_*(X)$ cycle

$\Rightarrow H^{\mathcal U}_*(C) \longrightarrow H_*(X)$ surjective

$[S^n(c)] \longmapsto S^n_*[c] = [c]$ by ②

③ $\Rightarrow$ $\forall$ bdry $c = \partial b$, $\exists n$ s.t. $S^n(b) \in C^{\mathcal U}_*(X)$

claim: $H^{\mathcal U}_*(C) \longrightarrow H_*(X)$ injective

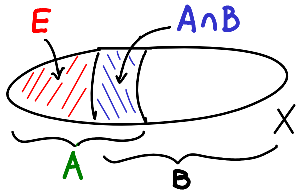
suppose $[c] \longmapsto 0$ then $c = \partial b$ for $b \in C_*(X)$

now $S^n c, S^n b \in C^{\mathcal U}_*(X)$ for large n

$\Rightarrow \partial S^n b = S^n \partial b = S^n c$ in $C^{\mathcal U}_*(X)$

$\Rightarrow [c] \underset{②}{=} S^n_*[c] = [S^n c] = [\partial S^n b] = 0$ in $H^{\mathcal U}_*(X)$ ✓ □

Proof of excision theorem



Let $B = X \setminus E$

use $\mathcal U = \{A, B\}$

so $C^{\mathcal U}_*(X) = C_*(A) + C_*(B) \subseteq C_*(X)$

$$\Rightarrow C_*(X\setminus E) \big/ C_*(A\setminus E) = C_*(B) \big/ C_*(A\cap B) \cong C_*(B) \big/ C_*(A)\cap C_*(B) \cong C^{\mathcal U}_*(X) \big/ C_*(A)$$

2nd isomorphism theorem for groups

$\Rightarrow$ Compare LES's:

$$\begin{tikzcd}
 & & H_*(X\setminus E, A\setminus E) \arrow[d, phantom, "\cong"'] \arrow[d, phantom, "\leftarrow \text{by above isos}"] & & \\
H_*(A) \arrow[r] \arrow[d, equal] & H_*(C^{\mathcal U}_* X) \arrow[r] \arrow[d, "\text{locality}"', "\cong"] & H_*(C^{\mathcal U}_* X / C_* A) \arrow[r] \arrow[d, "\text{iso by 5-lemma}"] & H_{*-1}(A) \arrow[r] \arrow[d, equal] & H_{*-1}(C^{\mathcal U} X) \arrow[d, "\text{locality}"', "\cong"] \\
H_*(A) \arrow[r] & H_*(X) \arrow[r] & H_*(C_* X / C_* A) \arrow[r] \arrow[d, equal] & H_{*-1}(A) \arrow[r] & H_{*-1}(X) \\
 & & H_*(X, A) & &
\end{tikzcd}$$

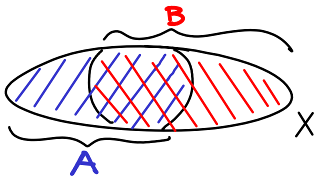
(we are using naturality of LES's induced by SES's)

□

6. MAYER-VIETORIS SEQUENCE ← Key computational tool

$X = A \cup B$ s.t. $X = A^\circ \cup B^\circ$

any subspaces



MV Theorem ∃ LES :

$$\boxed{\cdots \to H_*(A\cap B) \xrightarrow{i_*} H_*(A)\oplus H_*(B) \to H_*(X) \to H_{*-1}(A\cap B) \xrightarrow{i_*[-1]} \cdots}$$

& same holds for $\tilde{H}_*$ provided $A\cap B\neq\emptyset$.

Pf SES $0 \to C_*(A\cap B) \to C_*(A)\oplus C_*(B) \to C_*^{\mathcal{U}}(X) \to 0$ $\quad \mathcal{U}=\{A,B\}$

$\sigma \longmapsto (\sigma, -\sigma)$

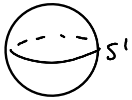
$(\alpha, \beta) \longmapsto \alpha+\beta$

⇒ induces the LES (using locality $H_*^{\mathcal{U}}X \cong H_*X$). □

Exercise connecting map is $\delta: H_*(X) \to H_{*-1}(A\cap B)$

$[\alpha+\beta] \longmapsto [\partial\alpha] = -[\partial\beta]$

Example S^2



$A \simeq pt$

$B \simeq pt$

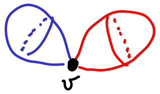
$A\cap B \simeq S^1$

$$\cdots \to \underset{\overset{\|}{0}}{H_2(pt)\oplus H_2(pt)} \to \underset{\overset{\uparrow}{\text{hence } \mathbb{Z}}}{H_2S^2} \to \underset{\overset{\wr\|}{\mathbb{Z}}}{H_1(S^1)} \to \underset{\overset{\|}{0}}{H_1(pt)\oplus H_1(pt)} \to \cdots$$

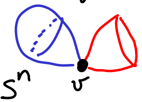
Exercise Compute $H_*S^n = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$ using MV

Example **Wedge sum** of X, Y with basepoints $x \in X$, $y \in Y$ $\qquad X \vee Y = X \times Y / x\sim y$

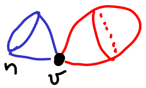
$X = S^n \vee S^n =$



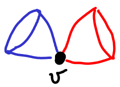
$A =$



$\simeq S^n$ $\quad B =$



$\simeq S^n$ $\quad A\cap B =$



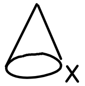
$\simeq pt$

$$0 \to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow[\cong]{} H_n(X) \to 0 \to \cdots \to 0 \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to H_0(X) \to 0$$

$1 \longmapsto (1,-1)$ $\qquad H_0(X) \cong \mathbb{Z}$

Similarly $\boxed{H_*(X\vee Y) \cong H_*(X)\oplus H_*(Y) \text{ for } * \neq 0}$ if ∃ contractible nbhds of $x\in X$, of $y \in Y$.

Cones and suspensions



$CX = (X\times[0,1]) / (x,s)\sim(y,t)$ iff equal or $s=t=1$

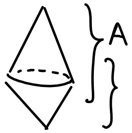
$\simeq pt$

$\Sigma X = (X\times[0,1]) / (x,s)\sim(y,t)$ iff equal or $s=t=0$ or $s=t=1$

Example $CS^n \cong D^{n+1}$, $\Sigma S^n \cong S^{n+1}$.

Lemma $\boxed{\tilde{H}_*(\Sigma X) \cong \tilde{H}_{*-1}(X)}$

Pf



$A \simeq pt$, $B \simeq pt$

$A\cap B \simeq X$ $\qquad$ now apply MV. □

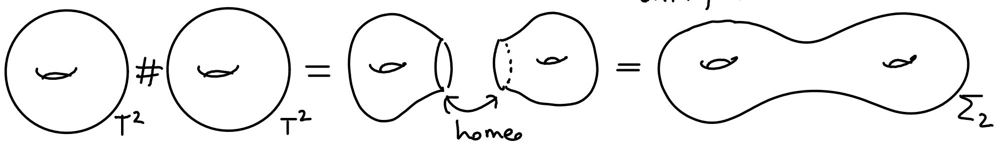
Rmk $\emptyset \neq A \subseteq X \Rightarrow \tilde{H}_*(X \underset{A}{\cup} CA) \overset{LES}{\cong} H_*(X \underset{A}{\cup} CA, CA) \overset{exc.}{\cong} H_*(X,A)$

identify $a\in A\subseteq X$ with $(a,0)\in CA$

Connected sum

M, N connected n-manifolds $\Rightarrow$ $M\#N = (M \setminus \text{open n-ball}) \cup (N \setminus \text{open n-ball})$

identify ∂balls via a homeo



Fact compact connected orientable surfaces are homeo to S^2 or $T^2\#\cdots\#T^2$

and " " non-orientable ones: $\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2$.

genus $g=0$ $\qquad$ $g = \#$ copies $\qquad$ called Σ_g

Exercise (Homework) For M, N compact connected

By MV, $\boxed{H_*(M\#N) \cong H_*(M)\oplus H_*(N) \text{ for } 1 \leq * \leq n-2}$

If M or N orientable: $* = n-1$ also works

If both non-orientable: $* = n-1$ one of $\mathbb{Z}/2$ summands becomes $\mathbb{Z}$

$H_0(M\#N)\cong\mathbb{Z}$ since connected

fact: $H_n(M\#N)$ is $\mathbb{Z}$ or 0

if M,N both orientable ← else

(see later in course)

Cor 1) $\chi(M\#N) = \chi(M) + \chi(N) - \underbrace{\begin{cases} 2 & \text{for n even} \\ 0 & \text{" " odd} \end{cases}}_{= \chi(S^n)}$

2) $H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}^{2g} & *=1 \\ \mathbb{Z} & *=2 \end{cases}$ ← genus g

7. DEGREE OF MAPS OF SPHERES

$$f : S^n \longrightarrow S^n \quad \Longrightarrow \quad \begin{tikzcd} f_n : H_nS^n \arrow[r] \arrow[d, phantom, "\cong" description] & H_nS^n \arrow[d, phantom, "\cong" description] \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \\ 1 \arrow[r, mapsto] & \deg(f) \in \mathbb{Z} \end{tikzcd}$$

$\Rightarrow$ $f_* : \tilde{H}_* S^n \to \tilde{H}_* S^n$ is $\deg(f) \cdot id$

Properties
1) $\deg(id) = 1$
2) $\deg(f \circ g) = \deg f \cdot \deg g$
3) $f \simeq g \Rightarrow \deg f = \deg g$
4) $f \simeq const \Rightarrow \deg f = 0$
5) f homeomorphism $\Rightarrow \deg f = \pm 1$ (sign depends on whether f is orientation-preserving or reversing)

Pf
$id_* = id$, $(f \circ g)_* = f_* \circ g_*$, $f \simeq g \Rightarrow f_* = g_*$, $const_* = 0$, f homeo $\Rightarrow f_n$ iso. ☐

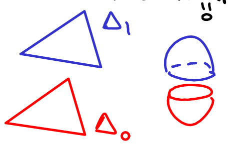
since $S^n \to pt \to S^n$ factors so $H^n S^n \to \underset{0}{H^n(pt)} \to H^n S^n$

Examples

1) $S^n = \Delta^n \times 1 \cup \Delta^n \times 0$ ← call this Δ_0

call this Δ_1

$(b,1) \sim (b,0)$ if $b \in \partial\Delta$



recall $H_nS^n = \mathbb{Z} \cdot (\Delta_1 - \Delta_0)$

reflection : $r : S^n \longrightarrow S^n$, $r(x,t) = (x, 1-t)$

so $\Delta_0 \leftrightarrow \Delta_1$ swapped by r, so $r_*(\Delta_1 - \Delta_0) = -(\Delta_1 - \Delta_0)$

$\Rightarrow \deg(r) = -1$

2) antipodal map $-id : S^n \to S^n$ viewing $S^n \subseteq \mathbb{R}^{n+1}$

$\Rightarrow$ $\boxed{\deg(-id) = (-1)^{n+1}}$

Pf $-id = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} \circ \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \circ \cdots \circ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ composition of $n+1$ reflections each homotopic to r. ☐

3) $A \in O(n) \Rightarrow A : S^{n-1} \to S^{n-1} \Rightarrow \deg A = \det A \in \{\pm 1\}$

Pf fact $SO(n)$ is path-connected so $A \in SO(n)$ is $\simeq id$ so $\deg A = \det A = +1$
The other path-component of $O(n)$ is $r \circ O(n)$ where r is any reflection ☐

4) $\boxed{f \text{ not surjective} \Longrightarrow \deg f = 0}$

Pf If $y \notin \mathrm{Im} f \Rightarrow$
$$\begin{tikzcd} H_n(S^n) \arrow[rr, "f_*"] \arrow[dr, "f_*"'] & & H_n(S^n) \\ & H_n(S^n \setminus y) \cong H_n(\mathbb{R}^n) = 0 \arrow[ur] & \end{tikzcd}$$
☐

Application to vector fields on S^n

$v : S^n \longrightarrow \mathbb{R}^{n+1}$ tangent vector field on S^n
so $v(x) \perp x$

Cor **Hairy ball theorem** $\boxed{\exists \text{ nowhere zero v.f. on } S^n \iff n \text{ odd}}$

(case $n=2$: "you cannot comb a ball of hair without creating a tuft")

Pf Suppose $v(x) \neq 0 \quad \forall x$

$\Rightarrow$ hpy $F : S^n \times [0,1] \longrightarrow S^n$
$$F(x,t) = \cos(\pi t)\, x + \sin(\pi t) \frac{v(x)}{\|v(x)\|}$$

$\Rightarrow F_0 = id, \quad F_1 = -id$

$\Rightarrow 1 = \deg F_0 = \deg F_1 = (-1)^{n+1}$

$\Rightarrow n$ odd

For n odd $\exists$ $v(x) = (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1})$ ☐

Cultural Remark Adams in 1962 proved using alg. topology:
(max # pointwise linearly independent vector fields on S^n) $= 2^b + 8a - 1$
where $n+1 = 2^{4a+b} \cdot$ (odd number), $0 \leq b \leq 3$, $a, b \in \mathbb{N}$, $n \geq 1$.
get 0 if n even $\Rightarrow$ Cor ✓

Local degree

$$\begin{tikzcd} f : S^n \arrow[r] & S^n \\ x \arrow[u, phantom, "\in" rotate=90] \arrow[r] & y = f(x) \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

⊛ Suppose points $\neq x$ near x do not map to y:
$\exists$ nbhds $x \in \underset{\underset{D^n}{\cong}}{U}$, $y \in \underset{\underset{D^n}{\cong}}{V}$ s.t. $\boxed{(U, U \setminus x) \xrightarrow{f} (V, V \setminus y)}$ ← call this $f|_x$ **local map at x**

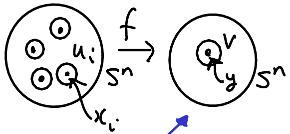
$$\Rightarrow \begin{tikzcd} (f|_x)_* : H_n(U, U\setminus x) \arrow[r, "f_*"] \arrow[d, "\cong", "\text{excise } S^n \setminus U"'] & H_n(V, V \setminus y) \arrow[ddd, equal] \\ H_n(S^n, \underbrace{S^n \setminus x}_{\simeq pt}) \arrow[d, phantom, "\cong" description] & \\ \tilde{H}_nS^n \arrow[d, phantom, "\cong" description] & \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \\ 1 \arrow[r, mapsto] & \deg_x f \end{tikzcd}$$

Lemma $f: S^n \to S^n$, $f^{-1}(y) = \{x_1, \dots, x_k\}$

$$\Rightarrow \boxed{\deg f = \sum \deg_{x_i} f}$$

Pf

$$\begin{tikzcd}
\tilde{H}_n S^n \arrow[r, "f_*"] \arrow[d, "\text{quotient}"'] & \tilde{H}_n(S^n) \arrow[d, "\text{quotient}"] \\
H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y) \\
\oplus H_n(U_i, U_i \setminus x_i) \arrow[u, "\text{exc. } S^n \setminus \cup U_i"', "\cong"] \arrow[r, "\oplus (f|_{x_i})_*"] & H_n(V, V \setminus y) \arrow[u, "\text{exc. } S^n \setminus V"']
\end{tikzcd}$$



Rmk can use same V for all i by taking $\tilde{V} = \cap V_i$, $\tilde{u}_i = f^{-1}(V) \cap U_i$

is:

$$\begin{tikzcd}
1 \in \mathbb{Z} \arrow[r, "\deg f"] \arrow[d] & \mathbb{Z} \arrow[d] \\
(1, \dots, 1) \in \oplus \mathbb{Z} \arrow[r, "\oplus \deg_{x_i} f"'] & \mathbb{Z}
\end{tikzcd}$$
$\square$

map to each summand is exc. of $S^n \setminus U_i$ so iso.

(the 2 squares commute: 1st: quotient is natural 2nd: excision is natural)

Example $p: \mathbb{C} \to \mathbb{C}$ polynomial $\overset{a_n \neq 0}{a_n} z^n + a_{n-1} z^{n-1} + \dots + a_0$, $a_i \in \mathbb{C}$

$\Rightarrow f: S^2 = \mathbb{CP}^1 \to \mathbb{CP}^1 = S^2$ (where view $\mathbb{CP}^1 = \mathbb{C} \cup \infty \cong S^2$)
$z \mapsto p(z)$ stereographic projection
$\infty \mapsto \infty$

$\Rightarrow$ hpy $F(z,t) = a_n z^n + t(a_{n-1} z^{n-1} + \dots + a_0)$

$F_0 = a_n z^n$ and $F_1 = f$

hpy is continuous at ∞ since $a_n z^n$ dominates other terms: $F^{-1}(\mathbb{CP}^1 \setminus K) = \mathbb{CP}^1 \setminus (\text{some compact set})$ $\forall$ compact K. } this would fail if you tried to homotope $t(a_n z^n) + a_{n-1} z^{n-1} + \dots$

$$\Rightarrow \deg f = \deg(a_n z^n) \overset{\text{Lemma}}{=\!=} \sum_{k=0}^{n-1} \underbrace{\deg_{\omega_k} \underbrace{a_n z^n}}_{=1} \leftarrow \omega_k = e^{\frac{2\pi i k}{n}}$$
$= n$
$= \text{degree of the poly } p.$

orient^n preserving homeo near ω_k ← holomorphic maps are always orientation preserving

Cor (Fundamental Thm of Algebra) $n \geq 1 \Rightarrow p$ has a root

Pf $p^{-1}(0) = \emptyset \Rightarrow f^{-1}(0) = \emptyset \Rightarrow \deg f = 0$ ↯ $\square$

Cultural Rmk For smooth $f: S^n \to S^n$

$\deg f = \left(\begin{array}{l}\text{the number of preimages} \\ \text{of a generic point.}\end{array}\right)$

(i.e. almost any point works)

Example $S^2 \to S^2$

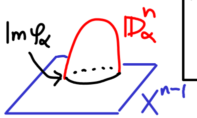
rotate by $\frac{2\pi}{d}$ about vertical axis

$\Rightarrow \deg = d = \#$ preimages of a point except if pick North/South pole

8. CELLULAR HOMOLOGY

Def **CW complex** X is sequence $\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq X^2 \subseteq \dots$
s.t. X^0 is any set

n-skeleton



$$\boxed{X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I_n} \mathbb{D}^n_\alpha \Big/ x \sim \varphi_\alpha(x)}$$

n-discs labelled by some index set I_n

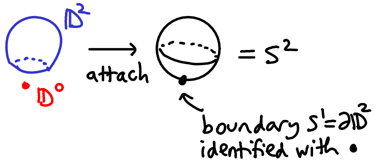
$\varphi_\alpha : \partial\mathbb{D}^n \to X^{n-1}$ **attaching map** (any continuous map, often not injective)

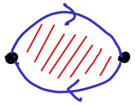
$\Rightarrow X = \bigcup_{n \geq 0} X^n$ top. space with **weak topology**:

$U \subseteq X$ open $\Leftrightarrow$ $U \cap X^n \subseteq X^n$ open $\forall n$.
($\Leftrightarrow$ $U \cap \mathbb{D}^n_\alpha \subseteq \mathbb{D}^n_\alpha$ open $\forall n, \alpha$)

Call X **n-dimensional** if $X = X^n$ and this is the least such n.

