C3.1 Algebraic Topology

Oxford 2019
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Please be aware there are likely typos in these notes: comments/corrections are welcome!

Course Book

- Hatcher, Algebraic Topology – Chp. 2 & 3

  This is also freely available from the author's website.
  You are expected to read chapters 2 & 3.

Other references

- Ulrike Tillmann's C3.1 notes – see course page
- The discussion of orientations in my B3.2 Geometry of Surfaces notes may be helpful.

Other books

Massey, A basic course in Algebraic Topology
(Not "Algebraic Topology: An Introduction" ← does not treat homology)

MORE BASIC but full of ideas:

Fulton, Algebraic Topology: a first course

MORE ADVANCED:

May, A concise course in Algebraic Topology

Davis & Kirk, Lecture notes in Algebraic Topology

Bredon, Topology and Geometry

Bott & Tu, Differential forms in Algebraic Topology

Classics by Spanier, Dold, also see references in May's book

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0. OVERVIEW OF THE COURSE

Motivation, category theory, functors H_* and H^*: some computations
why functors are useful: Invariance of dimension, Brower fixed pt thm

1. ALGEBRAIC PRELIMINARIES : CHAIN COMPLEXES

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chain complexes, chain maps, subcomplex, quotient complex
chain map induces map on homology
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5-Lemma, SES splits $\Leftrightarrow$ direct sum

2. Δ-COMPLEXES AND SIMPLICIAL HOMOLOGY

Δ^n, n-simplices, Δ-complex (structure), simplicial cx, triangulation
simplicial chain complex, $H_*^\Delta(S^n)$, $H_*^\Delta(T^2)$, remark about orientations
$H_*^\Delta(\sqcup \text{ conn.comp.}) \cong \oplus H_*^\Delta(\text{conn. comp.})$, $H_0^\Delta(X) \cong \mathbb{Z}^{\#\text{conn.comp.}}$

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naturality / functoriality, $H_*(\text{point})$

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chain homotopy, prism operator
homotopic maps $f \simeq g$ (relative A), homotopy equivalent spaces $X \simeq Y$
contractible space, homotopic maps induce equal maps on homology
homotopy equivalence induces iso on H_*, $H_*(\mathbb{R}^n) = H_*(\mathbb{D}^n) = H_*(pt)$
pairs of spaces, relative homology $H_*(X,A)$, LES in H_* for pair
reduced homology $\widetilde{H}_*(X)$, LES for $\widetilde{H}_*$, $H_*(\mathbb{D}^n, S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})$
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6. MAYER-VIETORIS SEQUENCE

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Univ. coeff. thm for PID R, Duality $H^*(X;\mathbb{F}) \simeq H_*(X;\mathbb{F})$ over fields
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Alexander duality, Knot complements, Jordan curve thm

0. OVERVIEW OF THE COURSE

Motivation

Space $X \xrightarrow{\text{associate}}$ Algebraic object $A(X)$
like numbers, groups, rings, ...

Isomorphism of spaces $X \cong Y$ $\Longrightarrow$ Isomorphism $A(X) \cong A(Y)$

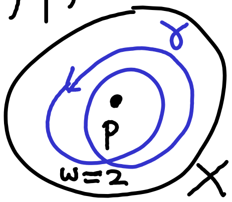
PAY-OFF Can tell spaces apart:
compute $A(X), A(Y) \rightsquigarrow$ if $A(X) \not\cong A(Y)$ then $X \not\cong Y$

Examples

1) Set $X \longrightarrow A(X) = \# X \in \mathbb{N}$
(bijection $X \to Y$) $\Longrightarrow$ same size

2) Vector space $X \longrightarrow A(X) = \dim X \in \mathbb{N}$
(linear iso $X \to Y$) $\Longrightarrow$ same dim

3) Topological Space X
$\longrightarrow \# \pi_0(X) = $ **# path components** $\Big\}\in \mathbb{N}$
$\longrightarrow$ **#connected components** $\Big\}\in \mathbb{N}$
$\longrightarrow \chi(X) =$ **Euler characteristic** $\in \mathbb{Z}$
$\longrightarrow$ Function $X \times \overbrace{\mathcal{L}X}^{\text{loops} = C^0(S^1, X)} \longrightarrow \mathbb{Z}$ (for $X \subseteq \mathbb{R}^2$)
$(p, \gamma) \longmapsto w(\gamma; p)$
Winding number of γ around p.



(Homeomorphism $X \to Y$) $\longrightarrow A(X) = A(Y)$

CONVENTION for this course:

Space = topological space

Map of spaces = continuous map = cts map

For spaces, "$\cong$" means homeomorphism

"id" = identity map

All diagrams commute unless we say otherwise, e.g.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"'] & D \end{tikzcd}$$

means $\beta \circ \alpha = \delta \circ \gamma$

Category theory is the best language to phrase all this

Algebraic topology builds functors such as

$$\begin{tikzcd} & \{\text{Groups and group homs}\} \\ F : \{\text{Topological spaces and continuous maps}\} \arrow[ur] \arrow[r] \arrow[dr] & \{\text{Abelian groups and group homs}\} \\ & \{\text{Rings and ring homs}\} \end{tikzcd}$$

We will not use much category theory, just basic terminology:

Def A **category** C consists of the data:

$\mathrm{Ob}(C)$ = a collection of **objects**

$\mathrm{Hom}(A,B)$ = a set of **morphisms** between any $A,B \in \mathrm{Ob}\,C$ ("**arrows**")

- with **composition** rule $\mathrm{Hom}(B,C) \times \mathrm{Hom}(A,B) \xrightarrow{\circ} \mathrm{Hom}(A,C)$ which is associative.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

- with **identity** morphs $id_A \in \mathrm{Hom}(A,A)$ s.t. $f \circ id_A = id_B \circ f = f$ $\forall (f : A \to B) \in \mathrm{Hom}(A,B)$

Example
$\mathrm{Sets} = \{$ sets with all maps between sets $\}$
$\mathrm{Top} = \{$ topological spaces with continuous maps $\}$
$\mathrm{Gps} = \{$ groups with group homs $\}$

Def A (covariant) **functor** $F : C_1 \longrightarrow C_2$ is the data:

- an assignment $(A \in \mathrm{Ob}\, C_1) \longmapsto (F(A) \in \mathrm{Ob}\, C_2)$
- an assignment $\underset{\mathrm{Hom}_{C_1}(A,B)}{\underset{\cap}{(A \xrightarrow{f} B)}} \longmapsto \underset{\mathrm{Hom}_{C_2}(F(A),F(B))}{\underset{\cap}{(F(A) \xrightarrow{F(f)} F(B))}}$

Compatible with identities and compositions.

$$F(id_A) = id_{F(A)} \qquad F(g \circ f) = F(g) \circ F(f)$$

A **contravariant functor** is defined similarly except it reverses the direction of arrows: $(F(A) \xleftarrow{F(f)} F(B)) \in \mathrm{Hom}(F(\underline{B}), F(\underline{\underline{A}}))$
(so $F(g \circ f) = F(f) \circ F(g)$ reverses order of compositions)

Examples

1) $F : \text{Top} \longrightarrow \text{Sets}$, $A \longmapsto A$, $f \longmapsto f$ "forget the topology and continuity"

2) $F : \text{Sets} \longrightarrow \text{Gps}$, $A \longmapsto$ free abelian group generated by A

$$\mathbb{Z}\langle A\rangle = \left\{ \sum_{\text{finite}} n_i \cdot a_i \;:\; \begin{matrix} a_i \in A \\ n_i \in \mathbb{Z} \end{matrix} \right\}$$

$$(A \xrightarrow{f} B) \longmapsto \left(\begin{matrix} F(A) : \mathbb{Z}\langle A\rangle \longrightarrow \mathbb{Z}\langle B\rangle \\ \sum n_i \cdot a_i \longmapsto \sum n_i \cdot f(a_i) \end{matrix} \right)$$

When we say a construction is **natural** we mean functorial:

$$\begin{tikzcd}
X \arrow[r, "A"] \arrow[d, "f"'] \arrow[dd, bend right=60, "g\circ f"'] & A(X) \arrow[d, "A(f)"] \arrow[dd, bend left=60, "\begin{array}{c} A(g\circ f) \\ \shortparallel \\ A(g)\circ A(f) \end{array}"] \\
Y \arrow[r, "A"] \arrow[d, "g"'] & A(Y) \arrow[d, "A(g)"] \\
Z \arrow[r, "A"] & A(Z)
\end{tikzcd}$$

A : (a category of spaces) $\longrightarrow$ (a cat. of algebraic objects)
The algebraic objects we assigned are assigned compatibly with maps of spaces, and the compatibility maps $A(f)$ are also compatible w.r.t. composition.
So we made compatible choices in constructing A.

Not to be confused with natural transformations of functors (later) which is about relating two such constructions A_1, A_2 in a compatible way

Example of a functor in algebraic topology

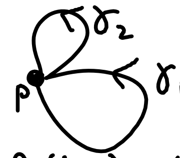
(see B3.5 Topology and Groups course)

$$\pi_1(X, p) = \text{Fundamental group} = \left\{ \begin{matrix} \gamma \in \mathcal{L}X \\ \gamma(1) = p \end{matrix} \right\} \Big/ \begin{matrix} \text{continuous deformations} \\ \text{of loops based at } p \end{matrix}$$

topological space

$p \in X$

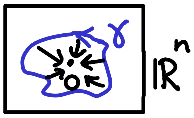
Group multiplication: concatenate loops $\gamma_1 * \gamma_2$



(each travelling twice as fast)

Examples

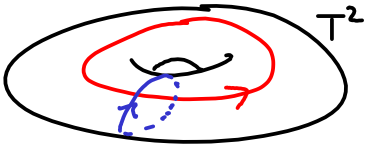
$\pi_1(\mathbb{R}^n) = 0$ ⟵ deform: $h : S^1 \times [0,1] \longrightarrow \mathbb{R}^n$, $h(t,s) = (1-s)\cdot\gamma(t)$



$\pi_1(S^1) \cong \mathbb{Z}$ ⟵ total # times wind around circle

$\pi_1(S^n) \cong 0 \quad n \geq 2$ (not obvious)

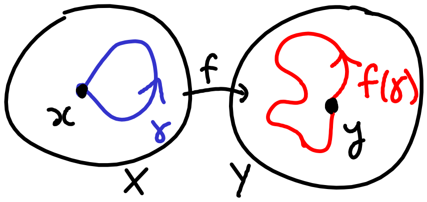
$\pi_1(\text{torus}) \cong \mathbb{Z}^2$



those loops generate π_1

FUNCTOR

$$\text{Based Top} = \left\{ \begin{matrix} \text{Topological spaces with choice of basepoint,} \\ \text{and continuous basepoint-preserving maps} \end{matrix} \right\} \xrightarrow{\pi_1} \text{Gps}$$



$$(X, p) \longmapsto \pi_1(X, p)$$

$$\left((X, x) \xrightarrow[f]{\text{cts}} (Y, y) \right) \longmapsto \left(\begin{matrix} \pi_1(X, x) \xrightarrow{\text{gp. hom.}} \pi_1(Y, y) \\ \gamma \longmapsto f \circ \gamma \end{matrix} \right)$$

Lemma Functors map isomorphisms to isomorphisms (iso. means $\exists$ inverse w.r.t. composition)

Pf $$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "id"'] & B \arrow[r, "g"] & A \end{tikzcd} \Rightarrow \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[rr, bend right, "F(id)=id"'] & FB \arrow[r, "Fg"] & FA \end{tikzcd}$$, similarly for $\begin{tikzcd} B \arrow[r, "g"] \arrow[rr, bend right, "id"'] & A \arrow[r, "f"] & B \end{tikzcd}$. □

Def **Natural transformation** $\alpha : F \longrightarrow G$ between functors $\begin{tikzcd} C_1 \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & C_2 \end{tikzcd}$

is an association $(A \in Ob\, C_1) \longmapsto (\alpha_A : F(A) \longrightarrow G(A))$
$\in Hom_{C_2}(F(A), G(A))$

such that $(A \xrightarrow{f} B) \Rightarrow$
$\in$ $Hom_{C_1}(A,B)$

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"] & G(B) \end{tikzcd}$$ (commutes)

It is called a **natural isomomorphism** if each α_A is an isomorphism in C_2

Example of a natural transformation in algebraic topology

Let $H_1(X,p)$ = abelianisation of $\pi_1(X,p)$ (want to identify $ab=ba$ so quotient by $\langle aba^{-1}b^{-1}\rangle$)
↖ **commutators**

$\Rightarrow$ natural trans. $(\text{Based Top} \xrightarrow{\pi_1} \text{Gps}) \xrightarrow{\alpha} (\text{Based Top} \xrightarrow{H_1} \text{Gps})$

which associates $\underset{\in \text{BasedTop}}{(X,p)} \longmapsto (\alpha_{(X,p)} : \pi_1(X,p) \xrightarrow{\text{quotient}} H_1(X,p))$

Cultural Rmk **higher homotopy groups** $\pi_n(X,p) = \left\{ \begin{matrix} S^n \xrightarrow{cts} X \\ \text{basept} \mapsto p \end{matrix} \right\} \Big/ \text{cts deform}^n$

FACT abelian for $n \geq 2$, but hard: e.g. $\pi_k(S^n)$ not all known.

We will not study these in this course.

We will study simpler invariants called HOMOLOGY groups $H_n(X)$

FACT (Hurewicz) $\exists$ natural transformation $\pi_n \longrightarrow H_n$
which will make sense at the end of course:
$f : S^n \xrightarrow{cts} X$ gives rise to a class $f_*[S^n] \in H_n(X)$.

Exercise to practice these notions from category theory:
Summarise your undergraduate linear algebra as follows:

1) $\exists$ functor $F: \underbrace{\left\{ \begin{matrix} \mathbb{R}^n \text{ for } n \in \mathbb{N} \\ Hom(\mathbb{R}^n, \mathbb{R}^m) = \left\{ \begin{matrix} m \times n \\ \text{matrices} \end{matrix} \right\} \end{matrix} \right\}}_{\text{Mat}} \to \underbrace{\left\{ \begin{matrix} \text{finite dimensional real vector spaces} \\ \text{and linear maps} \end{matrix} \right\}}_{\text{Vect}}$

2) A choice of basis for each vector space V determines a functor $G : \text{Vect} \to \text{Mat}$

3) Construct natural isomorphisms $G \circ F \xrightarrow{\alpha} Id_{Mat}$, $F \circ G \xrightarrow{\beta} Id_{Vect}$

When functors satisfying such natural isos exist, the categories are called **equivalent** (not isomorphic). So Mat, Vect are equivalent categories.