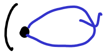
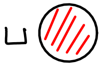
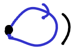
Example $S^n = (\mathbb{D}^0 \sqcup \mathbb{D}^n) / (\mathbb{D}^0 \sim \partial\mathbb{D}^n)$



Example $X = \mathbb{RP}^2 =$ 

$X^0 = \bullet = \mathbb{D}^0$

$X^1 =$  $= S^1 = (\mathbb{D}^0 \sqcup \mathbb{D}^1) / (\bullet \sim \varphi_1(x))$, $\partial\mathbb{D}^1 = S^0 = \{0,1\} \xrightarrow[\varphi_1]{} X^0 = \bullet$

$X^2 = ($  $\sqcup$  $) / ($wrap ∂ of  twice around $)$

$= (X^1 \sqcup \mathbb{D}^2) \Big/ \left(\overset{\overset{\partial\mathbb{D}^2 = S^1}{\Downarrow}}{z} \sim z^2 \in X^1 = S^1 \right)$ $\quad \partial\mathbb{D}^2 = S^1 \xrightarrow{\varphi_2} S^1$, $\varphi_2(z) = z^2$

Fact If we homotope φ_α, we get a homotopy equivalent space

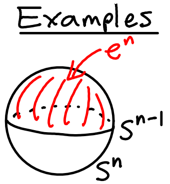
Example If use another degree 2 map φ_2 above, get $X \simeq \mathbb{RP}^2$.

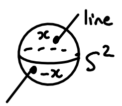
X is partitioned as a set by interiors of **n-cells** $\boxed{e^n_\alpha = \text{Image}(\mathbb{D}^n_\alpha \to X)}$

$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I_\alpha} \overset{\circ}{e}{}^n_\alpha$$
$$= \left(\bigsqcup_{\alpha \in I_0} e^0_\alpha \right) \sqcup \left(\bigsqcup_{\alpha \in I_1} \overset{\circ}{e}{}^1_\alpha \right) \sqcup \left(\bigsqcup_{\alpha \in I_2} \overset{\circ}{e}{}^2_\alpha \right) \sqcup \dots$$

← **Rmk** interior $\overset{\circ}{\mathbb{D}}{}^0 = \mathbb{D}^0$ so $\overset{\circ}{e}{}^0_\alpha = e^0_\alpha$

Examples **real projective space** $\mathbb{R}P^n = S^n/(\mathbb{Z}/2\text{-action by } \pm id)$ so $x \sim -x$





$X^k = \mathbb{R}P^k$ inductively

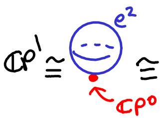
$X^n = X^{n-1} \cup e^n$ with $\varphi : S^{n-1} \to X^{n-1} = \mathbb{R}P^{n-1} = S^{n-1}/\pm id$, $x \longmapsto [x] = [-x]$

complex projective space

$\mathbb{C}P^n = (S^{2n+1} \subseteq \mathbb{C}^n)/(S^1\text{-action by } \lambda \cdot \mathrm{Id})$ $x \sim \lambda x$ for $\lambda \in S^1 \subseteq \mathbb{C}^*$

$X^0 = X^1 = pt = \mathbb{C}P^0$

$X^2 = X^3 = X^0 \cup e^2 = \mathbb{C}P^1$, $\varphi : S^1 \to pt$



$X^4 = X^5 = X^2 \cup e^4 = \mathbb{C}P^2$, $\varphi : S^3 \to \mathbb{C}P^1 = S^3/(S^1\text{-action})$, $x \longmapsto [x] = [\lambda x], \forall \lambda \in S^1$

$X^{2n} = X^{2n-2} \cup e^{2n} = \mathbb{C}P^n$, $\varphi : S^{2n-1} \to X^{2n-2} = \mathbb{C}P^{n-1}$, $x \longmapsto [x]$

In coordinates: $\mathbb{C}P^n = \{[z_0 : \dots : z_n] : \text{not all } z_j \in \mathbb{C} \text{ are } 0\}$ and $[z] \sim [\lambda z]$, $\forall \lambda \in \mathbb{C}^*$

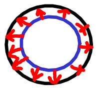
Can rescale so that $\sum |z_j|^2 = 1$ so $z \in S^{2n-1}$ and left with rescaling by $\lambda \in S^1 \subseteq \mathbb{C}^*$.

$\mathbb{C}P^{n-1} \cong X^{n-2} = \{[z_0 : \dots : z_{n-1} : 0]\} \subseteq \mathbb{C}P^n = X^n$ and

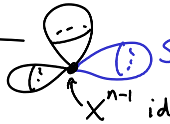
$e^{2n} : \mathbb{D}^{2n} = \{(w_0, \dots, w_{n-1}) : \sum |w_j|^2 \leq 1\} \longrightarrow X^n$ via $[w_0 : \dots : w_{n-1} : \sqrt{1 - \sum |w_j|^2}]$ ← notice this $=0$ if $w \in S^{2n-1} \cong \partial\mathbb{D}^{2n}$

Observe: For X CW complex, for $n \geq 1$: ← (For $n=0$: $(X^0, X^{-1}) = (X^0, \emptyset)$, $X^0/X^{-1} = X^0$)

- (X^n, X^{n-1}) is a good pair ← (since $\exists$ nbhd of $\partial\mathbb{D}^n$ that deformation retracts to $\partial\mathbb{D}^n$)



- $\boxed{X^n/X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n}$ ←



$S^n_\alpha = \mathbb{D}^n_\alpha / \partial\mathbb{D}^n_\alpha$

X^{n-1} identified to a point

Def **Cellular complex** for X a CW cx,

$$\boxed{\begin{aligned} C_n^{CW}(X) &= H_n(X^n, X^{n-1}) \cong H_n(X^n/X^{n-1}) \\ &= \text{free abelian gp gen. by the } n\text{-cells } e^n_\alpha \end{aligned}}$$

since $\Delta^n \cong \mathbb{D}^n \longrightarrow (e^n \subseteq X^n) \longrightarrow \mathbb{D}^n_\alpha/\partial\mathbb{D}^n_\alpha = S^n_\alpha$ generate

↖ as usual we use the standard orientations of $\Delta^n, \mathbb{D}^n, S^n$.

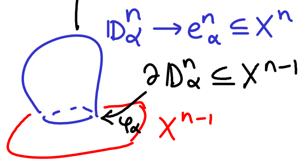
Will build **cellular differential** d, prove $d \circ d = 0$,

$\Rightarrow$ get $\boxed{H_*^{CW}(X) = H_*(C_*^{CW}(X), d)}$

$$\boxed{d\, e^n_\alpha = \sum_{\beta \in I_{n-1}} d^n_{\alpha\beta} \cdot e^{n-1}_\beta}$$

now describe the coefficients $d^n_{\alpha\beta} \in \mathbb{Z}$ and why that is a finite sum.

$$\begin{tikzcd}
d : C_n^{CW} \arrow[rr] \arrow[d, equal] & & C_{n-1}^{CW} \arrow[d, equal] \\
H_n(X^n, X^{n-1}) \arrow[r, "\delta^n_*"', "\text{connecting map of LES}"] & H_{n-1}(X^{n-1}) \arrow[r, "q^{n-1}_*"', "\text{quotient map}"] & H_{n-1}(X^{n-1}, X^{n-2}) \\
e^n_\alpha \arrow[r, mapsto] & (\partial\mathbb{D}^n_\alpha \to \varphi_\alpha(S^{n-1})) \arrow[r, mapsto] & \sum d_{\alpha\beta}\, e^{n-1}_\beta
\end{tikzcd}$$



Recall LES

$$\begin{tikzcd}
H_*(X^{n-1}) \arrow[rr] & & H_*(X^n) \arrow[dl, "q^n_*"] \\
& H_*(X^n, X^{n-1}) \arrow[ul, "\delta^n_*", "{[-1]}"'] &
\end{tikzcd}$$

here it is important that we chose identifications $\Delta^n \cong \mathbb{D}^n$, $S^n \cong \mathbb{D}^n/\partial\mathbb{D}^n$ compatibly with orientations.

Therefore:

Quotient by $\bigvee_{I_{n-1} \setminus \beta} S^{n-1}$

$$\boxed{d^n_{\alpha\beta} = \deg\Big(\underset{\partial\mathbb{D}^n_\alpha}{\overset{\|}{S^{n-1}}} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \cong \bigvee_{I_{n-1}} S^{n-1} \longrightarrow \underset{\mathbb{D}^{n-1}_\beta/\partial\mathbb{D}^{n-1}_\beta}{\overset{\|}{S^{n-1}}}\Big)}$$

Rmk Only finitely many $d^n_{\alpha\beta} \neq 0$ (for fixed α) because φ_α, q are continuous and S^{n-1} compact, so get a compact image in $\bigvee_\beta S^{n-1}$, therefore cannot surject onto ∞ many S^{n-1}_β.

↖ recall if don't surject then deg=0

Lemma $d \circ d = 0$

Pf $d_n = q^{n-1}_{n-1} \circ \delta^n_n$

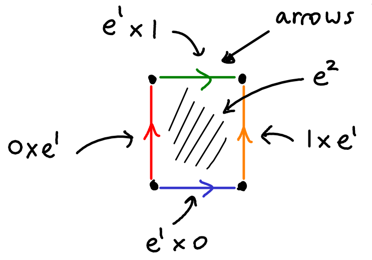
$$d_{n-1} \circ d_n = q^{n-2}_{n-2} \circ \overbrace{\delta^{n-1}_{n-1} \circ q^{n-1}_{n-1}}^{=0 \text{ by LES}} \circ \delta^n_n \qquad \square$$

Cor $\boxed{\operatorname{rank} H_n^{CW}(X) \leqslant \#n\text{-cells}}$

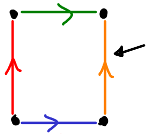
Pf $\#n\text{-cells} = \operatorname{rank} C_n^{CW}(X) \geqslant \operatorname{rank} \operatorname{Ker} d_n^{CW} \geqslant \operatorname{rank} H_n^{CW}(X)$ $\square$

Example $I\times I$ $\quad I=[0,1]$ $\quad \mathbb{D}^1=[-1,1]$

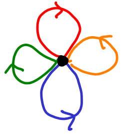
arrows here tell us how we map $[-1,1]\to$ edge (so orientation)



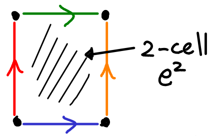
$X^0 = \; :: \; = 4$ 0-cells



$X^1 =$ ← 4 1-cells $\quad e^1\times 0$, $1\times e^1$, $e^1\times 1$, $0\times e^1$

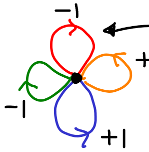


$X^1/X^0 =$ orientations of cells tell us how to orient the circles



$X^2 =$ 2-cell e^2

$e^2 : \mathbb{D}^2 \cong ▨ \longrightarrow X^1$



$\partial e^2 : S^1 \cong \square \longrightarrow X^1/X^0 =$

degree -1 because top edge of $\square$ maps to ○ by an orientation-reversing homeomorphism.

$$\Rightarrow \partial e^2 = +e^1\times 0 + 1\times e^1 - e^1\times 1 - 0\times e^1$$

$$\left(= (\partial e^1)\times e^1 - e^1\times(\partial e^1)\right.$$ ← we come back to this later)

Example $\mathbb{RP}^n$ $\quad$ recall: 1 cell in each dim, $\varphi : S^k \to X^k = \mathbb{RP}^k$, $x \longmapsto [\pm x]$

$S^{k-1} =$

$$\Delta_1 \xrightarrow{\varphi} X^{k-1}/X^{k-2} = \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \overset{\Delta_1/\partial\Delta}{\cong} S^{k-1} \qquad \deg = +1$$

$$\Delta_2 \longrightarrow -\mathrm{id}(\Delta_1) \qquad \deg = (-1)^k$$

$$\Rightarrow d^k_{\alpha\beta} = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd}\end{cases}$$

$$C^{CW}_*(\mathbb{RP}^n) \quad 0 \to \underset{k=n}{\mathbb{Z}} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{2} \underset{k=1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \to \underset{-1}{0}$$

2 if n even, 0 " odd $\qquad$ 0, 2 resp.

$$H^{CW}_*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & *=0, \text{ also } *=n \text{ if odd} \\ \mathbb{Z}/2 & *=1,3,5,\ldots < n \\ 0 & \text{else}\end{cases}$$

Example S^n: $\quad C^{CW}_*(S^n)$: $n\geq 2$: $0 \to \mathbb{Z}\cdot\mathbb{D}^n \xrightarrow{0} 0 \to \cdots \to 0 \to \mathbb{Z}\cdot\mathbb{D}^0 \to 0$

$n=1$: $0 \to \mathbb{Z}\mathbb{D}^1 \xrightarrow{0} \mathbb{Z}\mathbb{D}^0 \to 0$

$\deg\varphi = 0$

$$\Rightarrow H^{CW}_*(S^n) \cong \begin{cases}\mathbb{Z} & *=0,n \\ 0 & \text{else}\end{cases}$$

$H_1(S^1,pt) \xrightarrow{\delta} H_0(pt) \xrightarrow{q} H_0(pt,\emptyset)$

$(\Delta^1 \cong [0,1] \to S^1) \longmapsto \partial\sigma = pt - pt = 0$

$\sigma =$ quotient $0\sim 1$

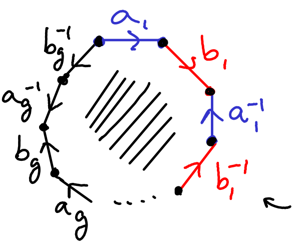
if you work with degrees, need to remember orientations:

$\partial\mathbb{D}^1 \cong \partial[0,1] = [1]-[0] \to$ point

so degree $= +1-1=0$

Example $\Sigma_g =$

genus g surface



boundary identifications

$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$

← Notice all vertices are identified, call vertex v

$\partial a_i = v - v = 0$

$\partial b_i = v - v = 0$

$$0 \longrightarrow \underset{\mathbb{Z}\cdot\mathbb{D}}{\overset{\cong}{\mathbb{Z}}} \xrightarrow{\partial} \underset{\mathbb{Z}\langle a_1,b_1,\ldots,a_g,b_g\rangle}{\overset{\cong}{\mathbb{Z}^{2g}}} \xrightarrow{\partial = 0} \underset{\mathbb{Z}\cdot v}{\overset{\cong}{\mathbb{Z}}} \longrightarrow 0$$

$$\mathbb{D} \longmapsto -a_1 - b_1 + a_1 + b_1 + \cdots - a_g - b_g + a_g + b_g = 0$$

← signs: compare edge orientation with anticlockwise orientation of $\partial\mathbb{D}$

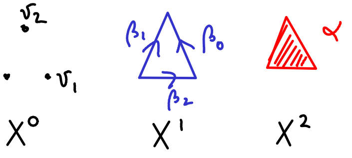
$$H_*(\Sigma_g) \cong \begin{cases}\mathbb{Z} & *=0,2 \\ \mathbb{Z}^{2g} & *=1 \\ 0 & \text{else}\end{cases}$$

Lemma X Δ-cx structure $\Longrightarrow$ induces CW-cx structure on X and $(C^{CW}_*(X), d^{CW}) \cong (C^\Delta_*(X), d^\Delta)$

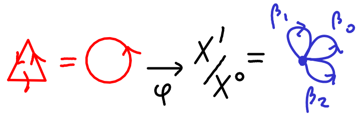
$$\Longrightarrow \boxed{H^{CW}_*(X) \cong H^\Delta_*(X)}$$

Pf $X^n = \underset{\mathbb{D}^n}{\overset{\cong}{\bigcup}}$ n-simplices of X and degrees are ± 1 depending on orient^n so can identify d^{CW} and d^Δ. □

Example $X =$ triangle $= \Delta^2$



$$\Rightarrow d^\Delta\alpha = \beta_0 - \beta_1 + \beta_2$$



$= ○ \xrightarrow{\varphi} X^1/X^0 =$

$d_{\alpha\beta_2} = d_{\alpha\beta_0} = +1$, $\quad d_{\alpha\beta_1} = -1$

$$\Rightarrow d^{CW}\alpha = d^\Delta\alpha \quad ✓ \quad □$$

Theorem X CW cx (or Δ-cx) $\Longrightarrow$ $\boxed{H^{CW}_*(X) \cong H_*(X)}$

$\Rightarrow$ H^Δ_*, H^{CW}_* independent of choice of CW-cx/Δ-cx structure.

Pf ① $H_*(X^n, X^{n-1}) \cong \tilde{H}_*(X^n/X^{n-1}) \cong \tilde{H}_*(\bigvee_\alpha S^n) \cong \bigoplus_\alpha \tilde{H}_* S^n$ $= 0 \Longleftrightarrow * \neq n$ — lives in degree n

LES for (X^n, X^{n-1}) $\Rightarrow$ $H_*(X^{n-1}) \to H_*(X^n)$ iso for $* < n-1$, $* > n$

② for $*<n$: $H_*(X^n) \cong H_*(X^{n+1}) \cong H_*(X^{n+2}) \cong \dots \cong H_*(X)$ — by ① ; by compactness each sing. chain lands in X^N, some N

③ for $* > n$: $H_*(X^n) \cong H_*(X^{n-1}) \cong H_*(X^{n-2}) \cong \dots \cong H_*(X^{-1}) = 0$

④ LES: $\dots \to \underset{\overset{\|}{0}\ \text{by ③}}{H_n(X^{n-1})} \to H_n(X^n) \xrightarrow[q^n_n]{} H_n(X^n, X^{n-1}) \to \dots$

$\Rightarrow$ q^n_n injective $\forall n$

⑤ LES: $\dots \xrightarrow{q^{n+1}_{n+1}} H_{n+1}(X^{n+1}, X^n) \xrightarrow{\delta^{n+1}_{n+1}} H_n(X^n) \to H_n(X^{n+1}) \to \overset{①}{0}$

UPSHOT

$$H_n(X) \overset{②}{\cong} H_n(X^{n+1}) \overset{⑤}{\cong} H_n(X^n)/\mathrm{im}\,\delta^{n+1}_{n+1} \overset{④}{\cong} \underbrace{(q^n_n H_n(X^n))}_{\substack{\| \\ \mathrm{im}\, q^n_n \\ \| \\ \mathrm{Ker}\,\delta^n_n}} / \mathrm{im}\ \underbrace{q^n_n \circ \delta^{n+1}_{n+1}}_{\substack{\| \\ d^{CW}_{n+1}}} \cong H^{CW}_n(X)$$

1st iso thm

exactness LES

$\mathrm{Ker}\,\delta^n_n \overset{④}{=} \mathrm{Ker}\ \underbrace{q^{n-1}_{n-1} \circ \delta^n_n}_{\substack{\| \\ d^{CW}_n}}$ □

Rmk by ① $\boxed{H_k \text{ not affected if attach } (k+2)\text{-cells or higher}}$

by ② $\boxed{\text{Inclusion } X^n \to X \text{ induces iso } H_*(X^n) \to H_*(X) \text{ for } * < n}$

Cor X n-dimensional cell cx $\Longrightarrow$ $H_*(X) = 0$ for $* > n$

Axioms for homology: Eilenberg-Steenrod axioms

It was no accident that H^Δ_*, H^{CW}_*, H_* all agreed.

Def A **generalised homology theory (GHT)** is a **functor** F: $\mathrm{TopPairs} = \left(\begin{array}{l}\text{Category of pairs} \\ \text{of spaces, and} \\ \text{maps of pairs}\end{array}\right) \longrightarrow$ Graded Abelian Gps with a **natural transformation** $\delta : F_*(X,A) \longrightarrow \underbrace{F_{*-1}(X,\emptyset)}$ satisfying : ← abbreviated: $F_{*-1}(X)$

1) **homotopy invariance**: $f \simeq g \Rightarrow F(f) = F(g)$

2) **exactness**: $\exists$ LES $\boxed{\dots \to F_*(A) \to F_*(X) \to F_*(X,A) \xrightarrow{\delta} F_{*-1}(A) \to \dots}$

$F(\mathrm{incl}: A \to X)$ $\quad$ $F(\mathrm{incl}: (X,\emptyset) \to (X,A))$

3) **additivity**: $(X,A) = \bigsqcup (X_i, A_i)$, $\mathrm{incl}_i : (X_i, A_i) \longrightarrow (X,A)$

then $\boxed{\sum F(\mathrm{incl}) : \bigoplus F(X_i, A_i) \xrightarrow{\cong} F(X,A)}$

4) **excision**: $\boxed{\overline{E} \subseteq A^\circ \subseteq X \Rightarrow F(X \setminus E, A \setminus E) \underset{F(\mathrm{incl})}{\xrightarrow{\cong}} F(X,A)}$

Remark (4) $\Longleftrightarrow$ $X = A^\circ \cup B^\circ$, $\mathrm{incl} : (B, A \cap B) \longrightarrow (X,A)$

then $\boxed{F(\mathrm{incl}) : F(B, A \cap B) \xrightarrow{\cong} F(X,A)}$

Pf $B = X \setminus E$, $E = X \setminus B$ noticing that $(X \setminus E)^\circ \cup A^\circ = X$

$E = A \setminus B$ noticing that $\overline{E} \subseteq \overline{A} \setminus B^\circ \subseteq A^\circ \setminus B^\circ \subseteq A^\circ$. □

$X = A^\circ \cup B^\circ$
so $\partial B \subseteq A^\circ$

Rmk In (3), the topology on the disjoint union $\bigsqcup (X_i, A_i)$ is defined by : $U \subseteq \bigsqcup (X_i, A_i)$ open $\Leftrightarrow$ $U \cap X_i \subseteq X_i$ open $\forall i$

FACT **Theorem**

a) $(F, \delta_F), (G, \delta_G)$ GHTs, $\alpha : F \to G$ a natural transformation commuting with δ_F, δ_G such that $\alpha_{\mathrm{point}} : F(\mathrm{point}) \to G(\mathrm{point})$ is an iso, then α is an iso.

b) If (F, δ_F) GHT satisfies (5) **dimension**: $\boxed{F_*(\mathrm{point}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}}$

Then $\exists$ natural iso $F \cong H_*$ (such an F is called a **homology theory**)

Rmk In (b) if require $F_0(\mathrm{point}) = G$ an abelian group (instead of $\mathbb{Z}$)

$\Longrightarrow$ $F(X,A) \cong H_*(X,A;G) =$ (homology with coefficients in G) ← later in course

9. COHOMOLOGY

(C_*, ∂_*) chain cx s.t. C_* free $\mathbb{Z}$-module ← $C_* \cong \bigoplus_\alpha \mathbb{Z}$

Def n-cochains $\boxed{C^n = \mathrm{Hom}(C_n, \mathbb{Z})}$

coboundary map $\partial^n : C^n \longrightarrow C^{n+1}$

(this is the dual of ∂) $\partial^n(\phi) = \phi \circ \partial_{n+1}$

$$\begin{tikzcd} C_n \arrow[d, "\phi"'] & C_{n+1} \arrow[l, "\partial_{n+1}"'] \arrow[dl, "\partial^*\phi = \phi\circ\partial"] \\ \mathbb{Z} & \end{tikzcd}$$

Notice ∂^* is degree +1 map (not −1)

$$\boxed{H^m(C_*, \partial_*) = \frac{\mathrm{Ker}\,\partial^m}{\mathrm{Im}\,\partial^{m-1}}}$$

$\mathrm{Ker}\,\partial^m$ ← cocycles

$\mathrm{Im}\,\partial^{m-1}$ ← coboundaries

(Note $\partial^* \circ \partial^* = 0$: $\partial^*\partial^*\phi = \phi \circ \underbrace{\partial \circ \partial}_{=0} = 0$)

Rmk If use negative grading, (C^{-*}, ∂^{-*}) is a chain complex with homology so many results from H_* carry over to H^*. So abusively write

$$\boxed{H^*(C_*, \partial_*) = H_*(\mathrm{Hom}(C_*, \mathbb{Z}), \partial^*)}$$

Remarks about dualisation

$\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$ generated by projection maps $\pi_i(x_1, \dots, x_n) = x_i$ ← this is the dual of the standard basis: $\pi_i = e_i^* : e_i \to 1$, $e_k \to 0$, $k \neq i$

$$\begin{tikzcd} \alpha : \mathbb{Z}^n \arrow[r] & \mathbb{Z}^m & \Rightarrow & \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \arrow[d, phantom, "\cong"] & & \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z}) \arrow[ll, "\text{dual}"', "\alpha^*"] \arrow[d, phantom, "\cong"] \\ x \arrow[r, mapsto] & A \cdot x & & \mathbb{Z}^n & & \mathbb{Z}^m \arrow[ll, "\text{transpose}(A)"] \end{tikzcd}$$

A ← m×n matrix

$\alpha^*\phi = \phi \circ \alpha$

Def X space ⇒ singular cohomology $\boxed{H^*(X) = H^*(C^*(X), \partial^*)}$

similarly define H^*_Δ, H^*_{CW} — dualise $C_* = C_*(X)$

Example $\mathbb{RP}^3$: $C^{CW}_*(\mathbb{RP}^3)$: $0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

dualise : $C^*_{CW}(\mathbb{RP}^3)$: $0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longrightarrow 0$

$$H^*(\mathbb{RP}^3) \cong H^*_{CW}(\mathbb{RP}^3) = \begin{cases} \mathbb{Z} & * = 0,3 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

← notice $H_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$ has moved to grading 2.

Functoriality

$f : X \to Y \Rightarrow f_* : C_*X \to C_*Y$ ← called pull-back

$\Rightarrow C^*X \xleftarrow{f^*} C^*Y$ is dual : $\boxed{f^*\phi = \phi \circ f_*}$

Lemma f^* is a cochain map (meaning $\partial^* \circ f^* = f^* \circ \partial^*$)

$\Rightarrow \boxed{f^* : H^*Y \to H^*X}$

Pf
$$\begin{aligned} (\partial^* \circ f^*)(\phi) &= \partial^*(\phi \circ f_*) \\ &= (\phi \circ f_*) \circ \partial \\ &= (\phi \circ \partial) \circ f_* \quad \text{as } f_* \text{ chain map} \\ &= f^* \circ (\phi \circ \partial) \\ &= f^* \circ (\partial^* \phi) \\ &= (f^* \circ \partial^*)(\phi) \end{aligned}$$

Properties
- $\mathrm{id}^* = \mathrm{id}$
- $(f \circ g)^* = g^* \circ f^*$ notice order!

$\Longrightarrow \boxed{H^* : \mathrm{Top} \longrightarrow \mathrm{GradedAbGps}}$ contravariant functor

Exercise $H^0(X) = \prod_{\pi_0 X} \mathbb{Z}$ where $\pi_0 X = \{\text{path-components of } X\}$

Homotopy invariance

Lemma $f_*, g_* : C_* \longrightarrow \tilde{C}_*$ chain h'pic (free) $\Rightarrow f^* = g^* : H^*\tilde{C} \to H^*C$

Pf $f_* - g_* = \tilde{\partial} \circ h + h \circ \partial$ some $h : C_* \longrightarrow \tilde{C}_*[1]$

$f^* - g^* = h^* \circ \tilde{\partial}^* + \partial^* \circ h^*$ for dual $h^* : \tilde{C}^* \longrightarrow C^*[-1]$.

(notice degree −1, not +1) □

Def h^* called cochain homotopy

Cor $f \simeq g : X \to Y \Rightarrow f^* = g^* : H^*Y \longrightarrow H^*X$ □

Algebra : dual of SES

Lemma $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ exact, A,B,C free

$\Rightarrow$ $0 \leftarrow A^* \xleftarrow{i^*} B^* \xleftarrow{j^*} C^* \leftarrow 0$ exact

Cultural Remark $\left(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}\right)^* = \prod_{n\in\mathbb{N}} \mathbb{Z}^*$ is not free. (Baer 1937) so A^*, B^*, C^* are not free unless A,B,C have finite ranks

Pf C free $\Rightarrow \exists$ splitting $B \underset{s}{\overset{j}{\rightleftarrows}} C$ $\quad j \circ s = id$

Pick preimages b_i for basis e_i of C, then $s(e_i) = b_i$

$\Rightarrow$ $A \oplus C \xrightarrow[i \oplus s]{\cong} B$

Rmk inverse is $B \cong A \oplus C$
$b \longmapsto i^{-1}(b - s(b)) \oplus j(b)$

dual $\Rightarrow$ $A^* \oplus C^* \xleftarrow[i^* \oplus s^*]{\cong} B^*$ and $s^* \circ j^* = id$
→ so i^* surj → so j^* inj

$\Rightarrow$ $0 \leftarrow A^* \xleftarrow{i^*} B^* \underset{s^*}{\overset{j^*}{\leftrightarrows}} C^* \leftarrow 0$

where $0 = (j \circ i)^* = i^* \circ j^*$ so $\operatorname{Im} j^* \subseteq \operatorname{Ker} i^*$

prove $\supseteq$: $i^* b = 0 \Rightarrow b - j^* s^* b \in \operatorname{Ker} i^* \cap \operatorname{Ker} s^* = \{0\}$ since $i^* \oplus s^*$ is iso.

$\Rightarrow b = j^* s^* b \in \operatorname{Im} j^*$ since $s^* j^* = id$

$\Rightarrow \operatorname{Ker} i^* = \operatorname{Im} j^*$ □

Excision, LES, Mayer–Vietoris

By previous Lemma get dual results :

Excision $\overline{E} \subseteq A^\circ \subseteq X \Rightarrow H^*(X \setminus E, A \setminus E) \xleftarrow[\cong]{i^*} H^*(X,A)$

LES for pair (X,A) $\quad \cdots \xleftarrow{q^*[+1]} H^{*+1}(X,A) \xleftarrow{\delta} H^*(A) \xleftarrow{i^*} H^*(X) \xleftarrow{q^*} H^*(X,A) \leftarrow \cdots$

M.V. $X = A^\circ \cup B^\circ \Rightarrow \cdots \leftarrow H^{*+1}(X) \leftarrow H^*(A\cap B) \xleftarrow[i_A^* \oplus -i_B^*]{} H^*(A) \oplus H^*(B) \xleftarrow[j_A^* \oplus j_B^*]{} H^*(X) \leftarrow \cdots$

where
$$\begin{tikzcd} & A \arrow[dr, "j_A"] & \\ A \cap B \arrow[ur, "i_A"] \arrow[dr, "i_B"'] & & X \\ & B \arrow[ur, "j_B"'] & \end{tikzcd}$$
are the obvious maps

Axioms for cohomology These are analogous to the axioms for homology except we reverse all arrows, and we change axiom (3): $\prod$ instead of $\oplus$

additivity : $(X,A) = \bigsqcup (X_i, A_i)$, $\mathrm{incl}_i : (X_i; A_i) \longrightarrow (X,A)$

then $\boxed{\prod F(\mathrm{incl}_i) : \prod F(X_i; A_i) \xleftarrow{\cong} F(X,A)}$

10. CUP PRODUCT

Theorem $H^*(X)$ is ① **unital** ② **graded-commutative** ring via
(space)

$\cup : H^k(X) \times H^\ell(X) \longrightarrow H^{k+\ell}(X)$ determined by

$$\boxed{\begin{array}{l} \cup : C^k(X) \times C^\ell(X) \longrightarrow C^{k+\ell}(X) \\ (\phi \cup \psi)(\sigma) = \phi\left(\sigma|_{[e_0,\dots,\underline{\underline{e_k}}]}\right) \cdot \psi\left(\sigma|_{[\underline{\underline{e_k}},\dots,e_{k+\ell}]}\right) \end{array}}$$

① $1 \in C^0(X)$ constant function $\Rightarrow$ $1 \cup \phi = \phi \cup 1 = \phi$

② $\phi \cup \psi = (-1)^{\deg\phi \cdot \deg\psi} \psi \cup \phi$

Useful trick

If X is CW-cx, then $C_*^{CW}(X) \underset{\simeq}{\xrightarrow{\text{inclusion}}} C_*(X)$, so $C^*_{CW}(X) \underset{\simeq}{\xleftarrow{\text{restriction}}} C^*(X)$. So to define/determine a class in $H^*(X)$ it is enough to define its values on CW chains (provided it is a CW-cocycle). So doing :
$\phi \circ \mathrm{incl} \longleftarrow\!| \ \phi$

$$\begin{tikzcd} H^k_{CW} \times H^\ell_{CW} \arrow[r, "\cong"] \arrow[d] & H^k \times H^\ell \arrow[d, "\cup"] \\ H^{k+\ell}_{CW} \arrow[r, "\cong"] & H^{k+\ell} \end{tikzcd}$$
we get an induced product

Proof of Theorem

$$\partial^*(\phi \cup \psi)(\sigma) = (\phi \cup \psi)(\partial\sigma)$$
$$= (\phi \cup \psi) \ \sum (-1)^i \ \sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_n]} \qquad n = k+\ell$$
$$= \sum_{i \leq k} (-1)^i \phi\left(\sigma|_{[e_0,\dots,\hat{e}_i,\dots,\underline{\underline{e_{k+1}}}]}\right) \cdot \psi\left(\sigma|_{[\underline{\underline{e_{k+1}}},\dots,e_n]}\right)$$
$$+ \sum_{i > k} (-1)^i \phi\left(\sigma|_{[e_0,\dots,e_k]}\right) \cdot \psi\left(\sigma|_{[e_k,\dots,\hat{e}_i,\dots,e_n]}\right) \cdot \underbrace{(-1)^{i-k}(-1)^{k-i}}_{1}$$
$$= \left((\partial^*\phi) \cup \psi(\sigma)\right) + (-1)^k \ \phi \cup \partial^*\psi$$

induces $[\phi] \cup [\psi] = [\phi \cup \psi]$:

well-defined : • cycles → cycle : $\partial(\phi \cup \psi) = \overset{=0}{\widetilde{(\partial\phi)}} \cup \psi \pm \phi \cup \overset{=0}{\widetilde{(\partial\psi)}} = 0$ ✓

• $[\phi] = [\phi + \partial\alpha]$ so need $[(\partial\alpha) \cup \psi] = 0$

$(\partial\alpha) \cup \psi \underset{\partial\psi = 0}{=} \partial(\alpha \cup \psi)$ ✓

• Similarly $[\phi] \cup [\partial\beta] = 0$

bilinear, associative, distributive: true at chain level unital: $(\partial 1)(\sigma) = 1(\sigma|_{[e_0]}) - 1(\sigma|_{[e_0]}) = 1-1=0$

$(1 \cup \psi)(\sigma) = 1(\sigma|_{[e_0]}) \cdot \psi(\sigma|_{[e_0,\ldots,e_n]}) = \psi(\sigma)$ $(\phi \cup 1 = \phi$ similar$)$

graded-comm. sketch proof: ⟵ **non-examinable**

Let $r : C_n(X) \longrightarrow C_n(X)$, $r(\sigma) = \varepsilon_n \overline{\sigma}$ where: $\varepsilon_n = (-1)^{\frac{n(n+1)}{2}}$

and $\overline{\sigma}|_{[v_0,\ldots,v_n]} = \sigma|_{[v_n,\ldots,v_0]}$ ⟵ reverse order of vertices: is product of $\underbrace{n+(n-1)+\cdots+1}_{n(n+1)/2}$ transpositions

(idea: each transposition is a reflection in hyperplane, so reverses orientation of simplex, so insert ε_n to compensate)

one checks:
- r chain map
- $\underbrace{r^*\varphi \cup r^*\psi}_{\varepsilon_k\varepsilon_\ell} = \underbrace{r^*(\psi \cup \varphi)}_{\varepsilon_{k+\ell}}$ differ by $(-1)^{k\ell}$
- $r \simeq \mathrm{id}$ so can drop $r^* = \mathrm{id}$ on cohomology

$$\left(\begin{array}{l} r - \mathrm{id} = P\partial + \partial P \text{ with} \\ P\sigma = \sum (-1)^i \varepsilon_{n-i} \, (\sigma \circ \pi)|_{[v_0,\ldots,v_i,w_n,\ldots,w_i]} \end{array}\right) \square$$

v_i, w_i like for prism operator

projection $\Delta^n \times I \xrightarrow{\pi} \Delta^n$

Naturality of cup product

Lemma $f : X \longrightarrow Y \implies f^* : H^*Y \longrightarrow H^*X$ hom of unital rings

Pf
$$\begin{aligned} f^*(\varphi \cup \psi)(\sigma) &= (\varphi \cup \psi)(f_*\sigma) \\ &= \varphi(f_*\sigma|_{[e_0,\ldots,e_k]}) \cdot \psi(f_*\sigma|_{[e_k,\ldots,e_n]}) \\ &= ((\varphi \circ f_*) \cup (\psi \circ f_*))(\sigma) \\ &= (f^*\varphi \cup f^*\psi)(\sigma) \end{aligned}$$

unital: $f^*(1) = 1 \circ f_* = 1$ $\square$

UPSHOT $\boxed{H^* : \text{Top} \longrightarrow \left\{\begin{array}{c}\text{Graded-commutative unital rings} \\ \text{with graded unital ring homs}\end{array}\right\} \text{ contravariant functor.}}$

Warning An (iso)morphism $H^*(Y) \longrightarrow H^*(X)$ of groups will also preserve the ring structure if f^* is induced by a map of spaces $X \to Y$ (by above Lemma).

$\implies$ Cor The excision theorem iso on cohomology is an iso of rings.

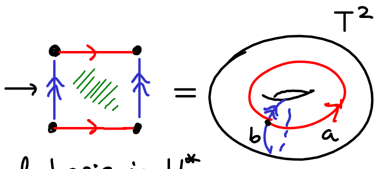
However the connecting hom in M.V. or LES cannot possibly be a ring hom since it drops gradings by 1 ($\Rightarrow \delta(a \cup b)$ and $\delta(a) \cup \delta(b)$ have different grading!)

Example $H^1(T^2) \times H^1(T^2) \xrightarrow{\cup} H^2(T^2)$ bilinear form $\mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ with matrix $\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$

Pf recall:

$*$		$H_*(T^2)$	$H^*(T^2)$
0	$\mathbb{Z}$	$\mathbb{Z}\cdot pt$	$\mathbb{Z}\cdot 1$
1	$\mathbb{Z}^2$	$\mathbb{Z}a \oplus \mathbb{Z}b$	$\mathbb{Z}a^* + \mathbb{Z}b^*$
2	$\mathbb{Z}$	$\mathbb{Z}\cdot D$	$\mathbb{Z}\cdot D^*$

where $D : \Delta^2 \cong$ square ⟶



$1, a^*, b^*, D^*$ are dual basis in H^*

Identify $H^*(T^2) \cong H^*_\Delta(T^2)$ so at chain level: ⟵ $X = T^2$

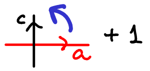
$a^* : C_1^{cw}(X) \longrightarrow \mathbb{Z}$, $a \longmapsto 1$, $b \longmapsto 0$

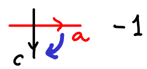
$b^* : C_1^{cw}(X) \longrightarrow \mathbb{Z}$, $a \longmapsto 0$, $b \longmapsto 1$

$D^* : C_2^{cw}(X) \longrightarrow \mathbb{Z}$, $D \longmapsto 1$

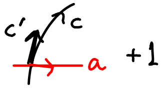
$\implies \underset{C_1^{cw}}{\overset{}{b^*(\underset{\cap}{c})}} = \#$ a intersects c counted with orientation signs

$\underset{C_1^{cw}}{a^*(\underset{\cap}{c})} = -\#$ b intersects c counted with signs.