Aim of the course: build a functor

HOMOLOGY $H_* : \textbf{Top} \longrightarrow$ **Graded abelian groups**

$$X \longmapsto H_*(X) \quad \longleftarrow \text{grading } * \in \mathbb{Z}$$

$$(X \to Y) \longmapsto \left(\begin{array}{c} H_*(X) \longrightarrow H_*(Y) \\ \text{grading preserving hom} \end{array} \right)$$

and a contravariant functor

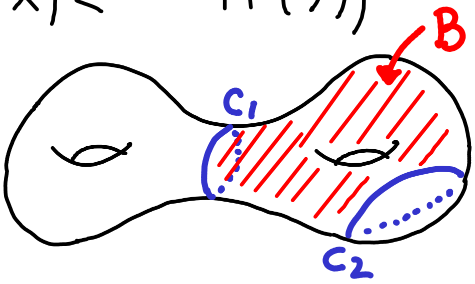
COHOMOLOGY $H^* : \textbf{Top} \longrightarrow$ **Graded rings**

$$X \longmapsto H^*(X)$$

$$(X \to Y) \longmapsto (H^*(X) \longleftarrow H^*(Y))$$

Rough idea:

$H_* X$ is generated by "nice" subspaces $C \subseteq X$ which have no boundary: $\partial C = \emptyset$, modulo identify C_1, C_2 if $C_1 \cup C_2$ arises as a boundary ∂B. Call such C_1, C_2 **homologous**.



Facts

- $H_0(X) \cong \bigoplus_{\pi_0 X} \mathbb{Z}$ ← $\pi_0 X = \{$path-connected components$\}$ ← generated by a point in each path-comp.
- $X = \bigsqcup X_i$ path-components $\Longrightarrow H_*(X) \cong \bigoplus H_*(X_i)$
- $\chi(X) = \sum_{d \geq 0} (-1)^d \operatorname{rank} H_d(X)$

χ ↑ **Euler characteristic**

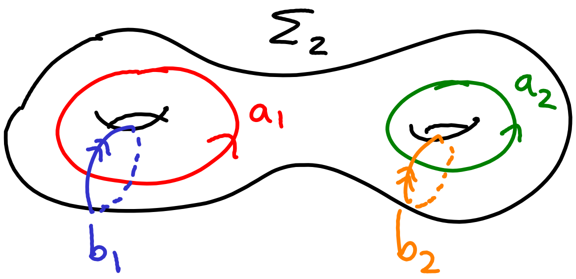
rank ↗ max # $\mathbb{Z}$-linearly independent elements

Example: compact surfaces

$$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{2g} & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$$

Σ_g ↑ orientable surface genus g

$\chi = 2 - 2g$

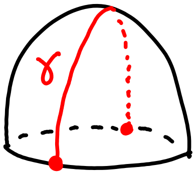


We will show that those 4 loops generate $H_1(\Sigma_2)$

$$H_*(N_h) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}^{h-1} & * = 1 \\ 0 & \text{else} \end{cases}$$

N_h ↑ non-orientable surface S^2 with h Möbius bands attached

$\chi = 2 - h$



$N_1 = \mathbb{R}P^2 = S^2/\pm \mathrm{Id}$

Notice γ is a loop. It generates $H_1(N_1)$

Examples of homology calculations

$$H_*(\mathbb{R}^n) \cong H_*(\mathbb{D}^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ 0 & \text{else} \end{cases}$$

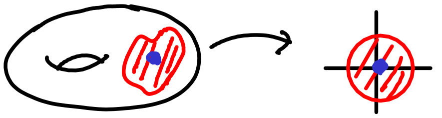
n-dimensional ball

$\mathbb{D}^n = \{x : \|x\| \leq 1\} \subseteq \mathbb{R}^n$

$$H_*(S^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$$

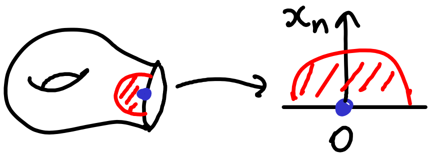
↗ $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ n-dim sphere

Hausdorff top. space s.t. each pt has an open neighbourhood homeo to an open ball in $\mathbb{R}^n$



$$H_*(X) = \begin{cases} 0 & \text{for } * < 0 \\ 0 & \text{for } * > n \text{ for } \textbf{n-dimensional manifolds } X \\ \mathbb{Z} & \text{for } * = n \text{ for connected orientable compact manifold} \\ 0 & \text{for } * = n \text{ for } \begin{cases} \text{non-orientable} & \text{"} \\ \text{non-compact} & \text{"} \\ \text{connected } \textbf{manifolds with boundary} \neq \emptyset \end{cases} \end{cases}$$

boundary point has an open nbhd homeo to open nbhd of $0 \in$ half-space: $\{x \in \mathbb{R}^n : x_n \geq 0\}$

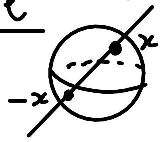


Rmk M compact connected n-mfd

$$\Rightarrow H_{n-1}(M) \cong \begin{cases} \mathbb{Z}^k & \text{some } k \geq 0 \text{ if orientable} \\ \mathbb{Z}^k \oplus \mathbb{Z}/2 & \text{" " non-orientable} \end{cases}$$

$$H_*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/2 & \text{odd } *=1,3,5,\dots<n \\ \mathbb{Z} & *=n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

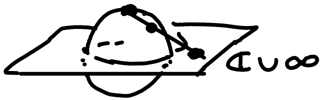
↗ $S^n/\pm\text{id}$ real projective space



$\mathbb{RP}^n$ orientable $\Leftrightarrow$ n odd (e.g. $\mathbb{RP}^1 \cong S^1$)

$$H_*(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \text{even } *=0,2,4,\dots,2n \\ 0 & \text{else} \end{cases}$$

e.g. $\mathbb{CP}^1 \cong S^2$ stereographic projection



↑ space of complex lines through $0 \in \mathbb{C}^{n+1}$

complex projective space

$$\left(\begin{aligned} &\cong (\mathbb{C}^n \setminus 0) / \mathbb{C}^*\text{-rescaling} \\ &= \{[z_0, z_1, \dots, z_n] : z_j \in \mathbb{C} \text{ not all } 0\} / [z] = [\lambda z] \text{ for } \lambda \in \mathbb{C}^* \end{aligned} \right)$$

Examples of cohomology calculations

$$H^0(X) = \prod_{\pi_0 X} \mathbb{Z}$$ ← if $\pi_0 X$ finite, then $\cong \bigoplus_{\pi_0 X} \mathbb{Z} \cong H_0 X$

but if infinite then not: here allow only finite sums

$$H^*(X) \cong \prod H^*(X_i)$$ ← X_i path-components of X

FACT If $H_n(X)$ finitely generated abelian gp, so

$$H_n(X) \cong \mathbb{Z}^{r_n} \oplus T_n \longleftarrow T_n = \text{torsion elements} = \text{elements of finite order}$$

Then $H^n(X) \cong \mathbb{Z}^{r_n} \oplus T_{\underline{\underline{n-1}}}$ as abelian groups

$\Longrightarrow$ $H^*(\Sigma_g)$, $H^*(\mathbb{R}^n)$, $H^*(\mathbb{D}^n)$, $H^*(S^n)$, $H^*(\mathbb{C}P^n)$ same as for H_*, but:

$$H^*(N_h) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{h-1} & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

$$H^*(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$ $(h=1)$

$$H_*(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * \text{ even} = 2, 4, \ldots \leq n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

and $H^n(\text{non-orientable compact } n\text{-mfd}) \cong \mathbb{Z}/2$.

$\Longrightarrow$ The interesting feature is the ring structure:

$$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/x^{n+1}$$

grading: $|x| = 2$

$\mathbb{Z}[x]$ = polynomials in x with $\mathbb{Z}$-coefficients

$$H^*(S^n) \cong \mathbb{Z}[x]/x^2 \qquad |x| = n$$

$$H^*(T^n) \cong \Lambda[x_1, \ldots, x_n] \qquad |x_i| = 1$$

$T^n = S^1 \times \cdots \times S^1$ **n-torus**

e.g. $n=2$ (torus)
$$\begin{cases} \mathbb{Z} \cdot 1 & * = 0 \\ \mathbb{Z}x_1 \oplus \mathbb{Z}x_2 & * = 1 \\ \mathbb{Z}\, x_1 \wedge x_2 & * = 2 \\ 0 & \text{else} \end{cases}$$

exterior algebra generated$/\mathbb{Z}$ by symbols $x_{i_1} \wedge \ldots \wedge x_{i_k}$ with $i_1 < \ldots < i_k$
product given by $\wedge$ using relations $x_i \wedge x_j = -x_j \wedge x_i$.

$$H^*(\mathbb{R}P^{2n}) \cong \mathbb{Z}/2[x]/x^{n+1} \qquad |x| = 2$$

$$H^*(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}/2[x]/x^{n+1} \oplus \underbrace{\mathbb{Z}[-2n-1]}$$ ← means: a copy of $\mathbb{Z}$ in degree $2n+1$

$$H^*(\Sigma_g) \cong \Lambda[a_1, \ldots, a_g, b_1, \ldots, b_g] / \langle a_i b_j \text{ for } i \neq j,\ a_i b_i - a_j b_j,\ a_i a_j,\ b_i b_j \rangle$$

$|a_i| = |b_i| = 1$ ← exterior alg. instead of poly. alg since $a_i b_i = -b_i a_i$

Why more information?

connected sum: remove a ball in each, glue along ∂ball

$S^2 \times S^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ have same $H_* = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & * = 2 \\ \mathbb{Z} & * = 4 \end{cases}$

but the rings H^* are not iso, hence $S^2 \times S^2 \not\simeq \mathbb{C}P^2 \# \mathbb{C}P^2$.

Example of why such functors are useful

Suppose $\exists$ F_* : $\mathrm{Top} \longrightarrow \mathrm{Gps}$ functors s.t.

① $F_*(S^n) \neq 0 \Leftrightarrow * = n$ and ② $F_*(\mathbb{D}^n) = 0$ all $*$

Rmk We'll build such an F_* : reduced homology $\widetilde{H}_*$ s.t. $\widetilde{H}_* = H_*$ for $* \neq 0$, and $\widetilde{H}_0 \cong \mathbb{Z}^{\#(\text{path-components})-1}$

Theorem Invariance of dimension

$$\boxed{\begin{array}{lll} S^n \cong S^m & \Longleftrightarrow & n=m \\ \mathbb{R}^n \cong \mathbb{R}^m & \Longleftrightarrow & n=m \end{array}}$$

by ①

"homeomorphisms preserve dimension"

Non-trivial result because there are space-filling curves. e.g. Peano (1890) $\exists$ cts surjection $\underset{\substack{\shortparallel \\ \text{interval}}}{[0,1]} \to \underset{\substack{\shortparallel \\ \text{square}}}{[0,1]^2}$

The theorem implies this is not injective.

(cts. bij. compact $\to$ Hausdorff $\Rightarrow$ homeo)

Pf Lemma $\Rightarrow$ $F_n(S^n \xrightarrow{\cong} S^m)$ is iso $\underset{\neq 0}{F_n(S^n)} \xrightarrow{\cong} \underset{\substack{\shortparallel \\ 0}}{F_n(S^m)}$ of gps. if $n \neq m$ ✓

If $\mathbb{R}^n \xrightarrow[\varphi]{\cong} \mathbb{R}^m$, then can extend φ to the one-point compactifications: $S^n \cong \mathbb{R}^n \cup \{\infty\} \xrightarrow{\cong} \mathbb{R}^m \cup \{\infty\} \cong S^m$, $\infty \mapsto \infty$. □

("Alexandroff extension")

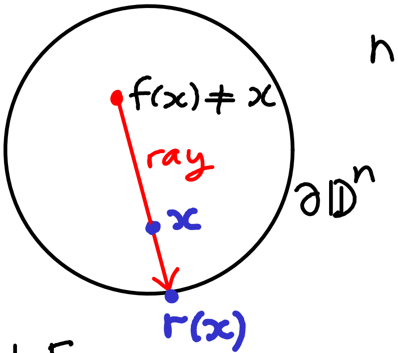
stereographic projection $(x_0, \dots, x_n) \longmapsto \dfrac{(x_1, \dots, x_n)}{1 - x_0}$

Rmk new open neighbourhoods at ∞ are $\{\infty\} \cup (\mathbb{R}^n \setminus C)$ where C is (closed &) compact. The extended map is cts since $\varphi^{-1}(C)$ is (closed &) compact since φ^{-1} is homeo.

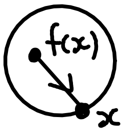
Theorem Brower fixed point thm by ① & ②

$f : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ continuous $\Longrightarrow$ f has a fixed point ($f(p) = p$ some p)

Proof Suppose not. Let $r(x) = (\text{ray from } f(x) \text{ to } x) \cap \partial\mathbb{D}^n$



notice:
- $r : \mathbb{D}^n \longrightarrow \partial\mathbb{D}^n = S^{n-1}$ continuous
- $r|_{\partial\mathbb{D}^n} = \mathrm{id}_{S^{n-1}}$

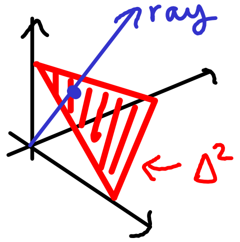


$$\begin{tikzcd} S^{n-1} = \partial\mathbb{D}^n \arrow[r, "\text{inclusion } i"] \arrow[rr, bend right, "r \circ i = \mathrm{id}"'] & \mathbb{D}^n \arrow[r, "r"] & S^{n-1} \end{tikzcd}$$

apply F_{n-1} $\Longrightarrow$ $F_{n-1}(r) \circ F_{n-1}(i) = \mathrm{id}$ $\Longrightarrow$ $F_{n-1}(i)$ injective $\overset{\neq 0}{F_{n-1}(S^{n-1})} \to \overset{\overset{0}{\shortparallel}}{F_{n-1}(\mathbb{D}^n)}$ ↯ □

Example (Brower) $A = n \times n$ matrix, $A_{ij} > 0$ real $\Longrightarrow$ $\exists$ evalue $\lambda > 0$ with real evector $(v_1, \dots, v_n)$ with $v_i \geq 0$

Hint $\mathbb{R}^3 \supseteq$



$X = \{\text{rays in "positive octant"}\}$ ← $x \in \mathbb{R}^n : x_i > 0 \ \forall i$

notice $AX \subseteq X$

notice $X \cong \Delta^n = \{x \in \text{octant} : \sum x_i = 1\} \cong \mathbb{D}^n$

ray $\longmapsto$ ray $\cap\, \Delta^n$

1. ALGEBRAIC PRELIMINARIES : CHAIN COMPLEXES

Graded abelian groups

Def A **$\mathbb{Z}$-graded abelian group** C is an abelian group together with a direct sum decomposition

$$\boxed{C = \bigoplus_{n\in\mathbb{Z}} C_n}$$

← abelian group

Convention: always grade by $\mathbb{Z}$ unless say otherwise.