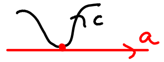
Fact Same holds for smooth singular 1-chains $c : \Delta^1 \cong I \longrightarrow T^2$ which intersect a transversely: velocity vectors a', c' span $\mathbb{R}^2$



Otherwise ill-defined:

c not smooth

and

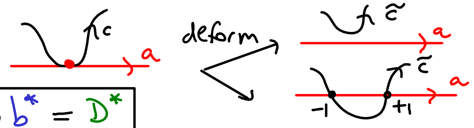


a, c not transverse (tangency)

are bad.

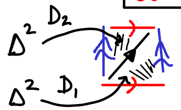
trick need first pick homologous representative which is smooth & transverse, by continuously deforming the chain (continuous map $\simeq$ id so id on H_*)

Example



both cases: $a^*(\tilde{c}) = 0$

Claim $\boxed{a^* \cup b^* = D^*}$



$$(a^* \cup b^*)(\underbrace{D_1 + D_2}_{\text{homologous to } D}) = a^*(\underbrace{D_1|_{[e_0,e_1]}}_{=}) \cdot b^*(\underbrace{D_1|_{[e_1,e_2]}}_{=}) + \text{same for } D_2$$

Notice we are using the "Useful Trick" (start of Sec. 10) We view D as the singular cycle $D_1 + D_2$.

$$\begin{aligned} &= a^*(a)\,b^*(b) + a^*(b)\,b^*(a) \\ &= 1 \end{aligned}$$

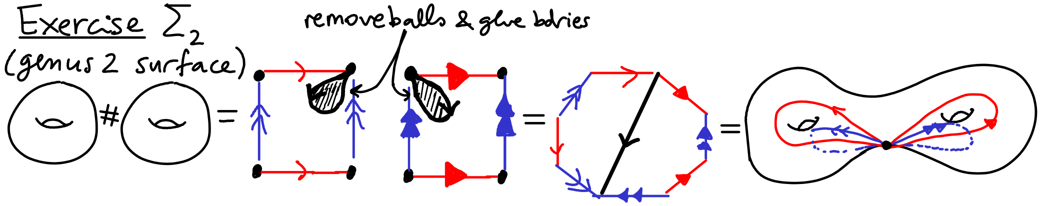
Graded-comm. $\Rightarrow b^* \cup a^* = -D^*$, $a^* \cup a^* = (-1)^{1\cdot 1} a^* \cup a^*$ so $=0$, sim'ly $b^* \cup b^* = 0$. $\square$

Idea $\cup$ just counts (signed) geometric intersection # of corresponding curves.

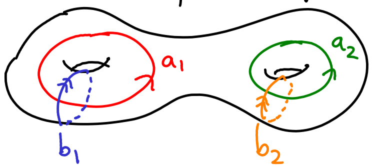
Why "$a \cap a = 0$"? Can deform a to make it disjoint from a:

Exercise Σ_2

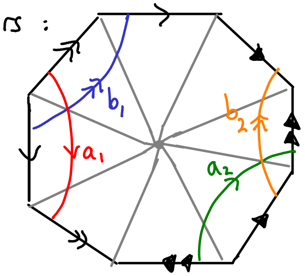
(genus 2 surface)



Make life simpler : deform generators :



=



		$H_*(\Sigma_2)$	$H^*(\Sigma_2)$
0	$\mathbb{Z}$	$\mathbb{Z}\cdot pt$	$\mathbb{Z}\cdot 1$
1	$\mathbb{Z}^4$	$\mathbb{Z}a_1+\mathbb{Z}b_1+\mathbb{Z}a_2+\mathbb{Z}b_2$	$\mathbb{Z}\langle a_1^*, b_1^*, a_2^*, b_2^*\rangle$ ← dual basis
2	$\mathbb{Z}$	$\mathbb{Z}\cdot D$	$\mathbb{Z}\cdot D^*$

Notice on $C_1^{CW}(\Sigma_2)$: $a_i^*(c) = -\#(b_i \text{ intersects } c)$ ← signed count

$b_i^*(c) = \#(a_i \quad \text{"} \quad \text{"})$

Exercise $a_i^* \cup b_j^* = \delta_{ij} \cdot D^* = -b_j^* \cup a_i^*$ $\quad \delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$

$a_i^* \cup a_i^* = b_i^* \cup b_i^* = 0$

so same as geometric intersection numbers of corresponding curves.

hint: D is homologous to the sum of ± triangles in last picture (orientation signs)

Cultural Rmk on general theory (Intersection Theory/Differential Topology)

M^m oriented m-mfd
$N^n \subseteq M^m$ oriented compact n-dim submfd $\Rightarrow$ $H_n(N) \xrightarrow{incl_*} H_n(M)$ ← see later in course
$\exists\, [N] \longmapsto [N]$

N, M also smooth (see Differential Geometry course) $\Rightarrow$ $\omega_N \in H^{m-n}(M)$ counts # intersecns with N ← with signs

$N_1^{n_1}, N_2^{n_2} \subseteq M$ compact oriented smooth submfds
$n_1+n_2=m$
(so complementary dimensions) $\Rightarrow$ $\omega_{N_1} \cup \omega_{N_2} = \underbrace{\#(N_1 \cap N_2)}_{\text{geometric intersection } \#} \cdot [M]^*$

may require ← deforming unless N_1, N_2 are already transverse

Fact (Thom 1954)

Not all $a \in H^j(M)$ arise as ω_N for connected compact oriented codim=j smooth submfd N

But $\exists\, N \in \mathbb{N}$ s.t. $N\cdot a$ does arise. They do arise for $H^*(M;\mathbb{Q}), H^*(M,\mathbb{R}), H^*(M;\mathbb{Z}/2)$

11. KÜNNETH FORMULA AND PRODUCT SPACES

Algebra : tensor products

R ring (comm. with 1) — e.g. abelian groups = $\mathbb{Z}$-mods, vector spaces/$\mathbb{F}$ = $\mathbb{F}$-mods

Def A, B R-modules $\Rightarrow$ **Tensor product** is R-module

$$\boxed{A \underset{R}{\otimes} B = \langle (a,b) : a\in A, b\in B\rangle / \text{relations of bilinearity \& rescaling}}$$

(or $A\otimes B$) R-mod generated; write $a\otimes b$ for its class

bilinearity : $(a_1+a_2)\otimes b = a_1\otimes b + a_2\otimes b$

$a\otimes(b_1+b_2) = a\otimes b_1 + a\otimes b_2$

rescaling : $r\cdot(a\otimes b) = (ra)\otimes b = a\otimes(rb)$ $\quad \forall r\in R$ ("can move $r\in R$ across the $\otimes$ symbol")

- So general element looks like $\sum a_k \otimes b_k$ (finite sum) ← NOT UNIQUELY!
- Don't confuse with $A\times B$: e.g. $0\otimes b = 0 \quad \forall b$

Rmk Can define $A\underset{R}{\otimes}B$ also by a universal property : for all R-mods C,

$$\boxed{\mathrm{Hom}_R(A\underset{R}{\otimes}B; C) \xrightarrow[\text{natural}]{\cong} \{R\text{-bilinear maps } A\times B \to C\}}$$

Using above description of $A\otimes B$: $\varphi \longmapsto (A\times B \to C, (a,b)\mapsto \varphi(a\otimes b))$

Example ($R=\mathbb{F}$) V, W v.s./$\mathbb{F}$ (basis v_i, basis w_j) $\Rightarrow$ $V\otimes W$ v.s./$\mathbb{F}$ basis $v_i\otimes w_j$

$\dim_{\mathbb{F}} V\otimes W = \dim V \cdot \dim W$

Exercise V, W finite $\dim/\mathbb{F} \Rightarrow V^*\otimes W \cong \mathrm{Hom}_{\mathbb{F}}(V,W)$

Hint $f: V\to\mathbb{F}$, $w\in W$, $f\otimes w \longmapsto (V\to W, v\longmapsto f(v)\cdot w)$

e.g. $(\mathbb{R}^n)^*\otimes\mathbb{R}^m \cong \mathrm{Mat}_{m\times n}(\mathbb{R}) \cong \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)$
$e_i^*\otimes e_j \longleftrightarrow$ matrix A with $A_{ji}=1$, 0 else.

Examples ($R=\mathbb{Z}$)

- $\mathbb{Z}^n \otimes \mathbb{Z}^m \cong \mathbb{Z}^{n\cdot m}$
- $\mathbb{Z}/n \otimes \mathbb{Z} \cong \mathbb{Z}/n \longleftarrow \sum a_i\otimes b_i = (\sum a_i b_i)\otimes 1$
- $\mathbb{Z}/2 \otimes \mathbb{Z}/3 = 0 \longleftarrow 1\otimes x = 3\otimes x = 1\otimes 3x = 0$
- $\mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong \mathbb{Z}/2 \longleftarrow \begin{cases}1\otimes 1 = 3\otimes 1 = 1\otimes 3\\ 1\otimes 2 = 2\otimes 1 = 0\end{cases}$

Examples

- $A\otimes B \cong B\otimes A$
- $(\bigoplus_i A_i)\otimes(\bigoplus_j B_j) \cong \bigoplus_i\bigoplus_j (A_i\otimes B_j)$
- $A\otimes R \cong A$ (so "$\cdot \underset{R}{\otimes} R$ does nothing")
- $A\otimes R/d \cong A/d\cdot A$

hence now know $A\otimes B$ for any f.g. R-mods A, B.

for example $\mathbb{Z}/2\otimes\mathbb{Z}/4 \cong (\mathbb{Z}/4)/2\cdot\mathbb{Z}/4 \cong \mathbb{Z}/2$ ← (**Rmk** $(\mathbb{Z}/n)/m\cdot\mathbb{Z}/n \cong \mathbb{Z}/\gcd(m,n)$)

More generally : $\begin{cases} R/I \underset{R}{\otimes} R/J \cong R/(I+J) \text{ for ideals } I,J\subseteq R.\\ A\otimes_R R/J \cong A/J\cdot A\end{cases}$

Warning $\cdot\otimes A$ often not an exact functor, i.e. does not preserve exact sequences

indeed it can ruin injectivity: $0\to\mathbb{Z}\xrightarrow{\cdot 2}\mathbb{Z}$ now take $\cdot\otimes\mathbb{Z}/_2$ get $0\to\mathbb{Z}/_2\xrightarrow{0}\mathbb{Z}/_2$.

Fact $\boxed{\cdot\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \text{ and } \cdot\underset{\mathbb{Z}}{\otimes}\mathbb{R} \text{ are exact functors on } \mathbb{Z}\text{-mods}}$ ← More generally $\cdot\underset{R}{\otimes}\text{Frac}(R)$ is exact on R-mods where Frac R is fraction field, and R is an integral domain "localisation is an exact functor"

example A f.g. $\mathbb{Z}$-mod $\Rightarrow A\cong\mathbb{Z}^r\oplus\mathbb{Z}/_{d_1}\oplus\dots\oplus\mathbb{Z}/_{d_k}$ some $d_i\neq 0$

$\Rightarrow A\otimes\mathbb{Q}\cong\mathbb{Q}^r \Rightarrow \boxed{r=\text{rank }A=\dim_{\mathbb{Q}}(A\otimes\mathbb{Q})}$

Tensor product of chain cxes

$C_*, \tilde{C}_*$ chain cxes of R-mods

$$(C_*\otimes\tilde{C}_*)_n = \bigoplus_{i+j=n} C_i\otimes\tilde{C}_j$$

$$\boxed{\partial(x\otimes y) = (\partial x)\otimes y + (-1)^{\deg x}\, x\otimes\tilde{\partial}y}$$ "Leibniz rule"

Think of ∂ as an operator of deg $=-1$ acting from left

since ∂ "jumps over x" get $(-1)^{\deg\partial\cdot\deg x}$

Exercise $\partial\circ\partial=0$ ← would fail without sign

recall $Z_*=\ker\partial$ = cycles, $B_*=\text{im}\,\partial$ = boundaries

$$Z_i\otimes\tilde{Z}_j\subseteq Z_{i+j}(C_*\otimes\tilde{C}_*) \text{ and } \left.\begin{matrix} Z_i\otimes\tilde{B}_j \\ B_i\otimes\tilde{Z}_j\end{matrix}\right\}\subseteq B_{i+j}(C_*\otimes\tilde{C}_*)$$

Cor $\exists$ natural maps
$$\boxed{\begin{matrix} H_i(C_*)\otimes H_j(\tilde{C}_*) \longrightarrow H_{i+j}(C_*\otimes\tilde{C}_*) \\ \sum[c_k]\otimes[\tilde{c}_k] \longmapsto \sum[c_k\otimes\tilde{c}_k]\end{matrix}}$$

FACT: **Algebraic Künneth Thm**

$C_*, H_*(C_*)$ f.g. free R-mods (PID (principal ideal domain)) (no assumption on $\tilde{C}_*$)

$$\Rightarrow \boxed{\bigoplus_{i+j=n} H_i(C_*)\otimes H_j(\tilde{C}_*) \cong H_n(C_*\otimes\tilde{C}_*)}$$ via

Algebra: Euler characteristic

C finitely generated graded abelian gp (so $\mathbb{Z}$-mod)
(more generally: R-mod for PID R)

Def **Euler characteristic** $\chi(C) = \sum(-1)^i\,\text{rank }C_i$

Example/Motivation X finite CW-cx then take $C=C_*^{CW}(X)$ to get

$$\boxed{\chi(X) = \#(0\text{-cells}) - \#(1\text{-cells}) + \#(2\text{-cells}) - \dots}$$

Lemma If C_* f.g. chain cx $\Rightarrow \boxed{\chi(C_*)=\chi(H_*(C_*))}$ $(=\sum(-1)^i\text{rank }H_i(C))$

Pf Observation: $\text{rank }C_i = \dim_{\mathbb{Q}}(C_i\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$ ← for R-mods, do $\dim_{\mathbb{F}}(C_i\underset{R}{\otimes}\mathbb{F})$ with $\mathbb{F}=\text{Frac}(R)$

$\Rightarrow$ WLOG assume C_i are vector spaces/field $\mathbb{F}$.

Abbreviate $|C_i|=\dim_{\mathbb{F}}C_i$.

Rank-nullity thm

$$0\to Z_i\to C_i\xrightarrow{\partial} B_{i+1}\to 0 \Rightarrow |C_i|=|Z_i|+|B_{i+1}|$$
$$0\to B_i\to Z_i\to H_i\to 0 \Rightarrow |H_i|=|Z_i|-|B_i|$$
$$\Rightarrow |C_i|-|H_i|=|B_{i+1}|-|B_i|$$

$$\Rightarrow \chi(C_*)-\chi(H_*) = \sum(-1)^i|B_{i+1}|-\sum(-1)^i|B_i| = \sum(-1)^i(-|B_i|+|B_i|)=0. \quad \square$$

Cor X space $\Rightarrow$
$$\boxed{\begin{aligned}\chi(X) &= \sum(-1)^i\,\text{rank }H_i(X) \\ &= \sum(-1)^i\,\text{rank }C_i(X)\end{aligned}}$$ ← if finite rank $H_*(X)$ ← if finite rank $C_*(X)$

So $\chi(X)$ is invariant up to hpy equivalence! **Example** $\chi\left(\begin{smallmatrix}\text{platonic}\\ \text{solid}\end{smallmatrix}\right)=\chi(S^2)=2$

Product spaces

X, Y CW-cxes $\Rightarrow$ $X\times Y$ CW-cx with cells $e_\alpha\times e_\beta$ attaching maps

e_α, e_β

$$\partial(\mathbb{D}^i_\alpha\times\mathbb{D}^j_\beta) = (\partial\mathbb{D}^i_\alpha)\times\mathbb{D}^j_\beta \cup \mathbb{D}^i_\alpha\times(\partial\mathbb{D}^j_\beta)$$

$$\begin{tikzcd} (\partial\mathbb{D}^i_\alpha)\times\mathbb{D}^j_\beta \arrow[d, "\varphi_\alpha\times id"] & \mathbb{D}^i_\alpha\times(\partial\mathbb{D}^j_\beta) \arrow[d, "id\times\varphi_\beta"] \\ X^{i-1}\times Y^j \arrow[dr] & X^i\times Y^{j-1} \arrow[d] \\ & (X\times Y)^{i+j-1} \end{tikzcd}$$

Cor $\boxed{\chi(X\times Y)=\chi(X)\cdot\chi(Y)}$ $\forall$ finite CW-cxes X, Y

Pf $\sum(-1)^k\,\text{rank }H_k^{CW}(X\times Y)$

$= \sum(-1)^k\,\text{rank }C_k^{CW}(X\times Y) = \sum_k\sum_{i+j=k}(-1)^{i+j}\,\text{rank }C_i^{CW}(X)\cdot\text{rank }C_j^{CW}(Y)$ $\square$

Lemma $d(e^i_\alpha\times e^j_\beta) = (de^i_\alpha)\times e^j_\beta + (-1)^i\, e^i_\alpha\times(de^j_\beta)$

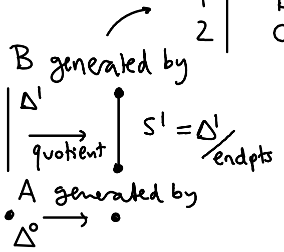
(proof later) hence $\boxed{C_*^{CW}(X\times Y)\cong C_*^{CW}(X)\otimes C_*^{CW}(Y)}$

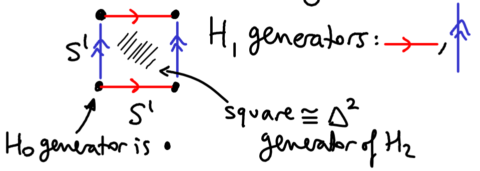
Hence if $H_*(Y)$ free then by Künneth $H_*(X\times Y)\cong H_*(X)\otimes H_*(Y)$.