Example $C = \mathbb{Z}[x]$ = integer polynomials in x, $C_n = \mathbb{Z}\cdot x^n$ ← so grading by degree

A graded ab.gp. A is a **graded subgp** of C if
- subgp
- $A_n \subseteq C_n$.

A **homomorphism** $h: C \longrightarrow D$ of gr.ab.gps is hom of gps s.t.

$$h(C_n) \subseteq D_n$$

A **hom of degree k** is hom with

$$h(C_n) \subseteq D_{n+k}$$

Shift by k: $\mathbb{Z}$-gr.ab.gp. $C[k]$ with

$$\boxed{C[k]_n = C_{k+n}}$$

Notice: $C[k]_0 = C_k$ is now in degree zero, so shifted down by k

$\Rightarrow$ Can view gr. hom of deg k as a gr. hom

$$h: C \longrightarrow D[k]$$

Abelian groups which are **finitely generated**

recall f.g. means $\exists$ surjection $\mathbb{Z}^m \to G$ for some m

FACT Finitely generated abelian groups are classified:

$$G \cong \underbrace{\mathbb{Z}^{r}}_{\text{free part}} \oplus \underbrace{\mathbb{Z}/p_1^{n_1} \oplus \cdots \oplus \mathbb{Z}/p_k^{n_k}}_{\text{torsion part}}$$

r called **rank G**

$n_i \in \mathbb{Z}$
p_i primes (possibly not distinct)

compare finite dimensional vectorspaces/field $\mathbb{F}$: $V \cong \mathbb{F}^r$ ← $r = \dim V$

Chain complexes

differential or **boundary homomorph**

Def A **chain complex** (C_*, ∂_*) is a gr. ab. gp. C together with a hom ∂ of degree -1 such that $\partial \circ \partial = 0$.

Thus:

$$\boxed{\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[rr, bend right, "\partial_n \circ \partial_{n+1} = 0"'] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r] & \cdots \end{tikzcd}}$$

hence $\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n$

$$\begin{array}{ccc} \operatorname{Im} \partial_{n+1} & \subseteq & \operatorname{Ker} \partial_n \\ \overset{..}{\|} & & \overset{..}{\|} \\ B_n & & Z_n \end{array}$$

n-boundaries **n-cycles**

n-chains = elements of C_n

Now consider "cycles modulo boundaries":

Def The **homology** of (C_*, ∂_*) is the gr. ab. gp.

$$\boxed{H_n(C_*, \partial_*) = \operatorname{Ker} \partial_n \big/ \operatorname{Im} \partial_{n+1}}$$

often abbreviate by $H_*(C)$

Two cycles are called **homologous** if they differ by a boundary

Def A **chain map** $h : (C_*, \partial_*) \longrightarrow (\tilde{C}_*, \tilde{\partial}_*)$ is a graded hom such that $\boxed{h \circ \partial_* = \tilde{\partial}_* \circ h}$

Example A chain **subcomplex** $C_* \subseteq \tilde{C}_*$ is a graded subgp with $\partial_* =$ restriction of $\tilde{\partial}_*$ to C_*.

So the inclusion $\mathrm{incl} : (C_*, \partial_*) \longrightarrow (\tilde{C}_*, \tilde{\partial}_*)$ is a chain map.

Also get **quotient complex** $\tilde{C}_*/C_*$

with $\tilde{\partial}_* [\tilde{c}] = [\tilde{\partial}_* \tilde{c}]$ (well-defined: $\tilde{\partial}_* C_* = \partial_* C_* \subseteq C_*$)

Lemma A chain map induces a hom on homology

$$H_*(h) = h_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$$

$$[x] \longmapsto [h(x)]$$

Proof $\operatorname{Ker} \partial_n \xrightarrow{\text{hom}} \operatorname{Ker} \tilde{\partial}_n$

$x \longmapsto h(x)$ since $\tilde{\partial}(h(x)) = h(\underbrace{\partial x}_{=0}) = 0$

Need $\operatorname{Im} \partial_n \longrightarrow \operatorname{Im} \tilde{\partial}_n$ to get well-defined hom

$$H_n(C) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1} \longrightarrow \operatorname{Ker} \tilde{\partial}_n / \operatorname{Im} \tilde{\partial}_{n+1} = H_n(\tilde{C})$$

Proof: $h(b) \underset{\uparrow\, b = \partial c \in \operatorname{Im}(\partial)}{=} h(\partial c) = \tilde{\partial} h(c) \in \operatorname{Im}(\tilde{\partial})$. □

The last step was a very simple example of a proof by "**diagram chasing**"

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "h_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "h_n"] & C_{n-1} \arrow[r] \arrow[d, "h_{n-1}"] & \cdots \\
\cdots \arrow[r] & \tilde{C}_{n+1} \arrow[r, "\tilde{\partial}_{n+1}"'] & \tilde{C}_n \arrow[r, "\tilde{\partial}_n"'] & \tilde{C}_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
c \arrow[r, "\partial"] \arrow[d, "h"'] & \partial c = b \arrow[d, "h"] \\
hc \arrow[r, "\tilde{\partial}"] & \tilde{\partial}(hc) = h\partial c = h(b)
\end{tikzcd}$$

□

Def (C_*, ∂_*) is **exact** (or **acyclic**) if $H_*(C) = 0$

so $\operatorname{Im} \partial_{n+1} = \operatorname{Ker} \partial_n$

In general **exact sequence** means $\boxed{\text{"Im(previous map)} = \text{Ker (next map)"}}$

A **short exact sequence (SES)** is an exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

Easy exercise

$$\left(\underset{\text{exact}}{0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0}\right) \Leftrightarrow \begin{cases} i & \text{injective} \\ \pi & \text{surjective} \\ B/_{i(A)} \cong C & \text{via } [b] \longmapsto \pi(b) \end{cases}$$

Examples

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r, "\text{mod } 2"] & \mathbb{Z}/_2 \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\text{inclusion}"] & \mathbb{Z} \oplus \mathbb{Z}/_2 \arrow[r, "\text{project}"] & \mathbb{Z}/_2 \arrow[r] & 0 \end{tikzcd}$$

Note A, C do not determine B.

Snake Lemma A SES of chain complexes and chain maps yields a **long exact sequence (LES) on homology**:

$$\boxed{\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{\pi_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*[-1]} \cdots}$$

↑ degree -1 map $H_*(C) \to H_*(A)[-1]$ called **connecting map**

(So **exact triangle**:

$$\begin{tikzcd} H_*(A) \arrow[rr] & & H_*(B) \arrow[dl] \\ & H_*(C) \arrow[ul, "{[-1]}"] & \end{tikzcd}$$
)

Pf simplify notation by identifying A with $i(A) \subseteq B$: $\overset{\in A}{a} \overset{i}{\hookrightarrow} \overset{\in B}{i(a)}$, $a \equiv i(a)$, $\partial a \equiv i \partial a = \partial i a$

$\Longrightarrow$ now $A_* \subseteq B_*$ inclusion of subcomplex:

$$0 \longrightarrow (A_*, \partial_*) \xrightarrow{\text{incl}} (B_*, \partial_*) \xrightarrow{\pi} (C_*, \tilde{\partial}_*) \longrightarrow 0$$

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} & \exists b \arrow[r, "\text{surj.}"] \arrow[d] & \overset{\text{cycle}}{c} = \pi(b) \arrow[d] \\ \partial b \arrow[r] & \partial b \arrow[r] & \tilde{\partial} c = 0 \end{tikzcd}$$

↖ lifts to A by exactness

Define $\delta : H_*(C) \longrightarrow H_*(A)[-1]$ (typically b is not in A, so ∂b need not be a bdry in A)

$c \longmapsto \partial b$ ← where $b \in \pi^{-1}(c)$

Well-defined? • $\pi^{-1}(c) = \{b+a : a \in A\}$ and $\partial(b+a) = \partial b + \underbrace{\partial a}_{\text{boundary in } A}$ ✓

- cycle → cycle : $\partial(\partial b) = 0$ ✓
- boundary → boundary :

$$\begin{tikzcd}
\exists \beta \arrow[r, "\text{surj.}"] \arrow[d] & x \in C_{n+1} \arrow[d] \\
\partial \beta \arrow[r] & \text{boundary } c = \tilde{\partial} x \arrow[d] \\
 & 0
\end{tikzcd}$$

$\Rightarrow$ can pick $b = \partial \beta$

$\Rightarrow \partial b = \partial\partial\beta = 0$ ✓

Exactness at $H_n(C)$ (exercise: check exactness at H_*A, H_*B):

Need $\mathrm{Im}\, \pi_* = \mathrm{Ker}\, \delta$:

$\subseteq$: $\delta(\pi_* \underset{\text{cycle}}{b}) = \partial b = 0$ ✓

$\supseteq$:

$$\begin{tikzcd}
\exists a \arrow[d] & & b \arrow[r] \arrow[d] & c = \pi_* b \arrow[d] \\
\partial a \overset{\text{assumption } \delta c = 0 \in H_*A}{=} \delta c = \partial b \arrow[rr] & & \partial b \arrow[r] & 0
\end{tikzcd}$$

b ← not necessarily cycle!

$c \in \mathrm{Ker}\, \delta$

$\pi_* A = 0$

$\pi_*(b-a) = c$

$\partial(b-a) = \partial b - \partial a = 0$

$\underbrace{b-a}_{\text{thus cycle!}}$

$\Rightarrow c = \pi_*(b-a) \in \mathrm{Im}\, \pi_*$ □

Rmk $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$ SES $\Rightarrow$ the connecting map of LES is

$$\boxed{\begin{array}{l} \delta : H_*(C) \longrightarrow H_*(A)[-1] \\ \qquad c \longmapsto i^{-1}(\partial b) \end{array}}$$

$\forall\, b \in B$ with $\pi(b) = c$.

Lemma The construction of δ is natural (i.e. functorial)

Pf

all chain maps →

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "\pi"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \tilde{A} \arrow[r, "\tilde{i}"] & \tilde{B} \arrow[r, "\tilde{\pi}"] & \tilde{C} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\delta c = a \arrow[r] \arrow[d, "f"] & \partial b \arrow[d, "g"] & b \arrow[r] \arrow[d, "g"] & c \arrow[d, "h"] \\
fa \arrow[r] & g\partial b & gb \arrow[r] & hc
\end{tikzcd}$$

$g\partial b = \tilde{\partial} g b = \delta h c$

$$\Rightarrow \delta h c = \tilde{i}^{-1} \tilde{\partial} g b = \tilde{i}^{-1} g \partial b = fa = f \delta c$$ □

Exercise Deduce the LES is natural, so

$$\begin{tikzcd}
\cdots \arrow[r] & H_*A \arrow[r, "i_*"] \arrow[d, "f_*"] & H_*B \arrow[r, "\pi_*"] \arrow[d, "g_*"] & H_*C \arrow[r, "\delta"] \arrow[d, "h_*"] & H_{*-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_*\tilde{A} \arrow[r] & H_*\tilde{B} \arrow[r, "\tilde{\pi}_*"] & H_*\tilde{C} \arrow[r, "\delta"] & H_{*-1}(\tilde{C}) \arrow[r] & \cdots
\end{tikzcd}$$

5-Lemma

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha", "\cong"'] & B \arrow[r] \arrow[d, "\beta", "\cong"'] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta", "\cong"'] & E \arrow[d, "\varepsilon", "\cong"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

exact rows $\Longrightarrow$ γ also iso.

Pf exercise (diagram chase) □

Splitting Lemma

Cor $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ SES of abelian gps

If
$$\begin{tikzcd}
B \arrow[r, "\beta"] & C \arrow[l, bend left, "\exists\gamma"]
\end{tikzcd}$$
s.t. $\beta \circ \gamma = id_C$ then the SES **splits**: $B \cong A \oplus C$

(converse is obvious)

Pf
$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & A\oplus C \arrow[r] \arrow[d, "\alpha+\gamma"] & C \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$
□

Exercise If
$$\begin{tikzcd}
A \arrow[r, "\alpha"] & B \arrow[l, bend left, "\exists\mu"]
\end{tikzcd}$$
s.t. $\mu \circ \alpha = id_A$ then it splits: $B \xrightarrow[\mu\oplus\beta]{\cong} A \oplus C$

Exercise If C is a free abelian group $(C \cong \bigoplus_{i\in I} \mathbb{Z})$ then the SES splits.

Rmk A free $\not\Rightarrow$ splits, e.g. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \longrightarrow 0$

Cultural Rmk Splitting Lemma generalises the **rank-nullity theorem** from linear algebra: $V \xrightarrow{\beta} W$ linear map of vector spaces $\Rightarrow \operatorname{Im}\beta \oplus \operatorname{Ker}\beta \cong V$

Pf $0 \to \operatorname{Ker}\beta \xrightarrow{incl} V \xrightarrow{\beta} \operatorname{Im}\beta \to 0$ is SES, and splits since $\operatorname{Im}\beta$ free.

2. Δ-COMPLEXES AND SIMPLICIAL HOMOLOGY

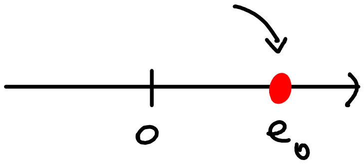
standard n-simplex

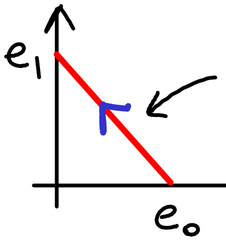
$$\Delta^n = \left\{ \underset{\textstyle \sum t_i e_i}{\underset{\shortparallel}{(t_0, \dots, t_n)}} \in \mathbb{R}^{n+1} : \begin{array}{l} t_i \geq 0 \\ \sum t_i = 1 \end{array} \right\}$$

↖ standard basis of $\mathbb{R}^{n+1}$ $e_0, \dots, e_n$ $(e_0 = (1,0,\dots,0), \dots$

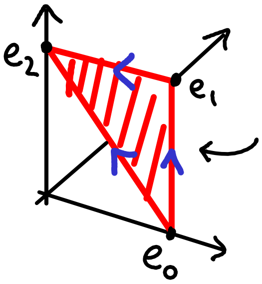
Examples

point $\Delta^0 \subseteq \mathbb{R}$





segment $\Delta^1 \subseteq \mathbb{R}^2$



triangle $\Delta^2 \subseteq \mathbb{R}^3$

Def For $\{v_0, \dots, v_n\} \subseteq \mathbb{R}^{n+k}$ ← any $k \geq 0$ s.t.