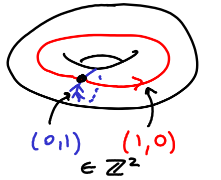
Example

$*$	$H_*(S^1)$
0	$A\cong\mathbb{Z}$
1	$B\cong\mathbb{Z}$
2	0

$*$	$H_*(S^1)\otimes H_*(S^1)\cong H_*(S^1\times S^1)$ ← torus	
0	$A\otimes A$	$\cong\mathbb{Z}$
1	$(A\otimes B)\oplus(B\otimes A)$	$\cong\mathbb{Z}^2$
2	$B\otimes B$	$\cong\mathbb{Z}$
3	0	







Pf $(\partial \mathbb{D}_\alpha^i)\times \mathbb{D}_\gamma^j \xrightarrow[\varphi_\alpha \times id]{} X^{i-1}\times Y^j \longrightarrow X^{i-1}\times Y^j \Big/ \underbrace{(X\times Y)^{i+j-2}\cap(X^{i-1}\times Y^j)}_{\overset{\text{easy check}}{\|}\; X^{i-2}\times Y^j \,\cup\, X^{i-1}\times Y^{j-1}}$

$\uparrow$ This proof is Non-examinable

$\bigstar :=$

$X^{i-1} = X^{i-2}\cup(\mathbb{D}_\beta^{i-1} \cup \dots)/\sim$

$Y^j = Y^{j-1}\cup(\mathbb{D}_\gamma^j \cup \dots)/\sim$

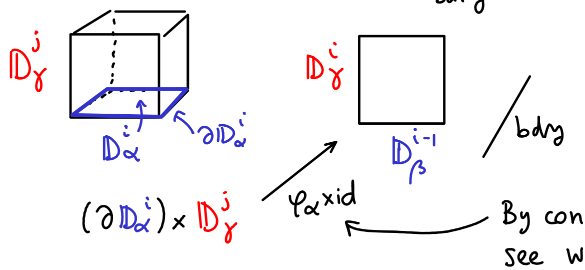
$\Big\}$ get $\sim$ from attaching maps

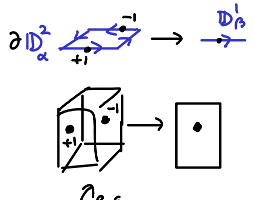
$X^{i-1}\times Y^j = X^{i-2}\times Y^j \;\cup\; X^{i-1}\times Y^{j-1} \;\cup\; (\mathbb{D}_\beta^{i-1}\times \mathbb{D}_\gamma^j \cup \dots)/\sim$

$\Rightarrow \bigstar = (\mathbb{D}_\beta^{i-1}\times\mathbb{D}_\gamma^j \cup \dots)$ / boundaries

$= \mathbb{D}_\beta^{i-1}\times \mathbb{D}_\gamma^j \Big/ \partial(\mathbb{D}_\beta^{i-1}\times\mathbb{D}_\gamma^j) \quad \vee \quad \dots$

$(\partial \mathbb{D}_\alpha^i)\times \mathbb{D}_\gamma^j \xrightarrow[\varphi_\alpha\times id]{} \mathbb{D}_\beta^{i-1}\times\mathbb{D}_\gamma^j \big/ \text{bdry} \quad \vee \dots$





e.g.

By considering local degrees now we see we get degree $= d_{\alpha\beta}$ for this.

$\Rightarrow$ get contribution $(d e_\alpha^i)\times e_\beta^j$ ✓

similarly

$\mathbb{D}_\alpha^i \times \partial\mathbb{D}_\gamma^j \xrightarrow[id\times\varphi_\gamma]{} \mathbb{D}_\alpha^i \times \mathbb{D}_\delta^{j-1}\big/\text{bdry}$

$\Rightarrow$ degree $(-1)^i d_{\gamma\delta}$

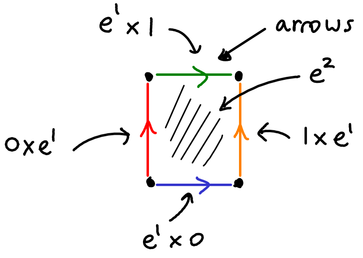
so get $(-1)^i e_\alpha^i \times d e_\beta^j$

$(-1)^i$ caused by orientations:

Could reorder factors: $\mathbb{D}_\alpha^i \times \mathbb{D}_\gamma^j \cong \mathbb{D}_\gamma^j \times \mathbb{D}_\alpha^i$ by $\begin{pmatrix} 0 & Id_j \\ Id_i & 0\end{pmatrix}$ whose $\det = (-1)^{ij}$. Then $\partial\mathbb{D}_\gamma^j \times \mathbb{D}_\alpha^i \to \mathbb{D}_\delta^{j-1}\times\mathbb{D}_\alpha^i/\text{bdry}$ gives degree $d_{\gamma\delta}$.

Swap factors $\mathbb{D}_\delta^{j-1}\times\mathbb{D}_\alpha^i/\text{bdry}$ by $\begin{pmatrix} 0 & Id_i \\ Id_{j-1} & 0\end{pmatrix}$, $\det = (-1)^{i(j-1)}$. Total sign $=(-1)^i$.

Example Recall after definition of H_*^{CW} we had example $I\times I$:



arrows here tell us how we map $[-1,1]\to$ edge (so orientation)

$\partial e^2 = +e^1\times 0 + 1\times e^1 - e^1\times 1 - 0\times e^1$

$= (\partial e^1)\times e^1 \;\; \overset{\uparrow}{-} \;\; e^1\times(\partial e^1)$ ✓

$(-1)^{\dim e^1}$

A further comment on orientation sign $(-1)^i$

$\mathbb{D}^i \times \mathbb{D}^j \cong \underset{\substack{\| \\ [v_0,\dots,v_i]}}{\Delta^i} \times \underset{\substack{\| \\ [w_0,\dots,w_j]}}{\Delta^j}$ $\leftarrow$ viewed in $\mathbb{R}^i, \mathbb{R}^j$

project $\mathbb{R}^{i+1}\to\mathbb{R}^i$

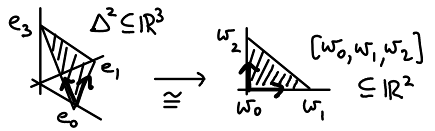
$(t_0,\dots,t_i)\mapsto(t_1,\dots,t_i)$

$\partial(\mathbb{D}^i\times\mathbb{D}^j) \cong \underbrace{\partial\Delta^i}_{\substack{\| \\ \sum_k (-1)^k[v_0,\dots,\hat{v}_k,\dots,v_i]}} \times \Delta^j \;\cup\; \Delta^i \times \underbrace{\partial\Delta^j}_{\substack{\| \\ \sum (-1)^k [w_0,\dots,\hat{w}_k,\dots,w_j]}}$

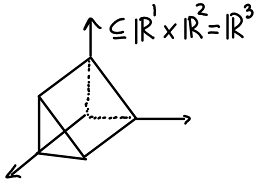
would be correct orientation sign for basis $w_1-w_0,\dots,\widehat{w_k-w_0},\dots,w_j-w_0$ but actually we have $[v_0,\dots,v_i]\times[w_0,\dots,\hat{w}_k,\dots,w_j]\subseteq\mathbb{R}^i\times\mathbb{R}^j$ and $(-1)^{i+k}$ is the orientation sign for the basis $v_1-v_0,\dots,v_i-v_0,w_1-w_0,\dots,\widehat{w_k-w_0},\dots,w_j-w_0$ for the hyperplane in $\mathbb{R}^{i+j+1}$ containing

$\Rightarrow$ need $(-1)^i$ to fix orientation sign.

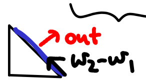
Example $\Delta^1\times\Delta^2$



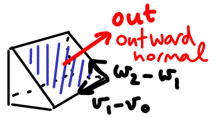
$\Delta^1\times\Delta^2 \cong$



$[v_0,v_1]\times\underbrace{[\hat{w}_0,w_1,w_2]}$



out, w_2-w_1 is positive $\mathbb{R}^2$-basis



out, v_1-v_0, w_2-w_1 is negative $\mathbb{R}^3$-basis

differ due to $(-1)^i$, $i=1$.

Projections
$$\begin{tikzcd} & X \\ X\times Y \arrow[ur, "p_X"] \arrow[dr, "p_Y"'] & \\ & Y \end{tikzcd}$$

FACT: **Künneth Theorem** If $H_n(Y)$ finitely generated, free $\forall n$

← no conditions on X

e.g. $Y \simeq$ finite CW cx

automatic if use field coefficients

$H_n(X\times Y)\cong \bigoplus_{i+j=n} H_i(X)\otimes H_j(Y)$	$H^n(X\times Y)\cong \bigoplus_{i+j=n} H^i(X)\otimes H^j(Y)$ $p_X^* a \cup p_Y^* b \longleftarrow a\otimes b$ (★)

Recall for cellular homology this on generators is: $e^i_\alpha \times e^j_\beta \longleftarrow e^i_\alpha\otimes e^j_\beta$

This is hom of rings if use following product
$$(a\otimes b)\cdot(\tilde a\otimes \tilde b) = (-1)^{|b|\cdot|\tilde a|}(a\cup\tilde a)\otimes(b\cup\tilde b)$$
gradings

think of it as "exchanging order of $b,\tilde a$"

Rmk

An indirect proof the Thm is to write down two generalised cohomology theories $F(X,A)=H^*(X,A)\otimes H^*(Y)$ and $G(X,A)=H^*(X\times Y, A\times Y)$, and consider the natural transformation $\alpha: F\to G$ given by (★), notice for $X=pt$, $A=\emptyset$ both F,G give $H^*(Y)$.

Example $X=S^n$, $Y=S^m$ $n\neq m$

$$H_*(S^n\times S^m)\cong\begin{cases}\mathbb{Z} & *=0,n,m,n+m\\ 0 & \text{else}\end{cases}\cong H^*(S^n\times S^m)$$
$a_n\cup a_m = a_{n+m}$ where $a_i = \text{dual}(e^i)$

$$H_*(S^n\times S^n)\cong\begin{cases}\mathbb{Z} & *=0,2n\\ \mathbb{Z}^2 & *=n\\ 0 & \text{else}\end{cases}\cong H^*(S^n\times S^n)$$
$a_n\cup a_n = a_{2n}$

Cor $H^k(T^n)\cong \Lambda^k[x_1,\dots,x_n]$ = free abelian gp. on gens. $\{x_{i_1}\wedge\dots\wedge x_{i_k} : i_1<\dots<i_k\}$

n-torus $S^1\times\dots\times S^1$

exterior algebra

where $x_i = p_i^*(\text{gen. of } H^1(S^1))$

$p_i: T^n\to S^1$ projections to factors.

so rank $=\binom{n}{k}$

Pf idea Künneth & induction $(T^n = T^{n-1}\times S^1)$ □

FACT cup product equals composition
$$\cup:\ H^i(X)\otimes H^j(X)\longrightarrow H^{i+j}(X\times X)\xrightarrow{\Delta^*} H^{i+j}(X)$$
$$(\Delta^i\xrightarrow{\sigma_1}X)\otimes(\Delta^j\xrightarrow{\sigma_2}X)\longmapsto(\underset{\underset{\Delta^{i+j}}{\cong}}{\Delta^i\times\Delta^j}\xrightarrow{\sigma_1\times\sigma_2}X\times X)$$
$\Delta =$ diagonal map $X\longrightarrow X\times X$, $x\longmapsto(x,x)$

12. UNIVERSAL COEFFICIENTS THEOREM

This proof is Non-examinable

(C_*,∂_*) chain cx
$$\Rightarrow 0\longrightarrow \underset{\partial=0\circlearrowleft}{Z_*=\operatorname{Ker}\partial_*}\xrightarrow{\text{incl}} C_*\xrightarrow{\partial_*}\underset{\partial=0\circlearrowleft}{B_{*-1}}=\operatorname{Im}\partial_{*-1}\longrightarrow 0 \text{ is SES}$$

FACT: Submodules of a free $\mathbb{Z}$-module are free
Rmk The same holds for R-mods if R is PID

($\mathbb{Z}$-module $\equiv$ abelian gp; free means: $\bigoplus_{\text{indexing set}}\mathbb{Z}$)

(PID = principal ideal domain = integral domain R s.t. every ideal $= R\cdot a$ some a)

Assume C_* free $\mathbb{Z}$-mod

FACT $\Rightarrow Z_*,B_*$ free (as $\operatorname{Ker}\partial^*,\operatorname{Im}\partial^*$ are submods of C_*)

$\Rightarrow$ SES splits, choose splitting $C_*\underset{s}{\overset{\partial_*}{\rightleftarrows}}B_{*-1}$ so $\partial_*\circ s = id$

$\Rightarrow$ dual SES
$$\begin{tikzcd}
0 & Z^* \arrow[l] & C^* \arrow[l, "\text{incl}^*"'] & B^{*-1} \arrow[l, "\partial^*"'] & 0 \arrow[l] \\
0 & Z^n \arrow[l] & C^n \arrow[l] & B^{n-1} \arrow[l, "\partial^*"'] & 0 \arrow[l] \\
0 & Z^{n-1} \arrow[l] \arrow[u, "\partial=0"] & C^{n-1} \arrow[l] \arrow[u, "\partial"] & B^{n-2} \arrow[l, "\partial^*"'] \arrow[u, "\partial=0"] & 0 \arrow[l]
\end{tikzcd}$$

note: incl^* = restrict to Z_* since $\text{incl}^*\circ\phi: Z_*\xrightarrow{\text{incl}}B_*\xrightarrow{\phi}\mathbb{Z}$

(Rmk Although $\partial^*=0: B^*\to B^{*+1}$ the map $\partial^*: B^{n-1}\to C^n$ need not $=0$
$\psi: B_{n-1}\to\mathbb{Z}$
$\Rightarrow \partial^*\psi=\psi\circ\partial: C_n\xrightarrow{\partial}B_{n-1}\xrightarrow{\psi}\mathbb{Z}$)

Connecting map $\delta: Z^{n-1}\to B^{n-1}$ of LES:
$$\begin{tikzcd}
0 & \partial^*\varphi & \varphi|_{B_*}=\phi|_{B_*} \arrow[l, "\partial^*"'] \\
\varphi|_{Z_*}=\phi \arrow[u, leftrightarrow] & \exists\varphi \arrow[l, leftrightarrow] \arrow[u] &
\end{tikzcd}$$
$B_*\subseteq Z_*$
$$\Rightarrow \boxed{\delta(\phi)=\phi|_{B_*}}$$

LES $\Rightarrow$ ($H^*B=B^*, H^*C=C^*$ since $\partial^*=0$)
$$\cdots\longleftarrow Z^n\longleftarrow H^nC\xleftarrow{\partial^*}B^{n-1}\xleftarrow{\delta^{n-1}}Z^{n-1}$$
$$\phi|_{B_{n-1}}\longleftarrow\!\shortmid\ \phi$$
$$\Rightarrow\quad 0\longleftarrow\operatorname{Ker}\delta^n\longleftarrow H^nC\longleftarrow B^{n-1}/\operatorname{Im}\delta^{n-1}\longleftarrow 0$$
$$\operatorname{Ker}\delta^n=\{\phi\in Z^n:\ \phi(B_n)=0\}\Longrightarrow \text{so: } \phi: Z_n\to\mathbb{Z}$$
$$\begin{tikzcd} Z_n \arrow[r] \arrow[dr] & \mathbb{Z} \\ & Z_n/B_n = H_n(C_*) \arrow[u] \end{tikzcd}$$
$= \operatorname{Hom}(H_n(C_*),\mathbb{Z})$

(★)

Universal Coefficients Thm:
$0\longrightarrow \underset{\underset{\textbf{Ext}^1(H_{n-1}(C);\mathbb{Z})}{\parallel}}{B^{n-1}/\operatorname{Im}\delta^{n-1}}\longrightarrow H^n(C)\longrightarrow\operatorname{Hom}(H_n(C_*),\mathbb{Z})\longrightarrow 0$ is SES
$[\varphi]\longmapsto(\varphi: H_n(C_*)\to\mathbb{Z})$

see next Lemma

and SES splits (but not naturally): $B^{n-1}/\operatorname{Im}\delta^{n-1}\underset{s^*}{\overset{\partial^*}{\rightleftarrows}}H^n(C)$ $\quad s^*\circ\partial^*=id$ (since $\partial\circ s=id \Rightarrow id=(\partial\circ s)^*=s^*\circ\partial^*$)

$\Rightarrow H^n(C)\cong\operatorname{Hom}(H_n(C),\mathbb{Z})\oplus\operatorname{Ext}^1(H_{n-1}(C);\mathbb{Z})$

Lemma $\mathrm{Ext}^1(H_{n-1}(C);\mathbb{Z}) \cong B^{n-1}/\mathrm{Im}\,\delta^{n-1}$ canonically

Algebra background on Extension groups $\mathrm{Ext}^i(M;\mathbb{Z})$

general case

M R-module, R ring (comm. with 1)

$\Rightarrow \exists$ free resolution :

$\cdots \longrightarrow P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \longrightarrow 0$ exact, P_i free R-mods

(pick gens x_α for $M \Rightarrow P_0 = \bigoplus_\alpha R \xrightarrow[\varphi_0]{} M,\ e_\alpha \mapsto x_\alpha$

" " y_β for $\mathrm{Ker}\,\varphi_0 \Rightarrow P_1 = \bigoplus_\beta R \xrightarrow[\varphi_1]{} \mathrm{Ker}\,\varphi_0,\ e_\beta \mapsto y_\beta$

continue inductively)

our case

$H_{n-1}(C_*)$ $\mathbb{Z}$-mod

$$\begin{tikzcd}
0 \arrow[r] & B_{n-1} \arrow[r, hook] \arrow[d, equal] & Z_{n-1} \arrow[r] \arrow[d, equal] & H_{n-1}(C) \arrow[r] \arrow[d, equal] & 0 \\
 & P_1 & P_0 & M &
\end{tikzcd}$$

Take $\mathrm{Hom}(\cdot\,;\mathbb{Z})$ and drop $\mathrm{Hom}(M;\mathbb{Z})$

$0 \longrightarrow \mathrm{Hom}(P_0;\mathbb{Z}) \xrightarrow{\varphi_1^*} \mathrm{Hom}(P_1;\mathbb{Z}) \xrightarrow{\varphi_2^*} \cdots$

Is cochain complex but not exact

$\Rightarrow$ take cohomology groups :

Def $\mathrm{Ext}^0(M;\mathbb{Z}) = \mathrm{Ker}\,\varphi_1^*$

Fact independent of choices P_i, φ_i

$\mathrm{Ext}^1(M;\mathbb{Z}) = \mathrm{Ker}\,\varphi_2^* / \mathrm{Im}\,\varphi_2^*$

...