$v_1 - v_0, \dots, v_n - v_0$ $\mathbb{R}$-linearly independent

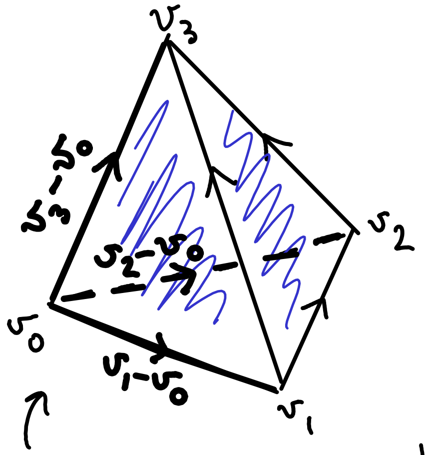
$[v_0, \dots, v_n]$ = **n-simplex** spanned by $v_0, \dots, v_n$

= convex hull of $v_0, \dots, v_n$

$= \{ \sum t_i v_i \ : \ t_i \geq 0 \text{ and } \sum t_i = 1 \}$

= Image of linear homeo $\sigma : \Delta^n \longrightarrow \mathbb{R}^{n+k}$

canonical homeomorphism $\qquad \sigma(e_i) = v_i$



(Solid prism: includes inside)

Will often blur the distinction between map σ and its image,

$$\sigma \equiv [\sigma e_0, \dots, \sigma e_n]$$

but the ordering of the v_j will be important (so the map σ is more precise)

we encode this extra data by orienting the edges $\overset{v_i}{\bullet} \longrightarrow \overset{v_j}{\bullet}$ if $i < j$

Def **d-dimensional faces** $[v_{i_0}, \dots, v_{i_d}]$ for $i_0 < \dots < i_d$

Example 0-dim faces are the vertices $v_0, \dots, v_n$

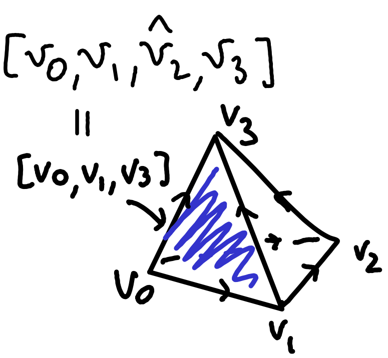
facets = $(n-1)$-dimensional faces

$= [v_0, \dots, \hat{v}_k, \dots, v_n]$ where we omit v_k

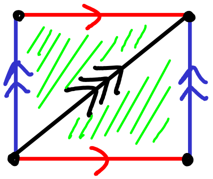
$= \{ \sum t_i v_i \in [v_0, \dots, v_n] \ : \ t_k = 0 \}$

$= \text{Image } \sigma\big|_{\Delta^{n-1}_k} : \underset{\{t \in \Delta^n : t_k = 0\}}{\overset{}{\Delta^{n-1}_k}} \longrightarrow \mathbb{R}^{n+k}$

$\Delta^{n-1}_k = \{t \in \Delta^n : t_k = 0\}$



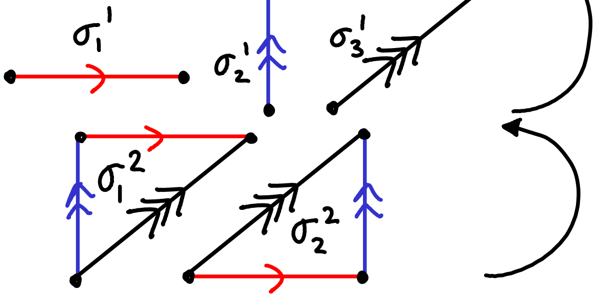
Example Can build a torus out of simplices:

$T^2 =$ 

1 0-simplex • σ_1^0

3 1-simplices

2 2-simplices

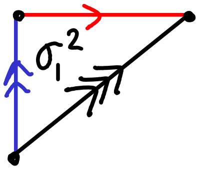
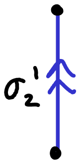


each facet is associated to another simplex, and we identify them linearly

The simplices here are abstract simplices: don't confuse them with their images in T^2

don't confuse the abstract simplices with their images in T^2=quotient space

$T^2 =$ quotient space $\bigsqcup \sigma_i^n \Big/$ canonical homeos associated to the facets

for example identify facet  of  with  via linear homeo (orientation-preserving)

Def **Δ-complex** is determined by data

- indexing set I_n, for each $n \in \mathbb{N}$
- choice of n-simplex σ_α^n (not necessarily standard) for each $\alpha \in I_n$
- gluing data: for each $\alpha \in I_n$, $0 \le i \le n$, associate some $\beta(\alpha,i) \in I_{n-1}$
- consitency condition (see later)

The Δ-complex is the quotient space

$$\boxed{X = \bigsqcup_{\substack{\alpha \in I_n \\ n \in \mathbb{N}}} \sigma_\alpha^n \Big/ \begin{array}{l} i\text{-th facet of } \sigma_\alpha^n \text{ is identified with } \sigma^{n-1}_{\beta(\alpha,i)} \\ \text{via the order-preserving canonical linear homeo} \end{array}}$$

(quotient topology: $U \subseteq X$ is open $\Leftrightarrow$ U intersects σ_α^n in an open set, $\forall \alpha, n$)

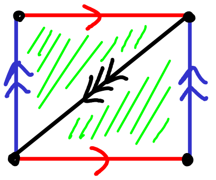
A **Δ-complex structure** on a top. space Y is a homeo from a Δ-cx $X \simeq Y$.

Explicit description of the facet identification

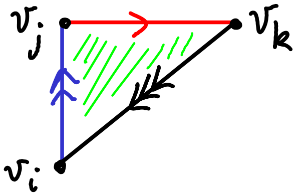
$$\begin{tikzcd}
\{\sum s_i w_i\} = [w_0,\dots,w_{n-1}] \arrow[r] & {[v_0,\dots,v_n] = \{\sum t_i v_i\}} & \\
 & & \rotatebox{90}{$\subseteq$} \\
 & & \{s_0v_0+\dots+s_{i-1}v_{i-1}+s_iv_{i+1}+\dots+s_{n-1}v_n\} = {[v_0,\dots,\hat{v}_i,\dots,v_n]} \\
\Delta^{n-1} \arrow[r] \arrow[uuu, "\sigma^{n-1}_{\beta(\alpha,i)}"] & \Delta^{n-1}_i \subseteq \Delta^n \arrow[uuu, "\sigma^n_\alpha"'] \arrow[uur, "\sigma^n_\alpha|_{\Delta^{n-1}_i}"'] & \\
(s_0,\dots,s_{n-1}) \arrow[r, mapsto] & (s_0,\dots,s_{i-1},0,s_i,\dots,s_{n-1}) &
\end{tikzcd}$$

Non-example

This decomposition for T^2 is not a Δ-complex. →



because:

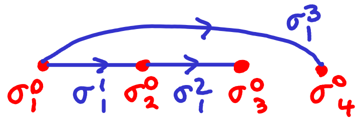


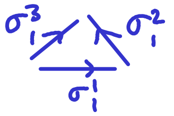
vertices are not totally ordered: $i<j<k<i$ ↯

Consistency condition

We want to additionally ensure that each point of X lies in the interior of exactly one σ_α^n, because we want to avoid unexpected identifications.

Example: $X =$ 



then glue $\sigma_1^2 =$  via 

notice how σ_3^0, σ_4^0 get identified in the quotient, but we only notice this after gluing σ_1^2

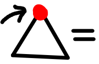
(If you try to run the definition of simplicial homology – defined later – you notice that the differential cannot satisfy $\partial_1 \circ \partial_2 = 0$)

Equivalently: the facet gluing maps are compatible under double restriction: $\forall i<j$

$$\begin{tikzcd}
{[v_0,\dots,v_n]} \arrow[r, rightsquigarrow, "\text{facet}"] \arrow[drr, rightsquigarrow, "\text{facet}"'] & {[v_0,\dots,\hat{v}_i,\dots,v_n]} \arrow[r, phantom, "\overset{\text{identify}}{\cong}"] & {[w_0,\dots,w_{n-1}]} \arrow[r, rightsquigarrow, "\text{facet}"] & {[w_0,\dots,\hat{w}_{j-1},\dots,w_{n-1}]} \arrow[dr, phantom, "\cong\ \text{identify}"] & \\
 & & & & {[x_0,\dots,x_{n-2}]} \\
 & {[v_0,\dots,\hat{v}_j,\dots,v_n]} \arrow[r, phantom, "\overset{\text{identify}}{\cong}"] & {[z_0,\dots,z_{n-1}]} \arrow[r, rightsquigarrow, "\text{facet}"] & {[z_0,\dots,\hat{z}_i,\dots,z_{n-1}]} \arrow[ur, phantom, "\cong\ \text{identify}"'] &
\end{tikzcd}$$

this ensures that $[v_0,..,\hat{v}_i,..,\hat{v}_j,...,v_n]$ is identified with the same $[x_0,..,x_{n-2}]$ whether we first restrict to $t_i=0$ (omit v_i) or first restrict to $t_j=0$ (omit v_j).

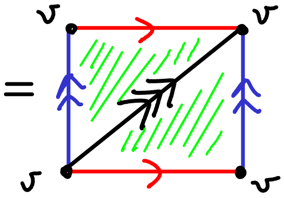
Another equivalent condition: can define the **k-th skeleton** of Δ-cx X, X^k = quotient space you get by gluing all simplices of dimensions $\leq k$. Consistency is the condition that the boundary of each σ_α^n should map continuously into X^{n-1}

(in the above Example consider the vertex  $=\partial\sigma_1^2$)

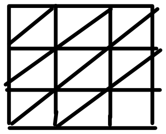
Rmk (see Part A Topology) A **simplicial complex** is a Δ-complex in which each d-dim face is uniquely determined by d distinct vertices. (more precisely, the "topological realisation" of a simpl. complex)

A homeo from such a complex to X is a **triangulation** of X.

Non-example

$T^2 =$ 

both 2-simplices have vertices v,v,v

whereas $T^2 =$  is a triangulation.

Simplicial chain complex

Def For a Δ-complex X, let X_n = set of n-simplices of X

$$C_n^\Delta(X) = \text{free abelian group generated by the set } X_n$$
$$= \left\{ \sum_{\alpha\in I_n} c_\alpha \cdot \sigma_\alpha^n : c_\alpha \in \mathbb{Z} \text{ and only finitely many } c_\alpha \neq 0 \right\}$$

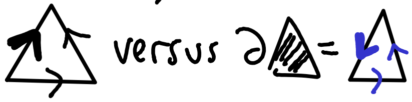
differential: $\boxed{\partial_n \sigma_\alpha = \sum (-1)^i \cdot \sigma_{\beta(\alpha,i)}}$

so: $\boxed{\partial_n [v_0,\dots,v_n] = \sum (-1)^i \cdot [v_0,\dots,\hat{v}_i,\dots,v_n]}$

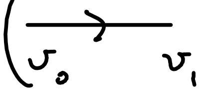
} and extend linearly

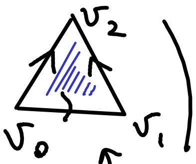
will show $\partial\circ\partial = 0$, so get **simplicial homology**: $\boxed{H_*^\Delta(X) = H_*(C_*^\Delta, \partial_*)}$

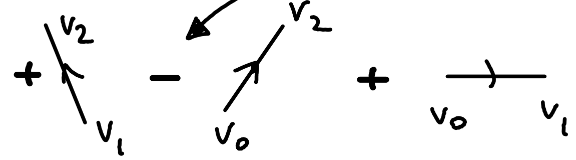
Later: The $(-1)^i$ signs keep track of whether the orientation agrees/disagrees with geometric boundary orientation, so



Examples

∂_1 () $= -v_0 \; + v_1$

∂_2 () =



$\partial_2\circ\partial_1$ (this) $= +(v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0$

$\partial\circ\partial = 0$ fails for  (not Δ-complex), try!