$0 \to B^{n-1} \longrightarrow Z^{n-1} \longrightarrow 0$

Proof of Lemma

By **Example 2**,

$$\mathrm{Ext}^1(H_{n-1}(C_*);\mathbb{Z}) = \left\{ \begin{tikzcd} 0 \arrow[r] & B_{n-1} \arrow[r] \arrow[d, "\phi"'] & Z_{n-1} \\ & \mathbb{Z} & \end{tikzcd} \right\}$$ modulo

those arising from restriction

$$\left\{ \begin{tikzcd} 0 \arrow[r] & B_{n-1} \arrow[r] \arrow[dr, "\phi|_{B_{n-1}}"'] & Z_{n-1} \arrow[d, "\phi"] \\ & & \mathbb{Z} \end{tikzcd} \right\}$$

Thus $B^{n-1}/\mathrm{Im}\,\delta^{n-1}$. □

Example 1 $\mathrm{Ext}^0(M;\mathbb{Z}) \cong \mathrm{Hom}(M,\mathbb{Z})$

$$\begin{tikzcd}
P_1 \arrow[r, "\varphi_1"] \arrow[dr, "0"'] & P_0 \arrow[r, two heads, "\varphi_0"] \arrow[d, "\phi"] & M \arrow[dl, dashed] \\
 & \mathbb{Z} &
\end{tikzcd}$$

descends : $m \longmapsto \phi(\varphi_0^{-1}m)$

well defined since $\phi(\underset{\mathrm{Im}\,\varphi_1}{\underset{\|}{\mathrm{Ker}\,\varphi_0}}) = 0$

Example 2 $\mathrm{Ext}^1(M;\mathbb{Z}) =$

$$\left\{ \phi : \begin{tikzcd} P_2 \arrow[r] \arrow[dr, "0"'] & P_1 \arrow[r] \arrow[d, "\phi"] & P_0 \\ & \mathbb{Z} & \end{tikzcd} \right\} \Big/ \left\{ \phi = \varphi \circ \varphi_1 : \begin{tikzcd} P_1 \arrow[r, "\varphi_1"] \arrow[dr, "\phi"'] & P_0 \arrow[d, "\varphi"] \\ & \mathbb{Z} \end{tikzcd} \right\}$$

Rmk If R PID, then $\mathrm{Ker}(P_0 \to M)$ is free (since submod of free mod P_0)

$\Rightarrow$ can pick $P_1 = \mathrm{Ker}(P_0 \to M)$, $P_k = 0$ for $k \geqslant 2 \Rightarrow \mathrm{Ext}^k(M;\mathbb{Z}) = 0$ $k \geqslant 2$

(Co)homology with coefficients in a ring/field/module

Motivation

So far we had (C_*, ∂_*) (group hom) chain cx of abelian groups } in graded sense

$\Rightarrow H_*(C_*) = \mathrm{Ker}\,\partial_* / \mathrm{Im}\,\partial_*$ abelian group (since $\mathrm{Ker}\,\partial$, $\mathrm{Im}\,\partial$ are)

We cannot use a chain cx of (non-abelian) groups, because $\mathrm{Im}\,\partial_*$ need not be a normal subgroup of $\mathrm{Ker}\,\partial_*$.

However, abelian groups can be thought of as $\mathbb{Z}$-modules, then given any **abelian group G**, define **homology with coeffs in G**

$$\boxed{H_*(C_*;G) = H_*(C_* \otimes_{\mathbb{Z}} G)}$$ with differential $\partial_* \otimes \mathrm{id}$

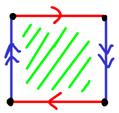
Def X space $\Longrightarrow$ $\boxed{H_*(X;G) = H_*(C_*(X) \otimes_{\mathbb{Z}} G)}$

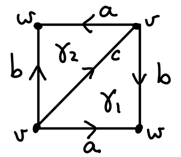
Explanation:

$C_k(X)$ free $\mathbb{Z}$-mod $\cong \bigoplus_{I_k} \mathbb{Z} \Rightarrow C_k(X) \otimes_{\mathbb{Z}} G \cong \bigoplus_{I_k} G$: just replace $\mathbb{Z}$ by G (as $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot \cong \cdot$)

Why care? We hope to get more/new invariants of spaces

Example $X = \mathbb{RP}^2 =$





$*$	$C_*^{\Delta}(\mathbb{RP}^2;G)$
0	$Gv \oplus Gw$
1	$Ga \oplus Gb \oplus Gc$
2	$G\gamma_1 \oplus G\gamma_2$

for $G = \mathbb{Z}/2$: $0 \to \mathbb{Z}/_2 \oplus \mathbb{Z}/_2 \xrightarrow[\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}]{\partial_2} \mathbb{Z}/_2 \oplus \mathbb{Z}/_2 \oplus \mathbb{Z}/_2 \xrightarrow[\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}]{\partial_1} \mathbb{Z}/_2 \oplus \mathbb{Z}/_2 \longrightarrow 0$

$$\Rightarrow H_*(\mathbb{RP}^2;\mathbb{Z}/_2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

compare: $H_*(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/_2 & * = 1 \\ 0 & \text{else} \end{cases}$ ($G = \mathbb{Z}$ case)

Form cochain complex using $\mathrm{Hom}_{\mathbb{Z}}(\cdot, G)$ (= group homs to G) in place of $\mathrm{Hom}(\cdot,\mathbb{Z})$

$$\boxed{\begin{aligned} H^*(C_*;G) &= H_*(\mathrm{Hom}_{\mathbb{Z}}(C_*,G)) \\ H^*(X;G) &= H^*(\mathrm{Hom}_{\mathbb{Z}}(C_*(X);G)) \end{aligned}}$$

with differential ∂^*: $\partial^*\phi = \phi \circ \partial_*$

X space

so: $H^*(C_*(X);G)$

Universal coefficients thm (Same proof using $\mathrm{Hom}_{\mathbb{Z}}(\cdot,G)$)

$$\boxed{\begin{aligned} 0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(C_*);G) \longrightarrow H^n(C_*;G) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_n(C_*),G) \longrightarrow 0 \\ [\varphi] &\longmapsto (\varphi : H_n(C_*) \to G) \end{aligned}}$$

__Example__ $X = \mathbb{R}P^2$, $G = \mathbb{Z}/2$, apply $\mathrm{Hom}_{\mathbb{Z}}(\cdot, G)$ to the complex in previous example

$$0 \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow[\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}]{\partial^1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow[\begin{pmatrix}1&1\\1&1\\0&0\end{pmatrix}]{\partial^0} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \leftarrow 0$$

$$\Rightarrow H^*(\mathbb{R}P^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases} \qquad \text{compare: } H^*(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

($G = \mathbb{Z}$ case)

__Can generalise further:__

C_* = chain cx of ...	coefficients in:	
abelian gps ($\mathbb{Z}$-mods)	abelian gp G ($\mathbb{Z}$-mod)	$H_*(C_*; G) = H_*(C_* \otimes_{\mathbb{Z}} G)$
R-modules ↖ ring (comm. with 1)	R-module M	$H_*(C_*; M) = H_*(C_* \otimes_R M)$

__Rmk__ $H_*(C; M)$ will be an R-module since $\ker \partial, \mathrm{Im}\, \partial$ are $\left(\partial_* \text{ is } R\text{-linear hom by assumption}\right)$

X space $\Rightarrow C_k(X; R) = C_k(X) \otimes_{\mathbb{Z}} R \cong \bigoplus_{I_k} R$: just replace $\mathbb{Z}$ by R (as $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot \cong \cdot$)

$$\Rightarrow \boxed{H_*(X; M) = H_*(C_*(X; R); M) = H_*(C_*(X; R) \otimes_R M)}$$ so just replace each $\mathbb{Z}$ by M in $C_*(X)$

Form __cochain complex__ using $\mathrm{Hom}_R(\cdot, M)$ $\left(= \begin{matrix}R\text{-linear}\\ \text{homs to } M\end{matrix}\right)$ in place of $\mathrm{Hom}(\cdot, \mathbb{Z})$

$$\boxed{\begin{aligned} H^*(C_*; M) &= H_*(\mathrm{Hom}_R(C_*, M)) \\ H^*(X; M) &= H^*(\mathrm{Hom}_R(C_*(X; R), M)) \end{aligned}}$$

with differential ∂^*: $\partial^* \phi = \phi \circ \partial_*$

X space

so: $H^*(C_*(X; R); M)$

__Rmk__ These are R-mods. If we use $M = R$, then they are also rings via cup product

__Universal Coefficients Thm__ For R any __PID__, C_* chain cx of R-mods,

$$0 \to \underset{\underset{B^{n-1}/\mathrm{im}\,\delta^{n-1}}{\rotatebox{90}{$\cong$}}}{\mathrm{Ext}^1_R(H_{n-1}(C_*); M)} \to H^n(C_*, M) \to \mathrm{Hom}_R(H_n(C_*), M) \to 0 \text{ is SES}$$

working over R using homs to M $\qquad [\varphi] \longmapsto (\varphi : H_n(C_*) \to M)$

and the SES splits but the splitting is not natural. $\boxed{\text{Same proof using } \mathrm{Hom}_R(\cdot, M)}$

__Example__ $R = \mathbb{F}$ __field__ $\Rightarrow C_*, H_*, H^*$ are vector spaces$/\mathbb{F}$.

__Rmk__ all $\mathbb{F}$-mods (i.e. vector spaces$/\mathbb{F}$) are free $\mathbb{F}$-mods $\cong \bigoplus \mathbb{F} b_i$
basis b_i
up to iso they are determined by $\dim_{\mathbb{F}}$ = cardinality of basis.

__Cor__ C_* = chain cx of $\mathbb{F}$-vector spaces $\Rightarrow \boxed{H^n(C_*; \mathbb{F}) \cong (H_n(C_*))^*}$ ← dual v.s.: $\mathrm{Hom}_{\mathbb{F}}(H_n(C_*), \mathbb{F})$

__Pf__ Pick any basis v_i for $\mathbb{F}$-v.s. B_{n-1}, extend it to a basis v_i, w_j of Z_{n-1} (also works in ∞ dim case).

$\Rightarrow$ can extend any $\mathbb{F}$-linear map $\psi : B_{n-1} \to \mathbb{F}$ to $\phi : Z_{n-1} \to \mathbb{F}$ just pick any values $\phi(w_j) \in \mathbb{F}$ e.g. $\phi(w_j) = 0$.

$\Rightarrow B^{n-1}/\mathrm{im}\,\delta^{n-1} = 0$ so $H^n(C_*; \mathbb{F}) \longrightarrow \mathrm{Hom}(H_n(C_*); \mathbb{F})$ iso □

__Cor__ $\boxed{H^n(X; \mathbb{F}) \cong H_n(X; \mathbb{F})^*}$ ← dual v.s. for any field $\mathbb{F}$.

__Cor__ $\boxed{H^n(X; M) \underset{\substack{\uparrow \\ \text{if } X \simeq \text{CW-cx}}}{\cong} H^n_{CW}(X; M) \underset{\substack{\uparrow \\ \text{if } X \simeq \Delta\text{-cx}}}{\cong} H^n_{\Delta}(X; M)}$

__Pf__ Cor holds for __homology__ and the isos are __natural__. ← i.e. functorial w.r.t. maps
The universal coeff. thm SES is __natural__. So result holds by 5-Lemma. □

__Algebra: Structure thm for f.g. abelian groups and R-mods__

__Fact 1__ A f.g. abelian gp $\Rightarrow A \cong \underbrace{\mathbb{Z}^r}_{\text{free part } F} \oplus \underbrace{\mathbb{Z}/p_1^{n_1} \oplus \cdots \oplus \mathbb{Z}/p_k^{n_k}}_{\text{torsion part } T}$

where $p_i \in \mathbb{Z}$ prime (need not be distinct)
Also r, k, p_i, n_i are unique (up to reordering)

__Example__ $\mathbb{Z}/4 = \mathbb{Z}/2^2 \not\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$ (Chinese Remainder Thm)

__Fact 2__ $T \cong \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_k$ with $d_1 | d_2 | \ldots | d_k$ ($d_i \in \mathbb{N}$ unique)

__Example__ $\mathbb{Z}/2 \oplus \mathbb{Z}/2^2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3 \cdot 2^2$ $\qquad d_1 = 2$, $d_2 = 12$

__Fact 3__ M f.g. R-mod, R __PID__, then:

$$\boxed{\begin{aligned} M &\cong F \oplus T \\ F &\cong R^r \\ T &\cong R/d_1 \oplus R/d_2 \oplus \cdots \oplus R/d_k \end{aligned}}$$

$r \in \mathbb{N}$ unique, called __rank__ of M

$d_1 | \cdots | d_k$ non-zero, not invertible
d_i called __invariant factors__
unique up to mult^n by invertible elements ↖ e.g. ± 1 if $R = \mathbb{Z}$

__Rmk__ $T = \{m \in M : rm = 0 \text{ some } r \neq 0 \in R\}$ = __torsion elements__
$F \cong M/T$

Torsion shift

Easy Exercise $\boxed{\mathrm{Ext}^*_R(\bigoplus_i M_i ; \prod_j N_j) \cong \prod_i \prod_j \mathrm{Ext}^*_R(M_i ; N_j)}$ ← any R-mods M_i, N_j

↓

Upshot To compute $\mathrm{Ext}^1_R(M;R)$ for $M = R^r \oplus R/d_1 \oplus \cdots$ just need:

$$\boxed{\begin{array}{ll} \mathrm{Ext}^1_R(R;R) & = 0 \\ \mathrm{Ext}^1_R(R/d;R) & \cong R/d \end{array}}$$

← since $0 \to R \xrightarrow{id} R \to 0$ (with $R = P_1$, $R = P_0$)

← since $0 \to R \xrightarrow{d} R \xrightarrow{1} R/d \to 0$, $\downarrow \phi$ to R ← so choice of $\phi(1) \in R$ modulo ϕ coming from

$$\begin{tikzcd} R \arrow[r, "d"] \arrow[d, "\phi"'] & R \arrow[dl, "\varphi"] \\ R & \end{tikzcd}$$

so $\phi(1) = d\cdot\varphi(1) \in d\cdot R$

$\Rightarrow \boxed{\mathrm{Ext}^1_R(M;R) \cong \mathrm{Torsion}(M)}$

Exercises

- $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n ; \mathbb{Z}/m) \cong \mathbb{Z}/\gcd(m,n)$
- G abelian gp $\Rightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z};G) = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/d;G) \cong G/d\cdot G$ ← $d \neq 0$
- R any ring (comm. with 1) $x \in R$ not zero divisor $\Rightarrow \mathrm{Ext}^*_R(R/(x); N) \cong \begin{cases} \{n \in N : x\cdot n = 0\} & *=0 \\ N/x\cdot N & *=1 \\ 0 & \text{else} \end{cases}$ (N ← R-mod)

Cor If $H_n(X;R)$ f.g. R-mod $\forall n$, R PID,

$\Rightarrow H_n(X;R) = R^{r_n} \oplus T_n$ (free & torsion parts)

$\Rightarrow \boxed{H^n(X;R) \cong R^{r_n} \oplus T_{n-1}}$ ← torsion moves up!

($\cong$ ↑ not natural)

Pf $0 \to T_{n-1} \to H^n(X;R) \to \mathrm{Hom}(R^{r_n} \oplus T_{n-1}; R) \to 0$

$\mathrm{Hom}(R^{r_n} \oplus T_{n-1}; R) \cong \underbrace{(\mathrm{Hom}(R;R))^{r_n}}_{} \oplus \underbrace{\mathrm{Hom}(T_{n-1};R)}_{0}$

$R \to R$, $1 \mapsto x$ ⟶ $\cong R^{r_n}$; x determines the hom

0 since $T_{n-1} \to R$, $1 \mapsto 0$ (R is integral domain, so no torsion elts $\neq 0$)

$\Rightarrow 0 \to T_{n-1} \to H^n(X;R) \to \underbrace{R^{r_n}}_{\text{free}} \to 0$

free, so can split the SES (pick lifts of basis). ← so not canonical □

Example

$*$	$H_*(\mathbb{RP}^3)$	$H^*(\mathbb{RP}^3)$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/2$	0
2	0	$\mathbb{Z}/2$
3	$\mathbb{Z}$	$\mathbb{Z}$

torsion moves up

Universal coefficients Theorem in homology

FACT Theorem C_* chain cx of free R-mods (R ← PID), M R-module

$$\Rightarrow \boxed{\begin{array}{l} \text{SES } 0 \to H_*(C_*) \otimes_R M \to H_*(C_* \otimes_R M) \to \mathrm{Tor}^R_1(C_{*-1}; M) \to 0 \\ \qquad [c] \otimes m \mapsto [c \otimes m] \qquad\qquad \nwarrow \text{defined below.} \\ \text{The SES splits, but the splitting is not natural.} \end{array}}$$

Torsion groups: A, B R-mods (R comm. ring with 1)

pick $\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} A \to 0$ free resolution ← exact sequence, P_i free R-mods

take $\cdot \otimes B$, omit $A \otimes B$ $\Rightarrow \cdots \to P_2 \otimes B \xrightarrow{\varphi_2 \otimes id} P_1 \otimes B \xrightarrow{\varphi_1 \otimes id} P_0 \otimes B \to 0$ not exact but is chain cx

$\mathrm{Tor}^R_k(A,B) = H_k$ (this complex) ← fact independent of choices of P_i, φ_i

Rmk R PID $\Rightarrow \ker\varphi_0$ free $\Rightarrow$ can pick $P_1 = \ker\varphi_0$, $P_k = 0$ for $k \geq 2$

$\Rightarrow$ only $\mathrm{Tor}^R_0, \mathrm{Tor}^R_1$ can be non-zero

Example $\mathrm{Tor}^{\mathbb{Z}}_*(\mathbb{Z}/a, \mathbb{Z}/b) = ?$

$0 \to \mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z} \xrightarrow[\text{quotient}]{\varphi_0} \mathbb{Z}/a \to 0$ free resolution

take $\cdot \otimes \mathbb{Z}/b$, drop $\mathbb{Z}/a \otimes \mathbb{Z}/b$ $\Rightarrow 0 \to \mathbb{Z}/b \xrightarrow{\cdot a} \mathbb{Z}/b \to 0$ (since $\mathbb{Z} \otimes_{\mathbb{Z}} G \cong G$ any G)

$\mathrm{Tor}^{\mathbb{Z}}_0(\mathbb{Z}/a, \mathbb{Z}/b) = (\mathbb{Z}/b)/a\cdot\mathbb{Z}/b \cong \mathbb{Z}/\gcd(a,b) \cong \mathbb{Z}/a \otimes \mathbb{Z}/b$

$\mathrm{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/a ; \mathbb{Z}/b) = \{x \in \mathbb{Z}/b : a\cdot x = 0\} \cong \mathbb{Z}/\gcd(a,b)$

Facts $\mathrm{Tor}^R_0(A,B) = P_0 \otimes B / \mathrm{Im}(\varphi_1 \otimes id) \cong A \otimes B$

$\mathrm{Tor}^R_*(A,B) \cong \mathrm{Tor}^R_*(B,A)$

Exercise $\mathrm{Tor}^R_*(\bigoplus A_i, \bigoplus B_j) \cong \bigoplus_i \bigoplus_j \mathrm{Tor}^R_*(A_i, B_j)$

$\mathrm{Tor}^R_*(A,B) = 0$ for $* \geq 1$ if A or B is free (use $M \otimes_R R \cong M$)

$\Downarrow$

deduce $\mathrm{Tor}^R_1(A,M)$ for f.g. R-mods A (R ← PID)

$\mathrm{Tor}^R_*(R/u; M) \overset{\uparrow \text{R PID}}{\cong} \begin{cases} M/u\cdot M & *=0 \\ \underline{u\text{-torsion}}(M) = \{x \in M : u\cdot x = 0\} & *=1 \\ 0 & \text{else} \end{cases}$

Example $H_*(\mathbb{RP}^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & *=0 \\ \mathbb{Z}/2 & *=1 \\ \mathbb{Z}/2 & *=2 \end{cases}$ $\quad H_*(\mathbb{RP}^2) \otimes \mathbb{Z}/2 \cong \begin{cases} \mathbb{Z} \otimes \mathbb{Z}/2 \\ \mathbb{Z}/2 \otimes \mathbb{Z}/2 \\ 0 \end{cases} \cong \begin{cases} \mathbb{Z}/2 \\ \mathbb{Z}/2 \\ 0 \end{cases}$

↖ caused by $\mathrm{Tor}^{\mathbb{Z}}_1(H_1(\mathbb{RP}^2); \mathbb{Z}/2) = \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$

Künneth Thm

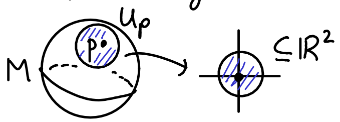
R PID $\Longrightarrow$ natural SES:

(C_* free ch. cx. R-mods, D_* any ch. cx. R-mods) → $\boxed{0 \to \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \to H_n(C_* \otimes D_*) \to \bigoplus_{i+j=n-1} \mathrm{Tor}^R_1(H_i(C), H_j(D)) \to 0}$

and the SES splits but the splitting is not natural. Example R = field, then this = 0.