Lemma $\boxed{\partial\circ\partial = 0}$

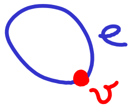
Pf $\partial_{n-1}(\partial_n[v_0,\dots,v_n]) = \sum(-1)^i \partial_{n-1}[v_0,\dots,\hat{v}_i,\dots,v_n]$

$$= \sum_{j<i} (-1)^i (-1)^j [v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]$$

↺ antisymmetric if swap i,j

$$+ \sum_{j>i} (-1)^i \underline{\underline{(-1)^{j-1}}} [v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]$$

$$= 0 \quad \square$$

Example $S^1 =$ 

Δ-cx: X_0: 1 0-simplex • $e_1^0 = e_{\beta(1,0)} = e_{\beta(1,1)}$

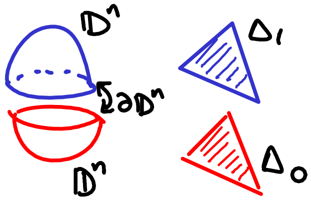
X_1: 1 1-simplex → e_1^1

$$\begin{tikzcd} 0 \arrow[r] & C_1^\Delta \arrow[r] \arrow[d, equal] & C_0^\Delta \arrow[r] \arrow[d, equal] & 0 \\ & \mathbb{Z}e & \mathbb{Z}v & \end{tikzcd}$$

$e \longmapsto v - v = 0$

$$\Rightarrow H_*^\Delta(S^1) = \begin{cases} \mathbb{Z} & * = 0,1 \\ 0 & \text{else} \end{cases}$$

Example Δ-cx structure on S^n:



$S^n = \Delta^n \sqcup \Delta^n /$ glue along $\partial\Delta^n$

↑ call this Δ_1 ↖ this Δ_0

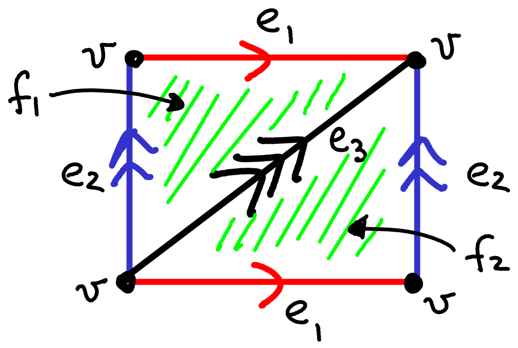
} One can deduce: ← but messy!

$$H_* S^n = \begin{cases} \mathbb{Z}\langle \text{point} \rangle \cong \mathbb{Z} & * = 0 \\ \mathbb{Z}\langle \Delta_1 - \Delta_0 \rangle \cong \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$$

↙ pick any vertex

Example

$X = T^2 =$



$$\begin{tikzcd}
0 \arrow[r] & C_2^\Delta \arrow[r] \arrow[d, equal] & C_1^\Delta \arrow[r] \arrow[d, equal] & C_0^\Delta \arrow[r] \arrow[d, equal] & 0 \\
& \mathbb{Z}f_1 + \mathbb{Z}f_2 & \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 & \mathbb{Z}v &
\end{tikzcd}$$

$$f_1 \mapsto e_1 - e_3 + e_2 \qquad e_1, e_2, e_3 \mapsto v - v = 0$$
$$f_2 \mapsto e_2 - e_3 + e_1$$

$$H_*^\Delta(T^2) = \begin{cases} \mathbb{Z}v & * = 0 \\ (\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3)/\mathbb{Z}(e_1 - e_3 + e_2) & * = 1 \\ \mathbb{Z}\cdot(f_1 - f_2) & * = 2 \\ 0 & \text{else} \end{cases}$$

$\leftarrow$ freely generated by e_1, e_2

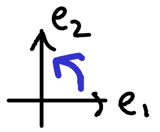
$$\cong \begin{cases} \mathbb{Z} & * = 0, 2 \\ \mathbb{Z}^2 & * = 1 \\ 0 & \text{else} \end{cases}$$

(Smith normal form of ∂_2:
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} \xrightarrow{\text{row op.}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{col. op.}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$
so after $\mathbb{Z}$-isos of C_2, C_1 we get $\mathbb{Z}^2 \to \mathbb{Z}^3$, $(a,b) \to (a,0,0)$)

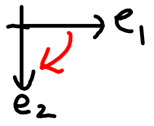
Remark about orientations (see also my B3.2 Geometry of Surfaces notes)

For vector space an **orientation** is a choice of basis modulo linear endomorphisms of $\det > 0$

Example $\mathbb{R}^2$



right-hand orientation (positive) $\xrightarrow[\det < 0]{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$



left-hand orientation (negative)

Fact $GL(n, \mathbb{R})$ has 2 path-components $<\begin{matrix} A : \det A > 0 \\ A : \det A < 0 \end{matrix}$ so can always continuously deform a basis to another within same orientation

Canonical orientation on $\mathbb{R}^n$: $e_1, \ldots, e_n$ standard basis $\leftarrow$ **"positive orientation"**

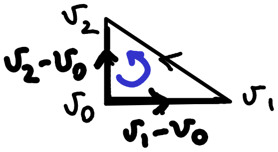
Example $[v_0, \ldots, v_n]$ simplex $\Rightarrow$ $v_1 - v_0, \ldots, v_n - v_0$ is a basis of vector subspace

$$V = \{\Sigma a_i v_i : \Sigma a_i = 0\} \subseteq \mathbb{R}^{n+k}$$

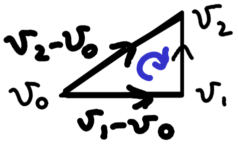
hence a choice of orientation of V, and each transposition of vertices $v_0, \ldots, v_n$ switches the orientation class.

If $v_0,..,v_n \in \underline{\underline{\mathbb{R}^n}}$ then $V=\mathbb{R}^n$ so simplex's orientation can be compared with "$\mathbb{R}^n$-orient".

Example In $\mathbb{R}^2$:

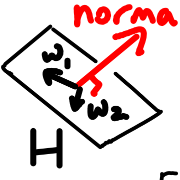


positively oriented



negatively oriented

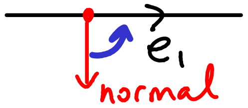
- No canonical choice of orientation for abstract vector space. Need choose basis $v_1,\to v_n$ then declare another basis positively oriented if the change of basis matrix has det >0.
- For hyperplane $H \subseteq \mathbb{R}^n$ with choice of normal can declare orientation



normal of basis $w_1,\to w_{n-1}$ of H positive if normal, $w_1,...,w_{n-1}$ is positive $\mathbb{R}^n$-basis

convention "outward normal first"

Example



$H \subseteq \mathbb{R}^2$ $\Rightarrow$ e_1 positive basis for H

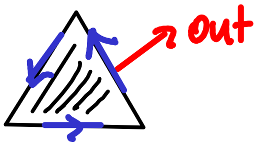
$(\text{normal}, e_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, det $= +1 > 0$

Example $\Delta^n \subseteq \mathbb{R}^{n+1}$ with normal $(1,1,..,1)$ is positively oriented.

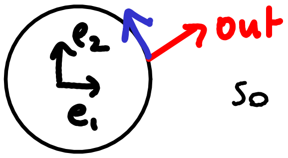
UPSHOT For an n-simplex $[v_0,.., \underline{\underline{v_n}}]$ in $\underline{\underline{\mathbb{R}^n}}$, each facet lies in a hyperplane and have canonical choice of normal: outward normal. Hence facets are canonically oriented.

Example

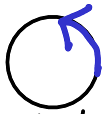
$\mathbb{R}^2 \supseteq$



in smooth world: D^2

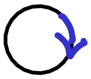


so $\partial D^2 = S^1$



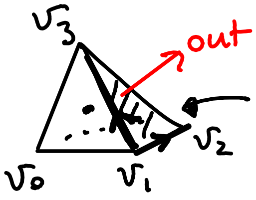
anticlockwise

Any reflection of $\mathbb{R}^n$ will swap orientation: after $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ get



clockwise

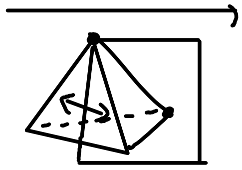
Example

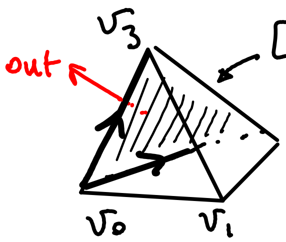


$[\hat{v}_0, v_1, v_2, v_3]$

out, v_2-v_1, v_3-v_1 positive $\mathbb{R}^3$-basis

reflect $v_0 \leftrightarrow v_1$





$[v_0, \hat{v}_1, v_2, v_3]$

out, v_2-v_0, v_3-v_0 negative $\mathbb{R}^3$-basis

UPSHOT $(-1)^i$ in $(-1)^i[v_0,\to \hat{v}_i,\to v_k]$ in definition of simplicial ∂ is there to ensure that orientations are consistent (crucial for $\partial\circ\partial=0$)

Lemma $H_*^\Delta(X) \cong \bigoplus H_*^\Delta(X_i)$ where X_i are the path-components of X.

Pf $\bigoplus C_*^\Delta(X_i) \longrightarrow C_*^\Delta(X)$, $\oplus c_i \longmapsto \sum c_i$

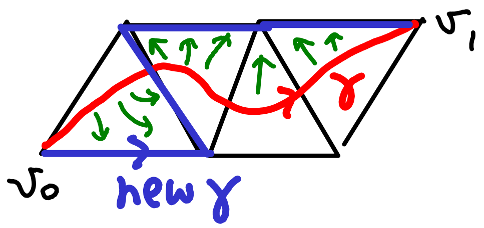
is chain isomorphism since any simplex $\sigma : \Delta^k \to X$ has path-connected image, so $\subseteq X_i$ some i. □

since Δ^k path-conn.

Theorem X has Δ-cx structure $\Longrightarrow$ $H_0^\Delta(X) \cong \bigoplus_{\text{path-conn. components}} \mathbb{Z}$

Pf By Lemma, WLOG X path-connected

- vertex v $\Rightarrow$ $v \in C_0(X) \xrightarrow{\partial} C_{-1}(X) \equiv 0$ $\Rightarrow [v] \in H_0(X)$
- vertices $v_0, v_1 \in X$ $\Longrightarrow$ $\exists$ path γ from v_0 to v_1



$\Rightarrow$ can homotope path so that go along edges (continuously deform)

$\Rightarrow$ γ is sum of 1-chains s.t. $\partial\gamma = v_1 - v_0$

$\Rightarrow$ $[v] \in H_0(X)$ independent of choice of v

$\Longrightarrow$ $H_0(X) = \langle [v] \rangle$

- $H_0(X) = \langle [v] \rangle \leftarrow \mathbb{Z}$ is injective?

  $nv \leftarrow\!\shortmid n$ Suppose $nv = \partial c$ some $c \in C_1(X)$

consider the augmentation hom

$$C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

$$\underbrace{\sum n_i \sigma_i}_{\text{0-simplices}} \longmapsto \sum n_i$$

notice composite is 0 since $\partial\left(\begin{matrix}\text{1-simplex}\\ \underset{\sigma_0}{\bullet}\!\longrightarrow\!\underset{\sigma_1}{\bullet}\end{matrix}\right) = \sigma_1 - \sigma_0 \xrightarrow{\varepsilon} 1-1=0$

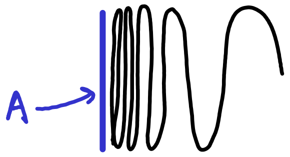
$\Longrightarrow$ $n = \varepsilon(nv) = \varepsilon\partial c = 0$. □

Rmk X top. space $\Longrightarrow$ path conn. component $\subseteq$ connected component since path-conn. $\Rightarrow$ connected. For Δ-cx, these are same (since connected + locally path-conn. $\Longrightarrow$ path-connected).

However for top. space they need not be the same, e.g. Topologist's sine curve

$\{(x, \sin\frac{1}{x}) : x \in (0,1]\} \cup \underbrace{0 \times [0,1]}_{A} \subseteq \mathbb{R}^2$

2 path-conn. components



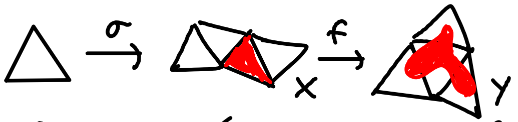
- connected
- not path-connected
- not locally path-connected

3. SINGULAR HOMOLOGY

Motivation Not obvious that H^Δ_* is functorial: $\Delta^k \xrightarrow{\sigma} X \xrightarrow{f} Y$

↖ continuous map

then $f\circ\sigma$ typically not a simplex:



Solution 1: only allow simplicial maps $f: X\to Y$ (so $f\circ\sigma$ simplex $\forall\sigma$)

Solution 2: show that any cts map $f: X\to Y$ can be approximated arbitrarily well by a simplicial map after having performed barycentric subdivision on X, Y enough times. Also any two such approximations induce the same map $H^\Delta_*(X)\to H^\Delta_*(Y)$

Solution 3: develop new homology $H_*(X)$ which allows any cts map $\Delta^n\to X$ and prove $H^\Delta_*(X)\cong H_*(X)$ for Δ-complexes X.

↑ WILL DO THIS.

X is any top. space

Def Singular n-simplex is any $\boxed{\text{continuous map } \sigma:\Delta^n \longrightarrow X}$

Singular n-chains $C_n(X)$ = free abelian group generated by ⤴

$$= \left\{ \sum_{\substack{\text{singular}\\ n\text{-simplices } \sigma}} c_\sigma\cdot\sigma \;:\; c_\sigma\in\mathbb{Z}, \text{ only finitely many } c_\sigma\neq 0\right\}$$

$$\boxed{\partial_n\sigma = \sum (-1)^i\cdot\sigma|_{\Delta^{n-1}_i}}$$ ← i-th facet

(and extend linearly)

Rmk Here $\Delta^{n-1}_i=[e_0,\dots,\hat e_i,\dots,e_n]$ is identified canonically with Δ^{n-1} (send $e_k\to e_k$, $k<i$; $e_k\to e_{k-1}$ for $k>i$)

Will show $\partial\circ\partial=0$, so get singular homology: $\boxed{H_*(X)=H_*(C_*,\partial_*)}$

For Δ-complex X have inclusion of subcomplex $C^\Delta_*\to C_*$

$\Longrightarrow$ induces $H^\Delta_*(X)\longrightarrow H_*(X)$ **Fact**: isomorphism (proof later, see cellular $H^{CW}_*\cong H_*$)

Corollary $H^\Delta_*(X)$ is independent of choice of Δ-cx structure on X

Lemma $\partial\circ\partial=0$

Proof

$$\partial_{n+1}(\partial_n\sigma) = \partial_{n+1}\left(\sum(-1)^i\sigma|_{\Delta^{n-1}_i}\right) \quad \leftarrow [e_0,\dots,\hat e_i,\dots,e_n]$$

$$= \sum_{j<i}(-1)^i(-1)^j\sigma|_{[e_0,\dots,\hat e_j,\dots,\hat e_i,\dots,e_n]}$$

$$+\sum_{j>i}(-1)^i\underline{\underline{(-1)^{j-1}}}\sigma|_{[e_0,\dots,\hat e_i,\dots,\hat e_j,\dots,e_n]}$$

$$=0$$

□

Example $X = \text{point} \implies C_n(X) = \mathbb{Z}\cdot(\sigma_n : \Delta^n \to X \text{ the constant map})$

$$\left.\begin{array}{l}\partial\sigma_n = \sum(-1)^i \sigma_n|_{\Delta_i^{n-1}} = \underbrace{\sum(-1)^i}_{\begin{cases}1 & \text{if } n \text{ even}\\ 0 & \text{'' odd}\end{cases}} \sigma_{n-1}\end{array}\right\} \begin{array}{l}\Rightarrow \cdots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \underset{C_1}{\underset{\wr\wr}{\mathbb{Z}}} \xrightarrow{0} \underset{C_0}{\underset{\wr\wr}{\mathbb{Z}}} \to 0 \\ \Longrightarrow H_*(pt) \cong \begin{cases}\mathbb{Z} & *=0\\ 0 & \text{else}\end{cases}\end{array}$$

Lemma $\boxed{H_*(X) \cong \bigoplus H_*(X_i)}$ where X_i are path-components of X

Pf Image of cts map $\Delta^n \to X$ is path conn. so lies in some X_i. □

Cor $\boxed{H_0(X) = \bigoplus_{X_i} \mathbb{Z}}$

← generators of $C_0(X)$

Pf By Lemma, WLOG X path-connected. $\Delta^0 = pt \to X$ is cycle since $C_{-1}(X) = \phi$

Given 2 points $x, y \in X$, a path $\Delta^1 = [0,1] \xrightarrow[\gamma]{} X$, $\gamma(0) = x$, $\gamma(1) = y$ is also a 1-chain!

So $x - y = \partial\gamma$, so x, y are homologous. Finally if $n\cdot[x] = 0 \in H_0(X)$ then $nx = \partial c$ some $c \in C_1(X)$ = generated by paths. Now run the augmentation hom. trick like we did for H_0^Δ: $n = \varepsilon(nx) = \varepsilon\partial c = 0$ as $\varepsilon\circ\partial = 0$. □

Naturality (i.e. functoriality)

Lemma $f : X \to Y$ continuous

$\Longrightarrow H_*(f) = f_* : H_*(X) \to H_*(Y)$ induced by chain map

$f_* : C_*(X) \to C_*(Y)$

induced map

$\boxed{f_*(\sigma) = f\circ\sigma}$ and extend linearly

$$\begin{tikzcd}\Delta^n \arrow[r, "\sigma"] \arrow[dr, "f_*\sigma"'] & X \arrow[d, "f"] \\ & Y\end{tikzcd}$$

Pf $\partial_n(f_*\sigma) = \sum(-1)^n f\circ\sigma|_{\Delta_i^{n-1}} = f_*\left(\sum(-1)^n \sigma|_{\Delta_i^{n-1}}\right) = f_*(\partial_n\sigma)$ □

Properties 1) $X \xrightarrow{f} Y \xrightarrow{g} Z \implies (g\circ f)_* = g_*\circ f_*$

2) $id_* = id$

Pf 1) $(g\circ f)_*\sigma = g\circ f\circ\sigma = g_*(f\circ\sigma) = g_*(f_*\sigma)$ ✓

2) $id_*\sigma = id\circ\sigma = \sigma$ ✓ □

Cor $H_* : \left\{\begin{array}{c}\text{topological spaces}\\ \&\\ \text{cts maps}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{graded abelian groups}\\ \&\\ \text{graded homs}\end{array}\right\}$ is a functor

Cor $X \cong Y \implies H_*(X) \cong H_*(Y)$

4. CHAIN HOMOTOPIES AND HOMOTOPY INVARIANCE

Algebra: chain homotopies

$f_*, g_* : (C_*, \partial_*) \longrightarrow (\tilde{C}_*, \tilde{\partial}_*)$ chain maps

Def f_*, g_* are **chain homotopic** if $\exists$ (degree +1) hom $h : C_* \longrightarrow \tilde{C}_*[1]$ s.t.

$$\boxed{\tilde{\partial} \circ h + h \circ \partial = f - g}$$

h is called a **chain homotopy**

Consequence $f_* = g_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$ on homology

Pf

$$\begin{tikzcd}
C_{n+1} \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[dl, "h_{n-1}"'] \\
\tilde{C}_{n+1} \arrow[r, "\tilde{\partial}_{n+1}"'] & \tilde{C}_n \arrow[r] & \tilde{C}_{n-1}
\end{tikzcd}$$

c cycle $\in C_n$

$$\Longrightarrow f_n(c) - g_n(c) = \underbrace{\tilde{\partial}_{n+1} \circ h_n(c)}_{\text{boundary}} + h_{n-1} \circ \underbrace{\partial_n(c)}_{\substack{\shortparallel \\ 0}}$$

$$\Longrightarrow f_n(c) = g_n(c) \text{ in } H_*(\tilde{C}) \qquad \square$$

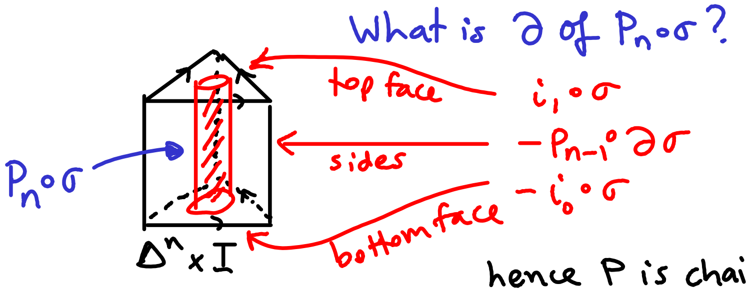
Theorem $i_0 : X \longrightarrow X \times I$, $i_0(x) = (x, 0)$ where $I = [0,1]$

$i_1 : X \longrightarrow X \times I$, $i_1(x) = (x, 1)$

$\Longrightarrow i_{0*}, i_{1*} : C_*(X) \rightarrow C_*(X \times I)$ are chain hpic.

Key idea Need "prism operator" which cuts $\Delta^n \times I$ into a sum Γ_n of $(n+1)$-simplices in $\Delta^n \times I$:

$$\begin{tikzcd}
(\sigma : \Delta^n \to X) \arrow[r, mapsto] \arrow[ddr, bend right, "\text{prism operator } P_n"'] & \sigma \times id : \Delta^n \times I \to X \times I \\
& \Delta^{n+1} \arrow[u, "\Gamma_n = \text{combo of maps}"] \\
& (\sigma \times id) \circ \Gamma_n : \Delta^{n+1} \to X \times I
\end{tikzcd}$$



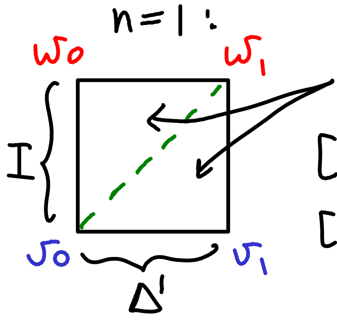
hence P is chain hpy

Pf ← Non-examinable

$$\left.\begin{array}{lll} \text{bottom facet} & \Delta^n\times 0 = [v_0,\dots,v_n] & \leftarrow v_i = e_i\times 0 \\ \text{top facet} & \Delta^n\times 1 = [w_0,\dots,w_n] & \leftarrow w_i = e_i\times 1 \end{array}\right\} \subseteq \Delta^n\times[0,1]\subseteq \mathbb{R}^{n+1}$$

Examples

$n=1$:

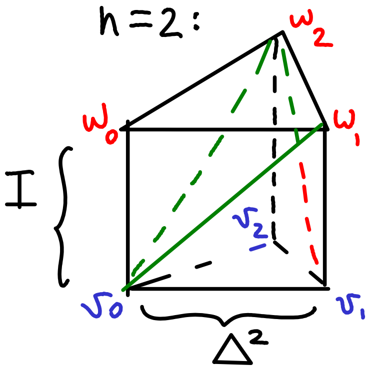


two 2-simplices

$[v_0,v_1,w_1]$

$[v_0,w_0,w_1]$

$n=2$:



three 3-simplices:

$[v_0,v_1,v_2,w_2]$

$[v_0,v_1,w_1,w_2]$

$[v_0,w_0,w_1,w_2]$

Let $s_i = [v_0,\dots,v_i,w_i,\dots,w_n]$

Claim The s_i cover $\Delta\times[0,1]$ and give Δ-cx structure on $\Delta^n\times I$

Pf $\displaystyle\sum_{k\le i} t_k v_k + \sum_{k\ge i} s_k w_k = (t_0,\dots,t_{i-1},\underline{t_i+s_i},s_{i+1},\dots,s_n,s_i+\dots+s_n)$

So given $(x_0,\dots,x_n,a)\in\Delta^n\times I$, equate and solve:

$t_0=x_0,\dots,t_{i-1}=x_{i-1},\ s_{i+1}=x_{i+1},\dots,s_n=x_n$, and $\begin{cases} s_i = a-x_{i+1}-\dots-x_n \\ t_i = x_i-s_i\end{cases}$

Note $x_k\ge 0$, $\Sigma x_k=1$, $a\in[0,1]$ hence $\Sigma t_k+\Sigma s_k=1$ ✓ $\begin{cases} t_k\ge 0 \text{ for } k<i ✓ \\ s_k\ge 0 \text{ for } k>i\end{cases}$

but $\left.\begin{array}{l} s_i\ge 0 \\ t_i\ge 0\end{array}\right\} \Leftrightarrow \begin{cases} a\ge x_{i+1}+\dots+x_n \\ x_i+x_{i+1}+\dots+x_n\ge a\end{cases}$. Thus a solution exists if we pick $i=\min\{k : a\ge x_{i+1}+\dots+x_n\}$

There are multiple solutions if $x_{i+1}=x_{i+2}=\dots=x_j=0$, but that is as expected: those points of $\Delta^n\times I$ belong to the faces of $s_i,s_{i+1},\dots,s_j$. □

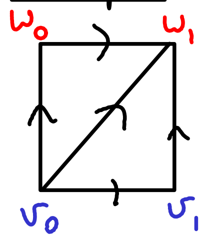
Def

$\Gamma_n = \sum(-1)^i s_i \in C_{n+1}(\Delta^n\times[0,1])$ ← geometrically this "represents" $\Delta^n\times I$ as a simplicial chain

$$\left.\begin{array}{l}\Rightarrow \partial\Gamma_n = \displaystyle\sum_i\sum_{j\le i}(-1)^i(-1)^j[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n] \\ \qquad + \displaystyle\sum_i\sum_{j\ge i}(-1)^i(-1)^{j+1}[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]\end{array}\right\}$$

geometrically this "represents" $\partial(\Delta^n\times I) = (\partial\Delta^n\times I)\sqcup(\Delta^n\times\partial I)$

Example



$\Gamma_1 = [v_0,w_0,w_1]-[v_0,v_1,w_1]$ "is the square"

$\partial\Gamma_1 = [w_0,w_1]-\cancel{[v_0,w_1]}+[v_0,w_0]-[v_1,w_1]+\cancel{[v_0,w_1]}-[v_0,v_1]$

"is ∂ of square"

"inside facets" cancel

Prism operator $P : C_n(X) \longrightarrow C_{n+1}(X \times [0,1])$

$$\boxed{P(\sigma) = (\sigma \times id)_*(\Gamma_n)}$$

$\sigma : \Delta^n \to X$

$\sigma \times id : \Delta^n \times [0,1] \to X \times [0,1]$

$(\sigma \times id)(x,t) = (\sigma(x), t)$

this abbreviated notation means the map

$$\underset{\Delta^n}{\overset{\cap}{(t_0,\dots,t_n)}} \mapsto (t_0\sigma e_0 + \dots + \widehat{t_j\sigma e_j} + t_j\sigma e_{j+1} + \dots + t_{i-1}\sigma e_i + $$
$$+ t_i\sigma e_i + \dots + t_n\sigma e_n, t_i + \dots + t_n) \in X \times I$$

$$\begin{aligned}
\partial P(\sigma) &= \partial(\sigma\times id)_*(\Gamma_n) \\
&= (\sigma \times id)_*(\partial\Gamma_n) \\
&= \sum_i \sum_{j\le i} (-1)^i(-1)^j [i_0\sigma e_0, \dots, \widehat{i_0\sigma e_j}, \dots, i_0\sigma e_i, i_1\sigma e_i, \dots, i_1\sigma e_n] \\
&\quad + \sum_i \sum_{j\ge i} (-1)^i(-1)^{j+1}[i_0\sigma e_0, \dots, i_0\sigma e_i, i_1\sigma e_i, \dots, \widehat{i_1\sigma e_j}, \dots, i_1\sigma e_n] \\
&= \underset{\substack{\uparrow \\ i=j=0 \\ 1^{st}\text{ sum}}}{i_{1*}\sigma} - \underset{\substack{\uparrow \\ i=j=n \\ 2^{nd}\text{ sum}}}{i_{0*}\sigma} - \underbrace{P\partial\sigma}_{((\partial\sigma)\times id)_*\Gamma_{n-1}}
\end{aligned}$$

⍟

$(\sigma \times id)(v_i)$
$= (\sigma \times id)(e_i, 0)$
$= (\sigma(e_i), 0)$
$= i_0(\sigma)(e_i)$

$(\sigma \times id)(w_i)$
$= (\sigma(e_i), 1)$
$= i_1(\sigma)(e_i)$

$\Gamma_{n-1} = \sum_i (-1)^i [v_0, \dots, v_i, w_i, \dots, w_{n-1}]$

now use ⍟ and

$\partial\sigma = \sum (-1)^k [\sigma e_0, \dots, \widehat{\sigma e_k}, \dots, \sigma e_n]$. $\square$

Homotopy invariance

$f_0, f_1 : X \longrightarrow Y$

Def **$f_0 \simeq f_1$ homotopic** if $\exists$ continuous map $F : X \times [0,1] \longrightarrow Y$ called **homotopy**

s.t. $f_0 = F \circ i_0$
$f_1 = F \circ i_1$.

Idea Think of this as a continuous family of maps

$f_t = F(\cdot, t) : X \to Y$ from f_0 to f_1.

Exercise $\simeq$ is an equivalence relation.

Homotopic **relative to $A \subseteq X$** if $F(a,t) = f_0(a) = f_1(a)$ all $a \in A$, all t.

write "$f \simeq g$ **rel A**"

Def $X \simeq Y$ **homotopy equivalent** spaces if $\exists$ maps

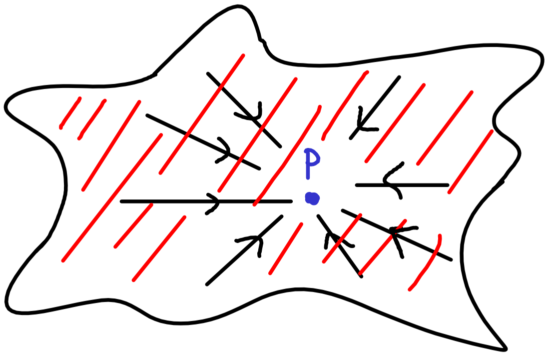
$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \quad \text{with} \quad \begin{array}{l} g \circ f \simeq id \\ f \circ g \simeq id \end{array}$$

Rmk homeo $\Rightarrow$ hpy equivalent

Def X **contractible** if $X \simeq pt$

equivalently $(X \xrightarrow{id} X) \simeq (X \xrightarrow{const} \text{point} \in X)$

Example
- $\mathbb{R}^n \simeq pt$
  $F(x,t) = tx$ then $f_0 \equiv 0$, $f_1 = id$.
- (star-shaped subsets of $\mathbb{R}^n$) $\simeq pt$



contains line segments to a specific point p

WLOG $p=0$ & use same F
↑
translate

(examples: $\mathbb{D}^n$, convex sets, ...)