13. MANIFOLDS : POINCARÉ-LEFSCHETZ DUALITY

- M **n-mfd** is Hausdorff topological space s.t. $\forall p \in M$ $\exists$ open neighbourhood $U_p \subseteq M$ homeomorphic to $\mathbb{R}^n$



(equivalently: to an open ball, or any open set in $\mathbb{R}^n$)

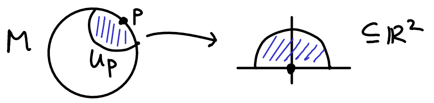
One also requires M **second countable** i.e. $\exists$ countable basis of open sets $\Longleftrightarrow$ M is covered by countably many such U_{p_i}
↖exercise

A **submanifold** $N \subseteq M$ is a mfd s.t. inclusion $N \to M$ is an embedding (i.e. a homeomorphism onto its image)

- M **n-mfd with boundary** if also allow $U_p \cong$ **upper half space $\mathbb{H}^n$** $= \{x \in \mathbb{R}^n : x_n \geq 0\} = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$, $p \longmapsto 0$

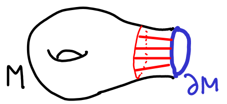
equivalently: any open nbhd of $0 \in \mathbb{H}^n$

Such p are called **boundary points** they form the **boundary ∂M** which is an $(n-1)$-mfd without boundary.



FACT (collar nbhd thm) $\partial M \subseteq M$ has an open neighbourhood $\cong \partial M \times (0,1]$
$\partial M \longrightarrow \partial M \times 1$

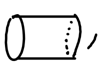
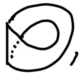
M is **closed** if compact without boundary.



Examples

closed mfds : S^n, $\mathbb{R}P^n$, $T^n = S^1 \times \cdots \times S^1$ (**n-torus**), $\mathbb{C}P^n$, $O(n)$, $SU(n)$

non-compact mfds: $\mathbb{R}^n$, $\mathrm{Mat}_{m\times n} \cong \mathbb{R}^{mn}$, $GL(n,\mathbb{R})$

mfds with bdry: $\mathbb{D}^n$, $\mathbb{D}^1 \times S^1 =$ , Möbius band $=$ , $T^2 \setminus$ open disc $=$ 

FACT (Milnor 1959) Any mfd is homotopy equivalent to a CW-cx

fact If M is a compact manifold then $H_*(M)$ are finitely generated

Rmk M **triangulable** if $M \simeq$ simplicial cx.
Not all mfds are triangulable, but most of those we encounter are.

Compact manifolds have f.g. homology

← Non-examinable proof

① X space is a **Euclidean neighbourhood retract** if $\exists$ **embedding** $j: X \longrightarrow \mathbb{R}^N$ some N, s.t. $i(X)$ is a retract of a nbhd $V \subseteq \mathbb{R}^n$.
↖(homeo onto image)

② X is **weakly locally contractible** if $\forall$ nbhd $x \in U \subseteq X$, $\exists$ nbhd $x \in V \subseteq U$ s.t. V is contractible inside U.

FACT compact $X \subseteq \mathbb{R}^n$ is ① $\Longleftrightarrow$ X is ②

Rmk If we find nbhd V as in ① with retraction $V \xrightarrow{f} X$ then any smaller nbhd V' also retracts using $f|_{V'} : V' \to X$. Similarly in ② $V' \subseteq V$ is contractible: restrict the hpy.

Lemma A X compact & ① $\Longrightarrow$ X is the retract of a finite simplicial cx

pf $i(X) \subseteq \mathbb{R}^n$ compact $\Rightarrow$ lies inside some large n-simplex $\Delta^n \to \mathbb{R}^n$
Apply barycentric subdivision until simplices have diameter $<$ dist$(X, \partial V)$.
Simpl.cx. $= \bigcup\{$subsimplices which intersect $X\}$ using the restriction of retraction $V \to X$. □

Rmk Also deduce X has f.g. homology since retractions are surjective on H_*.
$(\bigoplus \mathbb{Z} \longrightarrow\!\!\!\!\rightarrow H_*(\text{finite simpl. cx}) \overset{\text{retract}}{\longrightarrow\!\!\!\!\rightarrow} H_*(X)$ so get surjection from free $\mathbb{Z}$-mod, so f.g.)

Lemma B M compact mfd $\Longrightarrow$ M embeds into $\mathbb{R}^N$, some N.

Pf "Just do it proof":
$\forall p \in M$, $\exists$ homeo $\mathbb{D}^n \xrightarrow{\varphi_p}$ nbhd$(p \in M)$
Pick finite subcover of φ_{p_i} of $M = \bigcup_{p \in M} \varphi_p(\mathbb{D}^n)$. Say $i = 1, \ldots, k$
$\psi_{p_i} : M \xrightarrow{\varphi_{p_i}^{-1}} \mathbb{D}^n \to \mathbb{D}^n/_{\partial\mathbb{D}^n} \cong S^n \subseteq \mathbb{R}^{n+1}$ define embedding $(\psi_{p_1}, \ldots, \psi_{p_k}) : M \to \mathbb{R}^{k\cdot(n+1)}$
Finally use: a continuous bijection from a compact space to a Hausdorff space is $\cong$ □

Rmk Same works if M has boundary, just consider its **double** $\underset{\text{identify along } \partial M}{M \cup M}$ and apply the Lemma to the double.

Cor M compact mfd (possibly with bdry) $\Longrightarrow$ M has f.g. homology

Pf Mfds satisfy ② since locally ball $\simeq$ pt. M embeds in $\mathbb{R}^N$ by Lemma B.
① holds by FACT. Done by Lemma A. □

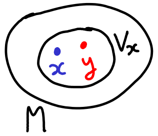
Def A **local orientation** of M at $x\in M$ is a choice of generator

excise complement of nbhd $V_x \cong \mathbb{D}^n$ — choice of homeo is not canonical!

$$\mu_x \in H_n(M, M\setminus x) \cong H_n(\mathbb{D}^n, \mathbb{D}^n\setminus 0) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}$$

Def An **orientation** of M is a **locally consistent** choice $x \mapsto \mu_x$

meaning:



$V_x \cong \mathbb{D}^n \simeq pt$

$$\begin{tikzcd} & H_n(M, M\setminus V_x) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ H_n(M,M\setminus x) \arrow[rr, "\cong"] & & H_n(M,M\setminus y) \end{tikzcd}$$

$$\begin{tikzcd} & \exists\, a \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ \mu_x \arrow[rr, mapsto] & & \mu_y \end{tikzcd}$$

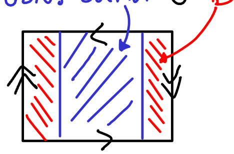
$\left(H_n(\mathbb{D}^n, \mathbb{D}^n\setminus x) \cong H_n(\mathbb{D}^n, \mathbb{D}^n\setminus y) \right)$ — $\mathbb{D}^n\setminus x \simeq \mathbb{D}^n \setminus y$

Def M **orientable** if $\exists$ orientation on M

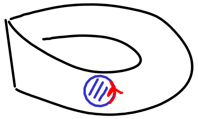
oriented if we chose an orientation

Examples S^n, $\mathbb{R}^n$, $\mathbb{C}P^n$, orientable surfaces Σ_g, $\mathbb{R}P^n$ ← odd n

Non-example $\mathbb{R}P^2$ = Möbius band $\cup\ \mathbb{D}^2$

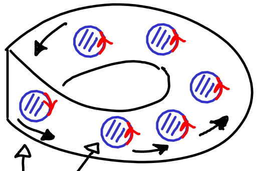


by local consistency can move disc continuously and preserves orientation



choice of μ_x is choice of orientation of boundary circle of small disc containing x

$\Rightarrow \mathbb{R}P^2$ not orientable



discs are differently oriented
$\Rightarrow$ contradicts local consistency.

The fundamental class [M]

FACT
Theorem For M closed n-mfd:

M orientble connected $\Rightarrow$ $H_n(M) \underset{\text{natural}}{\cong} H_n(M, M\setminus x) \underset{\text{choice}}{\cong} \mathbb{Z}$

$\Rightarrow$ $\exists\ [M] \longleftarrow\!\!\shortmid\ \mu_x$

↑ once we choose an orientation $(\mu_x)_{x\in M}$

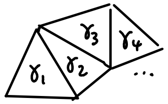
↑ called **fundamental class**

(if swap orientation: for $-\mu_x$ get $-[M]$)

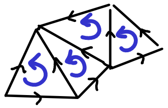
M not orientable connected $\Rightarrow$ $H_n(M) = 0$
$H_n(M; \mathbb{F}_2) = \mathbb{F}_2\cdot[M] \cong \mathbb{F}_2$ ← (or any field of characteristic 2)

Construction of [M] if M has Δ-complex structure

M compact $\Rightarrow$ finite # n-simplices $\gamma_1, \dots, \gamma_N$



M oriented $\Rightarrow$ pick orientations of $\gamma_1, \dots, \gamma_N$ to agree with given orientation of M: ↙ for $x\in \mathrm{Int}(\gamma_i)$



$$\mathbb{Z}\cdot\mu_x = H_n(M, M\setminus x) \underset{\text{exc.}}{\cong} H_n(\gamma_i, \gamma_i\setminus x) = \mathbb{Z}\cdot\gamma_i$$
$$\mu_x \longmapsto \gamma_i$$

$\Rightarrow$ $\boxed{[M] := \sum \gamma_i}$ satisfies $\partial[M] = 0$ ✓

$$\begin{tikzcd} H_n(M) \arrow[r] & H_n(M, M\setminus x) \arrow[r, "\cong"] & H_n(\gamma_i, \gamma_i\setminus x) \\ \left[M\right] \arrow[rr, mapsto, bend right] & \mu_x \arrow[r, mapsto] & \gamma_i \end{tikzcd}$$

Not difficult to see that $H_n^{CW}(X) = \mathbb{Z}\cdot[M]$, so $\Rightarrow H_n(M) \cong H_n(M, M\setminus x)$
$[M] \longmapsto \mu_x$

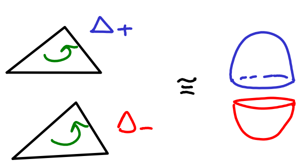
Also $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}\cdot[M]$ since $C_{n+1}(M) = 0$ ($\nexists (n+1)$-simplices since $\dim M = n$)

M non-orientble $\Rightarrow$ each facet of γ_i appears twice in $\partial\sum\gamma_i$
$\Rightarrow$ $\partial\sum\gamma_i = 0$ over $\mathbb{F}_2$ independently of choices of orientations of γ_i. ✓

Examples

1) $$S^n = \frac{\overset{\Delta_+}{\Delta^n} \cup \overset{\Delta_-}{\Delta^n}}{\text{glue bdries}}$$

S^2:

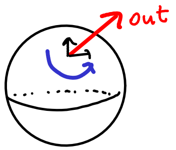


$[S^n] = \Delta_+ - \Delta_-$ if use canonical orientation we discussed

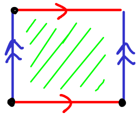
hence $\partial[S^n] = \partial\Delta - \partial\Delta = 0$

$\mathbb{D}^n \subseteq \mathbb{R}^n$ canonical orientation

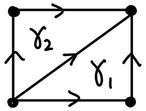
$\Rightarrow S^{n-1} = \partial\mathbb{D}^n$ " using outward normal first rule



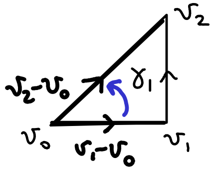
2) $T^2 =$



Δ-complex structure (compatibly with side identifications!)



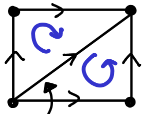
Want orientation induced by square $\subseteq \mathbb{R}^2$



$v_1 - v_0, v_2 - v_0$ positive $\mathbb{R}^2$-basis

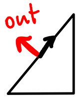
$\Rightarrow \gamma_1$ agrees with orientation

$\Longrightarrow$



$[T^2] = +\gamma_1 - \gamma_2$

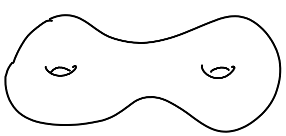
γ_2 orientation disagrees



Rmk general feature: if two simplices are identified along a facet then the 2 respective outward normals are related by reflection.

So consistency $\Rightarrow$
- either simplices are compatibly oriented and the two induced orientations on facet are opposite
- or not compatibly oriented but facet orient^n is same, then need sign like in example when build $[T^2]$

3) Recall $\Sigma_2 =$



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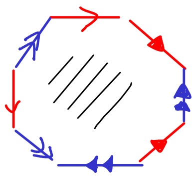
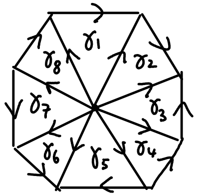


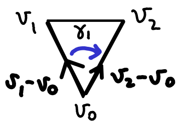
Δ-cx structure (compatible with side identifications!):



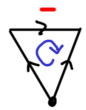
Use the orientation induced by polygon $\subseteq \mathbb{R}^2$

$\Rightarrow [\Sigma_2] = -\gamma_1 - \gamma_2 + \gamma_3 + \gamma_4 - \gamma_5 + \gamma_6 + \gamma_7 - \gamma_8$

Explanation:

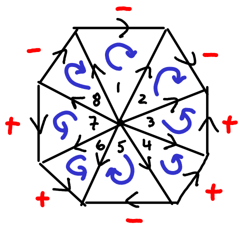


$v_1 - v_0, v_2 - v_0$ is negative $\mathbb{R}^2$-basis $\Rightarrow$



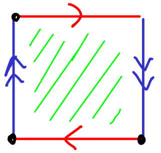
All simplices γ_i have $v_0 =$ centre of polygon

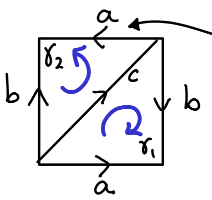
$\Rightarrow$ sign $<$ $-$ if outer edge clockwise / $+$ " anti "



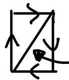
3) $\mathbb{R}P^2 =$

(non-orientable) example





won't get Δ-cx structure if you try



since get issue here

Use the orientation induced by square $\subseteq \mathbb{R}^2$

$$\Rightarrow [\mathbb{R}P^2] = -\gamma_1 + \gamma_2$$

$$\partial[\mathbb{R}P^2] = -(b - a + c) + (a - b + c)$$

$$= -2b + 2a$$

$$\neq 0$$ so not cycle in $C_*^{CW}(\mathbb{R}P^2)$

However, working modulo 2:

$$\partial[\mathbb{R}P^2] = 0 \in C_*^{CW}(\mathbb{R}P^2; \mathbb{F}_2)$$ since $2=0$ in $\mathbb{F}_2$

$$\Rightarrow [\mathbb{R}P^2] \in H^2(\mathbb{R}P^2; \mathbb{F}_2)$$

Degree

Def M, N oriented closed connected n-mfds, $f : M \to N$

$$f_* : H_n(M) \longrightarrow H_n(N)$$
$$[M] \mapsto \underbrace{\deg(f)}_{\in \mathbb{Z}} \cdot [N]$$

Lemma If $f^{-1}(y)$ finite,

then $\deg(f) = \sum_{x \in f^{-1}(y)} \deg(f_y)_*$

local degree

local map like in chapter 7

Pf

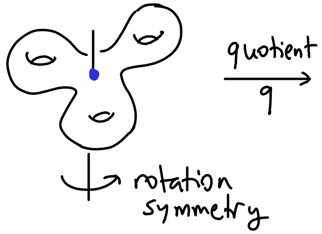
$$\begin{tikzcd}
{[M]} \arrow[dd, mapsto] & H_n(M) \arrow[r, "f_*"] \arrow[d, phantom, "\cong"] & H_n(N) \arrow[d, phantom, "\cong"] & {[N]} \\
 & \bigoplus_{x \in f^{-1}(y)} H_n(M, M \setminus x) \arrow[r, "(f_x)_*"] & H_n(N, N \setminus y) & \mu_y^N \arrow[u, mapsto] \\
\sum \mu_x^M \arrow[rr, mapsto] & & \left(\sum \deg(f_x)_*\right) \cdot \mu_y^N &
\end{tikzcd}$$

□

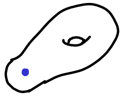
Examples

1) $S^1 \longrightarrow S^1,\ z \longmapsto z^n,\quad [S^1] \longmapsto n \cdot [S^1]$ so $\deg = n$

2) $\Sigma_3 =$



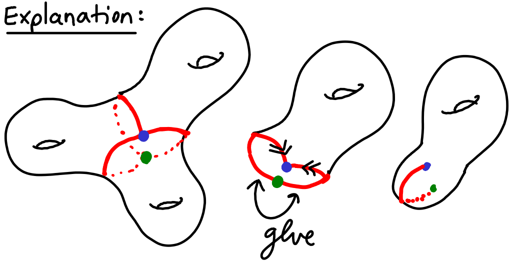
$\xrightarrow[q]{\text{quotient}} \Sigma_3 \big/ \mathbb{Z}/_3\text{-rotation action} =$



$= \Sigma_1$ torus

Easy check : $\deg(q) = 3$
(e.g. use local degrees)

Explanation:



Cultural Rmk

For M,N,f smooth, the $\deg f = \#$(preimages of a generic point of N)

Idea: $\deg f$ tells you how many times you cover N. (almost all points work)

Poincaré duality

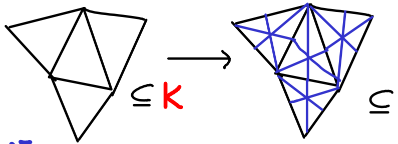
FACT Theorem For M closed n-mfd

M oriented $\Longrightarrow$ $\boxed{H^k(M) \cong H_{n-k}(M)}$ s.t. $\underset{H^0(M)\,\cong\,H_n(M)}{\overset{1\,\leftrightarrow\,[M]}{\cap \qquad \cap}}$

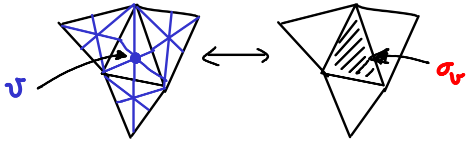
M non-oriented $\Longrightarrow$ same holds with $\mathbb{F}_2$ coefficients

Sketch proof when M is a simplicial complex K **(Non-examinable)**

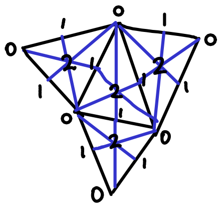
$S(K)$ = barycentric subdivision



1) simplex $\sigma = \sigma_v$ of K with barycentre v $\longleftrightarrow$ $v = v_\sigma$ vertex of S(K)