Theorem $f_0 \simeq f_1 \Rightarrow f_{0*} = f_{1*} : H_* X \to H_* Y$

Pf $f_{1*} - f_{0*} = F_* i_{1*} - F_* i_{0*}$ (where F = homotopy, i_0, i_1 as in previous Thm)

$= F_* (i_{1*} - i_{0*})$

previous Thm $\to$ $= F_* (\partial P + P \partial)$

F_* chain map $\to$ $= \partial \circ (F_* \circ P) + (F_* \circ P) \circ \partial$

$\Rightarrow F_* \circ P$ is chain hpy from f_{0*} to f_{1*} □

Cor $X \simeq Y \Rightarrow H_* X \cong H_* Y$

Pf $f_* g_* = id_*$, $g_* f_* = id_*$ □

Example X contractible $\Rightarrow H_* X \cong H_*(pt) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Cultural Rmk (Whitehead's theorem) For nice topological spaces ← (CW complexes - see later in course) if X, Y are simply connected and $\exists f: X \to Y$ inducing isomorphisms on H_* then $X \simeq Y$ are homotopy equivalent.

Relative homology

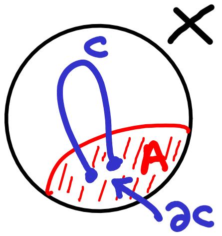
Def (X, A) *pair of spaces* if $A \subseteq X$ topological subspace

$\Rightarrow$ $i = \text{incl}: A \hookrightarrow X$ induces a subcx $i_*: C_*A \to C_*X$

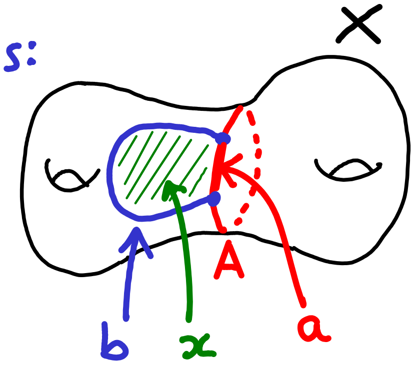
$\Rightarrow$ C_*X/C_*A quotient chain cx (recall $\partial[x] = [\partial x]$)

$$\boxed{H_*(X,A) = H_*(C_*X/C_*A)}$$

Idea: *relative cycles*:
$c \in C_*X$
s.t. $\partial c \in C_*A$



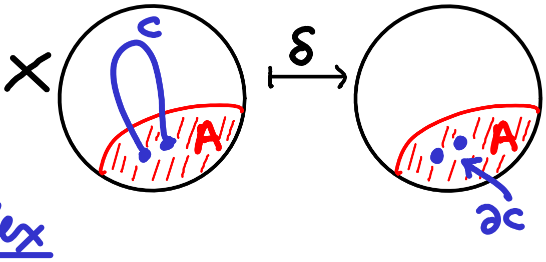
relative boundaries:
$b \in C_*X$
s.t. $\exists\, x \in C_{*+1}X$
$\partial x = b + \underset{\in C_*A}{a}$



$\Rightarrow$ $0 \to C_*A \to C_*X \to C_*X/C_*A \to 0$ SES

Cor $\boxed{\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} \cdots}$

LES for the pair



$\delta c = \underset{C_*A}{\underset{\Uparrow}{\partial c}}$

Need not be ∂a some $a \in C_*A$

Reduced homology

$\tilde H_* X = H_*$ of *augmented chain complex*

$$\cdots \to C_n(X) \to \cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

augmentation $\to$ $\varepsilon(\Sigma \underset{\in \mathbb{Z}}{n_i} \cdot \underset{\text{points } \in X}{p_i}) = \Sigma n_i$

can view $C_{-1}(X)$ $= \mathbb{Z}\cdot(\text{map } \emptyset \to X)$ where allow the empty simplex $\emptyset$

For $X \neq \emptyset$, $\tilde H_* X = \ker H_*X \to H_*(pt)$ induced by $X \to pt$

Example $\tilde H_*(\mathbb{R}^n) \cong \tilde H_*(\mathbb{D}^n) \cong \tilde H_*(pt) = 0$

Check · $H_*X = \tilde H_* X$ $* \neq 0$, and $H_0 X \cong \tilde H_0 X \oplus \mathbb{Z}$ for $X \neq \emptyset$

· $f: X \to Y \Rightarrow f_*: \tilde H_* X \to \tilde H_* Y$

Lemma (X,A) pair $\Rightarrow \exists$ LES

if $A = \emptyset$ we end with $\tilde H_{-1}A = \mathbb{Z}$

$$\boxed{\cdots \to \tilde H_n(A) \xrightarrow{i_*} \tilde H_n(X) \to H_n(X,A) \to \tilde H_{n-1}(A) \xrightarrow{i_*} \cdots \to H_0(X,A) \to 0}$$

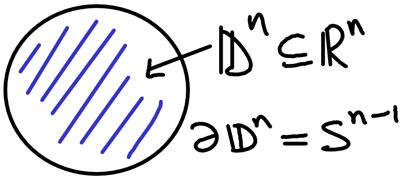
Pf use augmented ch. cx. and $\mathbb{Z}/\mathbb{Z} = 0$ □

Cor $H_*(X,pt) \cong \widetilde{H}_*(X)$

Pf $\widetilde{H}_*(pt)=0$. □

Example LES:

$$\begin{tikzcd} \widetilde{H}_*(S^{n-1}) \arrow[rr] & & \widetilde{H}_*(\mathbb{D}^n)=0 \arrow[dl] \\ & H_*(\mathbb{D}^n,S^{n-1}) \arrow[ul, "{[-1]}"] & \end{tikzcd}$$



$$\Longrightarrow \boxed{H_*(\mathbb{D}^n,S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})}$$

Naturality of the LES for pairs

Def A **map of pairs of spaces** $(X,A) \xrightarrow{f} (Y,B)$ means $f: X \to Y$ and $f(A) \subseteq B$.

Lemma

$$\begin{tikzcd} \cdots \arrow[r] & H_*A \arrow[r] \arrow[d, "f_*"'] & H_*X \arrow[r] \arrow[d, "f_*"'] & H_*(X,A) \arrow[r] \arrow[d] & H_{*-1}A \arrow[r] \arrow[d, "f_*"'] & \cdots \\ \cdots \arrow[r] & H_*B \arrow[r] & H_*Y \arrow[r] & H_*(Y,B) \arrow[r] & H_{*-1}B \arrow[r] & \cdots \end{tikzcd}$$

and similarly for $\widetilde{H}_*$.

Pf

$$\begin{tikzcd} 0 \arrow[r] & C_*A \arrow[r] \arrow[d, "f_*"'] & C_*X \arrow[r] \arrow[d, "f_*"'] & C_*X/C_*A \arrow[r] \arrow[d, "f_*"'] & 0 \\ 0 \arrow[r] & C_*B \arrow[r] & C_*Y \arrow[r] & C_*Y/C_*B \arrow[r] & 0 \end{tikzcd}$$

$\Longrightarrow$ claim follows by naturality of LES induced by SESs of chain cxes. □

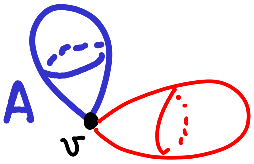
5. EXCISION THEOREM AND QUOTIENTS

(X,A) pair

Def $r: X \to X$ **retraction** onto A if $\begin{cases} r(X)=A \\ r|_A = id_A \end{cases}$

(equivalently $r^2=r$ then define $A=im(r)$)

Example



$X = \underbrace{S^2}_{=A} \vee S^2$ = two spheres glued at one point v (**wedge sum**)

$r: X \to A$ map second sphere to v

Example In Pf of Brower fixed pt thm we built a retraction r by contradiction

Cor r retraction $\Longrightarrow$ $r_*: H_*X \to H_*A$ surjective

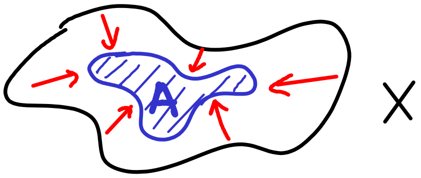
$incl_*: H_*A \to H_*X$ injective

Pf

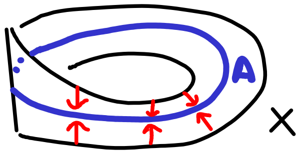
$$\begin{tikzcd} A \arrow[r, "incl"] \arrow[rr, bend right, "id_A"'] & X \arrow[r, "r"] & A \end{tikzcd}$$

now use H_* functorial □

Def $r : X \to X$ **deformation retraction** onto A if $\begin{cases} r \text{ retraction} \\ r \simeq \mathrm{id}_X \ \text{ rel } A \end{cases}$



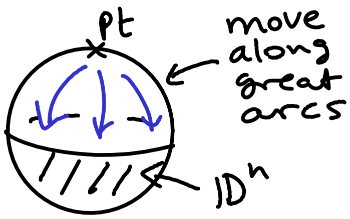
Example $X =$ Möbius strip
$A =$ equator



Lemma r def. retr. $\Rightarrow$ • $A \xrightarrow[\simeq]{\text{incl}} X$ is a homotopy equivalence.
• incl_* and r_* are isos on H_*, so $H_*A \cong H_*X$

Pf $$\begin{tikzcd} A \arrow[r, bend left, "\text{incl}"] & X \arrow[l, bend left, "r"] \end{tikzcd}$$ $\text{incl} \circ r = r \simeq \mathrm{id}_X$, $r \circ \text{incl} = r|_A = \mathrm{id}_A$ $\square$

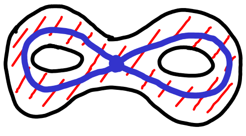
Example $S^n \setminus pt$ def. retracts to $D^n \cong$ lower hemisphere:



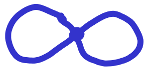
$\Rightarrow S^n \setminus pt \simeq D^n$

$\Rightarrow S^n \setminus \{2 \text{ points}\} \simeq D^n \setminus pt \cong D^n \setminus 0 \simeq S^{n-1}$

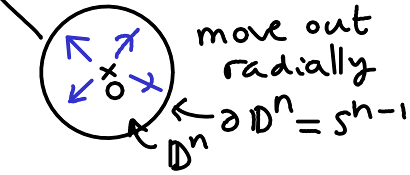
$\Rightarrow S^n \setminus \{3 \text{ points}\} \underset{\text{def.ret.}}{\simeq}$



$\underset{\text{ret.}}{\simeq}$



$S^{n-1} \vee S^{n-1}$



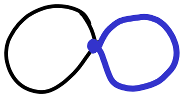
Excision theorem

$E \subseteq A \subseteq X$ subspaces with $\overline{E} \subseteq A^\circ$ $\Longrightarrow$ $(X \setminus E, A \setminus E) \longrightarrow (X, A)$ induces iso

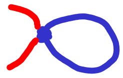
$$\boxed{H_*(X\setminus E, A \setminus E) \xrightarrow{\cong} H_*(X,A)}$$

Proof Later.

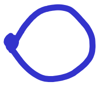
Example $X = S^1 \vee S^1 =$



$\supseteq A =$

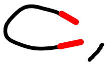


$\supseteq E =$



$\cong S^1$

$\Rightarrow H_*(X,A) \underset{\uparrow \text{exc. thm.}}{\cong} H_*($



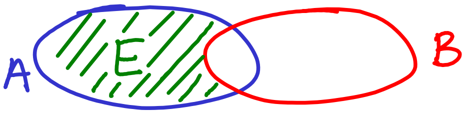
,

$) \underset{\text{hpy invce}}{\cong} H_*(D^1, \underbrace{\partial D^1}_{= S^0}) \cong \tilde{H}_0(\underbrace{S^0}_{= \text{2 points}}) \cong \mathbb{Z}$

Rephrasing of Excision Thm

$X = A^\circ \cup B^\circ$ $\Longrightarrow$ $\boxed{H_*(X,A) \cong H_*(B, A \cap B)}$ induced by inclusion $(X,A) \leftarrow (B, A\cap B)$

$(A, B \subseteq X$ subspaces$)$



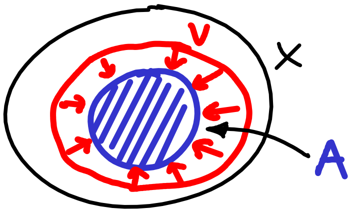
Pf Take $E = X \setminus B$ so $X \setminus E = B$ and $A \cap B = A \setminus E$. $\square$

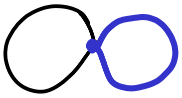
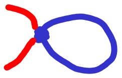
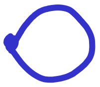
Idea why excision holds: $C_*(A) + C_*(B) \longrightarrow C_*(X)$ is a homotopy equivalence and $C_*(A) \cap C_*(B) = C_*(A \cap B)$. Idea $\uparrow$ can subdivide chains in X many times, and small enough chains lie either in A or in B (or in both).

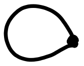
Good pairs and quotients

(X, A) pair

- **Quotient** $X/A = X/_{\sim}$ ← equiv. relation $x \sim y \iff x = y$ or $x, y \in A$
- (X, A) **good pair** if $\begin{cases} A \neq \emptyset \\ A \text{ closed} \\ A \text{ deformation retract of nbhd } V \text{ of } A \end{cases}$

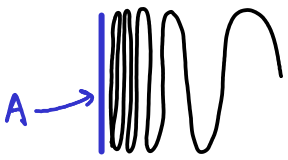


Example $X = S^1 \vee S^1 =$  $\supseteq V =$  $\supseteq A =$  $\cong S^1$

$X/A \cong$  ← all points of A are identified with the node

Non-example **Topologist's sine curve**

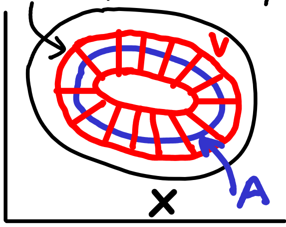
$\{(x, \sin\frac{1}{x}) : x \in (0,1]\} \cup \underbrace{0 \times [0,1]}_{A} \subseteq \mathbb{R}^2$



connected
not path-connected
not locally connected
not locally path-connected

Cultural Rmk

Smooth submanifold ⊆ smooth manifold is a good pair (tubular neighbourhood theorem)



Cor (X, A) good $\Rightarrow (X, A) \longrightarrow (X/A, pt)$ induces iso

$$\boxed{H_*(X, A) \longrightarrow H_*(X/A, pt) = \widetilde{H}_*(X/A)}$$

Pf good $\Rightarrow \exists$ nbhd V of A, and $A \xrightarrow[\text{incl}]{\simeq} V$.