2) $ht(v) = (\text{height of } v) = \dim \sigma_v$



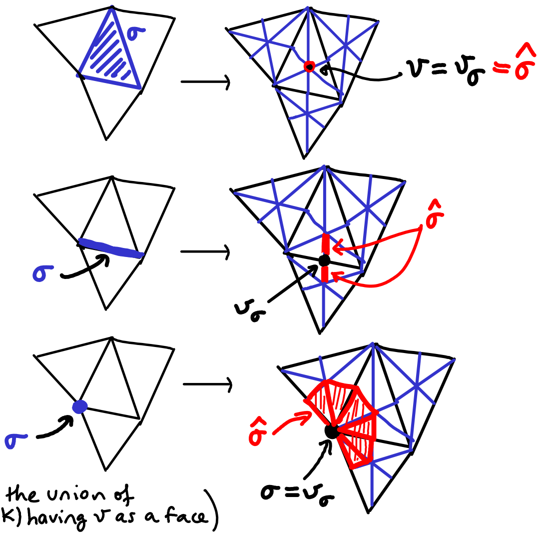
3) σ k-simplex of K
↓
dual simplex

$$\hat{\sigma} = \bigcup_{\substack{\tau \in S(K) \\ ht(v_\sigma) \text{ is min} \\ \text{of heights of} \\ \text{vertices of } \tau}} \tau$$

Rmk : $\cup \tau$ with $ht(v_\sigma)$ max will give back σ.
Thus $\hat{\sigma}, \sigma$ intersect transversely at v_σ.
One can also describe $\hat{\sigma}$ as

$$\hat{\sigma} = \bigcap_{\substack{\text{vertices} \\ v \in \sigma}} \overline{\text{Star}_{S(K)}(v)}$$

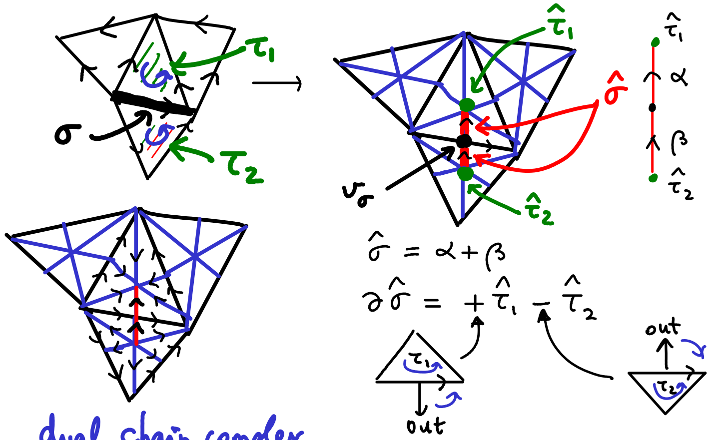
(closed star is the union of simplices of S(K) having v as a face)



FACTS

- $\dim \hat\sigma = n - \dim \sigma$
- dual cells $\hat\sigma$ give a cell decomposition of M ("polygonal" complex rather than Δ-cx)

(★) • $\partial\hat\sigma = \sum_{\substack{\text{facet}\\ \sigma \subsetneq \tau \\ \tau \in K}} \pm \hat\tau$ ← need compare orientations of σ, τ (+ if σ as a facet of τ has boundary orientation)



$\hat\sigma = \alpha + \beta$

$\partial\hat\sigma = +\hat\tau_1 - \hat\tau_2$

4) **dual chain complex**

D_{n-k} = free abelian group on dual chains $\hat\sigma$

$H_*(M) \cong H_*(D_*, \partial_*)$ (since $\hat\sigma$ give a cell decomp. of M)

5) $\varphi: D_{n-k} \longrightarrow C^k(M)$

$\hat\sigma \longmapsto \sigma^*$ where $\sigma^*(\alpha) = \begin{cases} 0 & \text{for } k\text{-cells } \alpha \neq \sigma \\ 1 & \text{for } \alpha = \sigma \end{cases}$

- φ linear bijection ✓
- chain map:

$\varphi(\partial\hat\sigma) = \varphi \underset{\swarrow \text{see (★)}}{\sum \pm \hat\tau} = \sum \pm \tau^*$

$\partial^*\varphi(\hat\sigma) = \partial^*\sigma^* = \left(\sigma^* \circ \partial : \tau \xmapsto{\partial} \sum_{\substack{\text{facet } \sigma_i \\ \text{of } \tau}} \pm\sigma_i \xmapsto{\sigma^*} \begin{cases} \pm 1 & \text{if one of } \sigma_i = \sigma \\ 0 & \text{else} \end{cases}\right)$

$= \sum \pm \tau^* = \varphi(\partial\hat\sigma)$ ✓

UPSHOT φ is chain iso so get iso:

$H_*(M) \underset{④}{\cong} H_*(D_*, \partial_*) \xrightarrow[\varphi]{\cong} H^{n-*}(M)$

Cor χ(odd dimensional closed orientable mfd) $= 0$

Pf b_i = rank $H_i(M)$ (**Betti numbers**)

$\chi(M) = b_0 - b_1 + \dots + b_{\dim M - 1} - b_{\dim M}$

equal by Poincaré duality □

(Poincaré–)Lefschetz duality

Theorem

M compact oriented n-mfd n-mfd with boundary

$\boxed{H^k(M) \cong H_{n-k}(M, \partial M)}$

$1 \in H^0(M) \longleftarrow [M, \partial M] \in H_n(M, \partial M)$

relative fundamental class

$\boxed{H_k(M) \cong H^{n-k}(M, \partial M)}$

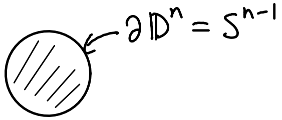
Non-oriented $\Longrightarrow$ same holds with $\mathbb{F}_2$ coefficients.

Pf basically same as Poincaré duality. □

Cor M compact, connected, $\partial M \neq \emptyset \Longrightarrow H^n(M) = H_0(M, \partial M) = 0$

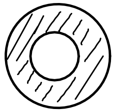
Examples

1) $\mathbb{D}^n$



$\mathbb{Z} \cong H^0 \mathbb{D}^n \cong H_n(\mathbb{D}^n, S^{n-1})$

2)



A = annulus $\subseteq \mathbb{R}^2$

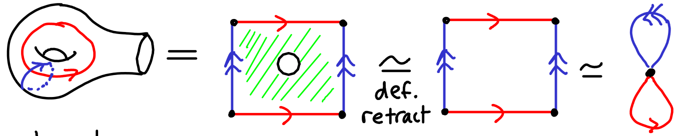
$\simeq S^1$

$\mathbb{Z} \cong H^0A \cong H_2(A, \partial A)$

$\mathbb{Z} \cong H^1A \cong H_1(A, \partial A)$

$0 \cong H^2A \cong H_0(A, \partial A)$

3) $M = T^2 \setminus$ open ball $=$



$\simeq S^1 \vee S^1$

$\Rightarrow H_*(M, \partial M) \cong H^{2-*}(S^1 \vee S^1) = \begin{cases} \mathbb{Z}^2 & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$ ← gen. by 2 loops ← gen. by $[M, \partial M]$

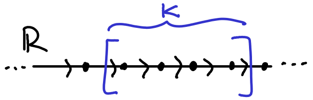
What happens in the non-compact case?

Locally finite homology (Borel-Moore)

$C^{lf}_*(X)$ allow infinite sums $\sum n_i \sigma_i$ ($n_i \in \mathbb{Z}$, σ_i ← generators of $C_*(X)$)
s.t. given any compact subset $K \subseteq X$,
$\#\{n_i \neq 0 : K \cap \mathrm{Im}\,\sigma_i \neq \emptyset\} < \infty$.

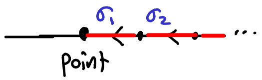
Examples

- $C_1^{lf}(\mathbb{R}) \ni \sum_{m\in\mathbb{Z}} \sigma_m$



$\Rightarrow$ get cycle $[\mathbb{R}] \in H_1^{lf}(\mathbb{R})$ $\qquad \sigma_m : I \cong [m, m+1] \subseteq \mathbb{R}$

- $C_0^{lf}(\mathbb{R}) \ni [\text{point}] \ni \partial(\sigma_1 + \sigma_2 + \cdots)$ is a boundary:



exercise $H_*^{lf}(\mathbb{R}) = \begin{cases} \mathbb{Z} & *=1 \\ 0 & \text{else} \end{cases}$ $(\cong H^{1-*}(\mathbb{R}))$

FACT Theorem M orientable n-mfd (possibly not compact) $\Rightarrow$ $\boxed{H^*(M) \cong H^{lf}_{n-*}(M)}$

cohomology with compact supports $H^*_c(X)$

$C^*_c(X)$: only allow cochains $\phi : C_*X \to \mathbb{Z}$ s.t. $\exists$ compact $K \subseteq X$ (depends on ϕ) with
$\phi(C_*(X\setminus K)) = 0$ (vanish on chains in $X\setminus K$)

Example $c \in C_*(X)$ cycle $\Rightarrow$ $\phi(\alpha) =$ signed # intersections of c with α
(geometric intersection #)
$\Rightarrow$ $\phi \in C^*_c(X)$ since $\phi(\alpha) = 0$ if $\alpha \subseteq X \setminus \mathrm{Im}(c)$

Thm M orientable n-mfd (possibly not compact) $\Rightarrow$ $\boxed{H_*(M) \cong H_c^{n-*}(M)}$

Warning H^{lf}_*, H^*_c are not homotopy invariant (indeed non-trivial for $\mathbb{R}^n$)
Caused because they are not functorial. They are however functorial for **proper maps**
Mayer-Vietoris holds for H^*_c but not for H^{lf}_*. (preimages of compact sets are compact)

Fact $H^*_c(X) \cong \varinjlim H^*(X, X\setminus K)$ where compacts $K_1 \subseteq K_2$ give $H^*(M, M\setminus K_1) \to H^*(M, M\setminus K_2)$
Direct limit $\varinjlim G_i$ via maps $G_i \to G_j$ means $\sqcup G_i /$ identify $g \in G_i$ with its images under those maps
(The indices are partially ordered & directed: $\forall i,j, \exists k \geqslant i,j$ so can compare G_i, G_j inside G_k (via $G_i \to G_k$, $G_j \to G_k$)
Fact $\varinjlim$ is an exact functor.

Cap product and Poincaré duality revisited

X space, $k \geqslant \ell$

$\cap : C_k(X) \times C^\ell(X) \longrightarrow C_{k-\ell}(X)$ **cap product**

$$(\sigma : \Delta^k \to X) \cap (\phi : C_\ell(X) \to \mathbb{Z}) = \underbrace{\phi(\sigma|_{[e_0,\dots,e_\ell]})}_{\substack{\text{"bottom face"}\\ \in \mathbb{Z}}} \cdot \underbrace{\sigma|_{[e_\ell,\dots,e_k]}}_{\substack{\text{"top face"} \cong \Delta^{k-\ell} \\ \in C_{k-\ell}(X)}}$$

(easy) Properties

- $\cap$ bilinear
- $\partial(\sigma \cap \phi) = (-1)^\ell (\partial\sigma \cap \phi - \sigma \cap \partial^*\phi)$
- cycle $\cap$ cycle is cycle
- boundary $\cap$ cycle, cycle $\cap$ boundary are boundaries

$\Rightarrow$ $\boxed{\cap : H_k(X) \times H^\ell(X) \longrightarrow H_{k-\ell}(X)}$ bilinear

Theorem (Poincaré duality) The map $\phi \longmapsto [M] \cap \phi$ gives following isos

① For M closed oriented n-mfd
$[M] \cap \cdot : H^*(M) \xrightarrow{\cong} H_{n-*}(M)$

② For M non-compact oriented n-mfd,
$[M] \cap \cdot : H^*_c(M) \xrightarrow{\cong} H_{n-*}(M)$ ⊛
$[M] \cap \cdot : H^*(M) \xrightarrow{\cong} H^{lf}_{n-*}(M)$

Sketch Pf of ① for smooth mfds (Non-examinable)

If M smooth $\Rightarrow$ $\exists$ "good cover" U_i of M meaning open cover s.t.
↑ FACT from Riemannian geometry ("convex neighbourhoods")
$U_i \cong \mathbb{R}^n$
$U_{i_1} \cap \dots \cap U_{i_k} \cong \mathbb{R}^n$ or $\emptyset$

Then compute $H^*_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$ and ⊛ holds for $\mathbb{R}^n$.

$\Rightarrow$ ⊛ holds $\forall U_i$

$\Rightarrow$ by naturality of ⊛ and of Mayer-Vietoris get ⊛ for $\cup U_i$ finite
↖ use 5-lemma

$\Rightarrow$ ⊛ for M, which is ①. □

General Pf of Poincaré duality ← Non-examinable

Step 1 : holds for $\mathbb{R}^n$

Pf $H^k_c(\mathbb{R}^n) \underset{\uparrow}{\cong} H^k(D^n, S^{n-1}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z} & k = n \end{cases} \cong H_{n-k}(\mathbb{R}^n)$

can make K larger by picking K = large ball
then $H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong H^k(K, \partial K)$

Pick Δ-cx structure for $\mathbb{R}^n$. So $[\mathbb{R}^n] = \sum \pm \sigma_i$ ← sum over n-simplices.
Say ∃ simplex $\sigma_0 : \Delta^n \to \mathbb{R}^n$. Define $\phi : C^{cw}_c(\mathbb{R}^n) \to \mathbb{Z}$, $\phi(\sigma_0) = \pm 1$ ⊛
$\Rightarrow \delta\phi = 0$ for dim reasons $\qquad \phi(\text{other simplices}) = 0$

$[\mathbb{R}^n] \cap \phi = \sum \phi(\pm\sigma_i) = \phi(\pm\sigma_0) = 1$ (pick sign in ⊛)

Step 2 holds for $A, B, A \cap C \Rightarrow$ holds for $A \cup B$

Pf Mayer-Vietoris for H^*_c, naturality, 5-lemma ✓

Step 3 holds for A_i, and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \Rightarrow$ holds for $\cup A_i$

Pf By applying $\underrightarrow{\lim}$: both sides of P.D. iso commute with limits ✓

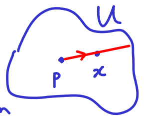
Step 4 holds for open subsets in $\mathbb{R}^n$

Pf Every such set is a union of convex open sets (e.g. balls)

By Step 3 enough to consider case of finite union.

By induction on # convex open sets :

1 convex set $U \cong \mathbb{R}^n$ via a proper homeomorphism,
now use Step 1 ✓



fix $p \in U$
$U \longrightarrow \mathbb{R}^n$
$x \longmapsto \dfrac{x - p}{\text{dist } p \text{ to } \partial U \text{ along ray to } x}$

2 convex sets : KEY TRICK convex set ∩ convex set is convex in $\mathbb{R}^n$!
⇒ use Step 2 & previous case

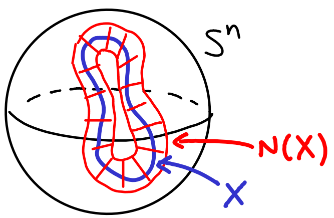
k+1 convex sets : $A = \bigcup\{\text{first k convex sets}\}$, B = last convex set
$\Rightarrow A \cap B \subseteq B$ is a union of k convex sets
} ⇒ use Step 2 & Inductive hypothesis ✓

Step 5 holds for mfd M

Consider open sets in M for which it holds.
By a Zorn's Lemma argument we get a maximal open subset U where holds.
If $U \neq M$ pick $p \in M \setminus U$ and nbhd $V \cong \mathbb{R}^n$ of p. Then holds for $U, V, U \cap V$
(note $U \cap V \subseteq V \cong \mathbb{R}^n$ open, so Step 4 applies) so by Step 2 holds for $U \cup V$
Contradicts maximality. ✓ □

Alexander duality

(in fact, enough to assume X is locally contractible)



$\emptyset \neq X \subsetneq S^n$ compact subset s.t.
∃ open neighbourhood N(X) which deformation retracts to X
such that $\overline{N(X)} \subseteq S^n$ is an n-mfd with boundary.

Theorem $\boxed{\tilde{H}_*(X) \cong \tilde{H}^{n-*-1}(S^n \setminus X)}$

Pf later

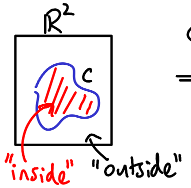
Example $X \subseteq S^3$ knot (i.e. $X = \text{image}(S^1 \xrightarrow[\text{onto the image}]{\text{homeomorphism}} S^3)$)
↖ embedding

$\Rightarrow N(X) \cong$ solid torus $\simeq S^1$

$$\Rightarrow \begin{array}{lll} \tilde{H}_0(X) = 0 & = & \tilde{H}^2(S^3 \setminus X) \\ \tilde{H}_1(X) = \mathbb{Z} & = & \tilde{H}^1 \quad " \\ \tilde{H}_2(X) = 0 & = & \tilde{H}^0 \quad " \end{array}$$

so the homology of a knot complement does not tell knots apart (always same)

Theorem (Jordan curve theorem)



$C \simeq S^1$ closed curve in $\mathbb{R}^2 \subseteq S^2$ ← e.g. by stereographic projection $S^2 \cong \mathbb{C} \cup \infty$
$\Rightarrow \mathbb{R}^2 \setminus C$ has 2 path-components (= connected components)
Similarly for $C \simeq S^n \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}$.

Pf $C \simeq S^n \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}$ $\Rightarrow \tilde{H}_n(S^n) \cong \mathbb{Z} \cong \tilde{H}^0(S^{n+1} \setminus C)$
$\Rightarrow H^0(S^{n+1} \setminus C) \cong \mathbb{Z}^2$
$\Rightarrow S^{n+1} \setminus C$ has 2 path components. □

Proof Alexander duality

$$Y := S^n \setminus N(X) \qquad (\simeq S^n \setminus X)$$

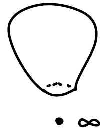
for $* < n-1$

$$\begin{aligned}\tilde H^{n-*-1}(Y) &= H^{n-*-1}(Y)\\ &\underset{\text{Lefschetz}}{\cong} H_{*+1}(Y,\partial Y)\\ &\underset{\text{exc.}}{\cong} H_{*+1}(S^n,\overline{N(X)})\\ &\underset{\substack{\text{LES}\\ \text{using } *<n-1}}{\cong} \tilde H_*(\underbrace{\overline{N(X)}}_{\simeq X})\end{aligned}$$

for $* = n-1$

$$\begin{aligned}\tilde H^0(Y)\oplus\mathbb{Z} &\cong H^0(Y)\\ &\underset{\text{Lef.}}{\cong} H_n(Y,\partial Y)\\ &\underset{\text{exc.}}{\cong} H_n(S^n,\overline{N(X)})\end{aligned}$$

$$\begin{tikzcd}
 & & \cong & \tilde H_{n-1}(\underbrace{\overline{N(X)}}_{\simeq X})\oplus\mathbb{Z} & \\
0 \arrow[r] & \tilde H_n(S^n) \arrow[r] \arrow[ur] \arrow[dr, "\cong"'] & H_n(S^n,\overline{N(X)}) \arrow[r] \arrow[d] & \tilde H_{n-1}(\overline{N(X)}) \arrow[r] & 0 \\
 & & H_n(S^n, S^n\setminus\infty)\cong\mathbb{Z} & & \\
 & & S^n=\mathbb{R}^n\cup\infty \arrow[u] & &
\end{tikzcd}$$



for $* = n$

$$H^{n-*-1}(Y) = H^{-1}(Y) = 0$$

$$H_n(X) \cong H_n(N(X)) \underset{\text{Lef.}}{\cong} 0 \quad \text{n-mfd with bdry} \neq \emptyset. \quad \square$$