LES for pairs & 5-Lemma since $A \simeq V$, $A/A \simeq V/A$

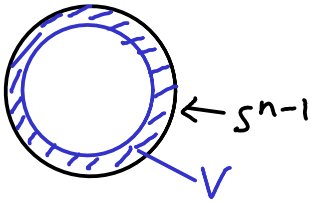
excision

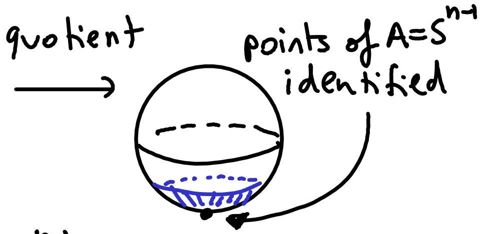
$$\begin{tikzcd}
H_n(X, A) \arrow[r, "\cong"] \arrow[d, "\text{quot.}"'] & H_n(X, V) \arrow[d, "\text{quot.}"'] & H_n(X \setminus A, V \setminus A) \arrow[l, "\cong"'] \arrow[d, "= \;", "id_* = \text{identity}"'] \\
H_n(X/A, \underbrace{A/A}) \arrow[r, "\cong"] & H_n(X/A, V/A) & H_n(X/A \setminus p, V/A \setminus p) \arrow[l, "\cong"']
\end{tikzcd}$$

← call this point p

Hence all arrows are isos. □

Example $\mathbb{D}^n \supseteq S^{n-1}$ good:





$$\Longrightarrow H_*(\mathbb{D}^n, S^{n-1}) \underset{\text{Cor}}{\cong} \widetilde{H}_*(\mathbb{D}^n/S^{n-1}) \cong \widetilde{H}_*(S^n) \leftarrow \mathbb{D}^n/S^{n-1} \cong S^n$$

Recall we proved $\widetilde{H}_*(\mathbb{D}^n, S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})$ (from LES & $\widetilde{H}_*(\mathbb{D}^n)=0$)

$\Rightarrow$ inductively, using Example

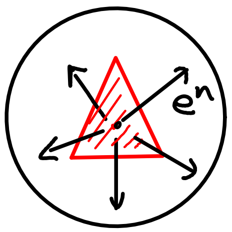
$$\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1}) \cong \cdots \cong \widetilde{H}_{k-n}(\underset{\text{2 points}}{\underset{\parallel}{S^0}}) = \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{else} \end{cases}$$

$H_0(2\text{pts}) = \mathbb{Z}\oplus\mathbb{Z}$

Generator of $H_n(S^n) \cong \widetilde{H}_n(\mathbb{D}^n/S^{n-1}) \cong H_n(\mathbb{D}^n, S^{n-1}) \cong \mathbb{Z}$

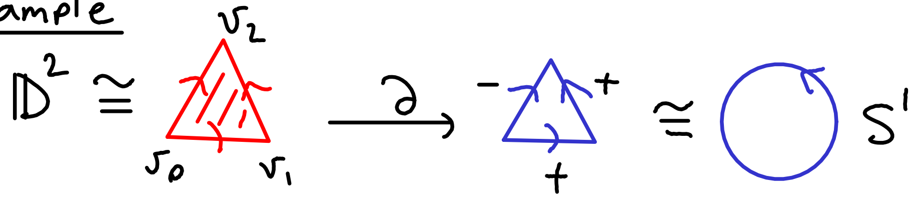
Observe $\exists$ homeo $e^n : \Delta^n \cong \mathbb{D}^n$ (homework) inducing Δ-cx structure on S^{n-1}:

$$\boxed{\partial\Delta^n \cong \partial\mathbb{D}^n = S^{n-1}}$$



stretch ctsly outwards from barycentre(Δ^n)

<u>Example</u>



<u>Upshot</u> $(n \geq 2)$

$$\boxed{\begin{aligned} H_n(\mathbb{D}^n, S^{n-1}) &= \mathbb{Z}\cdot e^n \\ H_{n-1}(S^{n-1}) &= \mathbb{Z}\cdot\partial e^n \\ \widetilde{H}_n(\mathbb{D}^n/S^{n-1}) &= \mathbb{Z}\cdot[e^n] \end{aligned}}$$

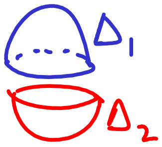
LES

for $n-1 \geq 1$, so $n \geq 2$

by Cor

($[e^n]$ really lives in $H_n(\mathbb{D}^n, S^{n-1}) \cong H_n(\mathbb{D}^n/S^{n-1}, S^{n-1}/S^{n-1})$)

<u>Exercise</u> Recall another Δ-cx structure on S^n:



$$\boxed{S^n = \Delta^n \sqcup \Delta^n / \text{glue along } \partial\Delta^n}$$

call this Δ_1 this Δ_0

then $\boxed{H_n S^n = \mathbb{Z}\cdot(\Delta_1 - \Delta_0)}$ and

$$\begin{array}{ccc} & & H_n(\mathbb{D}^n, \partial\mathbb{D}^n) \\ & & \wr\| \\ H_n(S^n, \Delta_0) & \overset{\text{exc.}}{\cong} & H_n(\Delta_1, \partial\Delta_1) \\ \Delta_1 - \Delta_0 & \to & \Delta_1 \end{array}$$

Another remark about orientations

<u>Fact</u> $\{\text{homeos } \Delta^n \to \mathbb{D}^n\}$ has 2 path-components

Above we chose a path-component by constructing e^n.

If r is any reflection in $\mathbb{R}^{n+1}$ then $e^n \circ r$ is in the other path-component

e.g. swap 2 coordinates in Δ^n

$$\begin{tikzcd} H_n(S^n) \cong H_n(\mathbb{D}^n, S^{n-1}) \arrow[r, "\cong"] & \mathbb{Z} \\ e^n \arrow[r, mapsto] & +1 \\ e^n \circ r \arrow[r, mapsto] & -1 \end{tikzcd}$$

We will see later in the course that this corresponds to a choice of orientation of $\mathbb{D}^n$ and S^n.

Our choice is consistent with the inclusion $\mathbb{D}^n \subseteq \mathbb{R}^n$ (with the positive (canonical) orientation of $\mathbb{R}^n$) and the inclusion

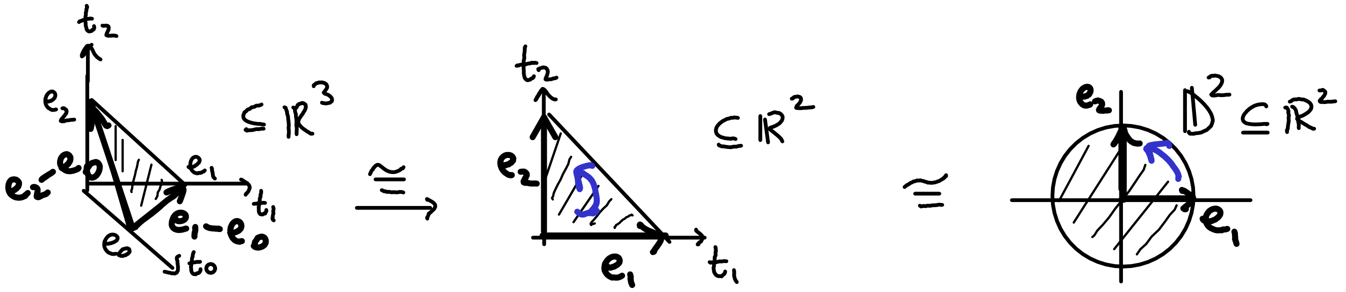
$$(\Delta^n \subseteq \underline{\underline{\mathbb{R}^{n+1}}}) \xrightarrow{\cong} \{(t_1,\dots,t_n) \in \underline{\underline{\mathbb{R}^n}} : t_i \geq 0, \sum_{i=1}^n t_i \leq 1\}$$

$$(\underline{\underline{t_0}},\dots,t_n) \longmapsto (\underline{\underline{t_1}},\dots,t_n)$$

$$t_i \geq 0, \sum t_i = 1$$

Example

$$\Delta^2 = [e_0,e_1,e_2]$$



e_1, e_2 positive $\mathbb{R}^2$-basis

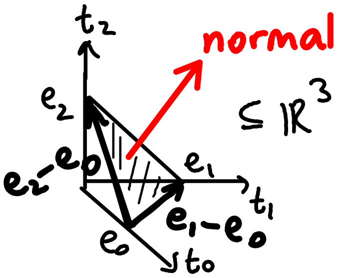
standard orientation

Our choice is also consistent with the "normal first" convention for orienting hyperplanes with a given choice of normal:

$\Delta^n \subseteq$ hyperplane $\{(t_0,\dots,t_n) : \sum t_j = 1\} \subseteq \underline{\underline{\mathbb{R}^{n+1}}}$ normal $(1,1,\dots,1)$ (so pointing to ∞ in positive quadrant)

Example

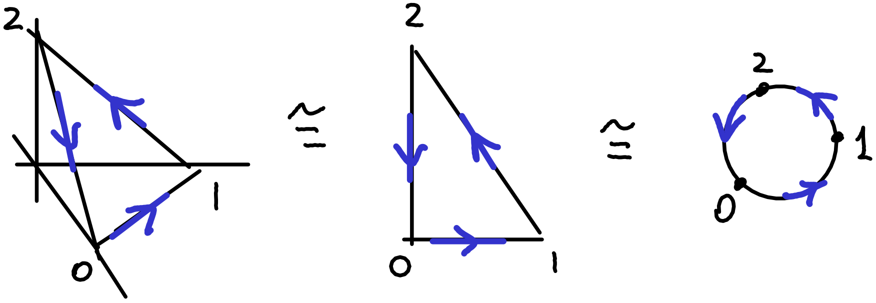
$$\Delta^2 = [e_0,e_1,e_2]$$



normal, e_1-e_0, e_2-e_0 positive $\mathbb{R}^3$-basis

Consistent also with the geometric boundary orientation (outward normal first convention)

$$\partial_{\text{geometric}} \Delta^2 =$$



$S^1 = \partial \mathbb{D}^2 \subseteq \mathbb{R}^2$ standard orientation

Compare $\partial \Delta = +[\hat{e}_0, e_1, e_2] - [e_0, \hat{e}_1, e_2] + [e_0, e_1, \hat{e}_2]$

This $-[e_0,e_2]$ is not equal to singular chain $[e_2,e_0]$ since they are different maps and we take free abelian group generated by maps.
But $[e_0,e_2] + [e_2,e_0]$ is homologous to 0 (Homework).

Locality (or "small simplices theorem")

$\mathcal{U} = \{$ subspaces $U_i \subseteq X\}$ whose interiors cover X:

$$X = \bigcup U_i^\circ$$

Def $C_*^{\mathcal{U}}(X) \subseteq C_*(X)$ subcx generated by n-simplices σ with $\sigma(\Delta^n) \subseteq U_i$ some i

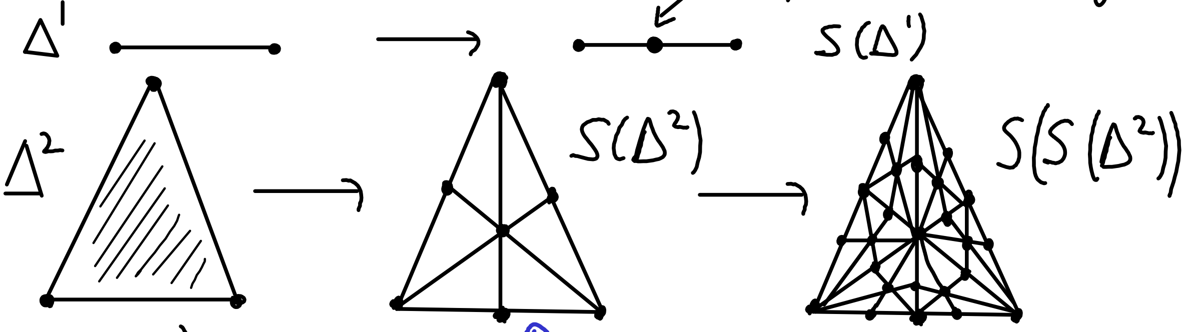
Theorem $\boxed{H_*(C_*^{\mathcal{U}}(X)) \cong H_*(C_*(X)) = H_* X}$

barycentre of $[v_0, .., v_n]$ is $\frac{1}{n+1}(v_0 + \dots + v_n)$

Sketch Pf ① Barycentric subdivision

Non-examinable

barycentre divides edge in 2



$\Rightarrow$ chain map $S: C_*(X) \longrightarrow C_*(X)$

$\sigma \longmapsto$ "$\sigma \circ S$"

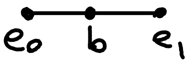
and $S(C_*^{\mathcal{U}}) \subseteq C_*^{\mathcal{U}}$

subdivide the boundary (inductively by dimension) then draw the new faces obtained by convex combinations involving the new vertices and the barycentre

Construction of "$\sigma \circ S$" is inductive:

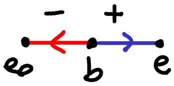
On linear simplices (then for maps σ you restrict $\sigma|_{...}$)

• $S[e_0] = [e_0]$

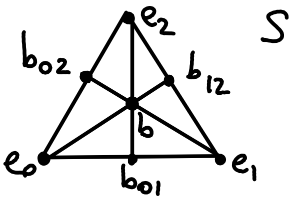


$S[e_0, e_1] = [b, e_1] - [b, e_0]$

geometrically



(= "$[b, S\partial[e_0, e_1]]$")



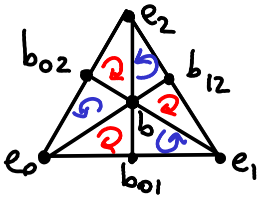
$S[e_0, e_1, e_2] =$ "$[b, S\partial[e_0, e_1, e_2]]$"

$=$ "$[b, S[e_1, e_2]] - [b, S[e_0, e_2]] + [b, S[e_0, e_1]]$"

$= ([b, b_{12}, e_2] - [b, b_{12}, e_1]) - ([b, b_{02}, e_2] - [b, b_{02}, e_0])$

$+ ([b, b_{01}, e_1] - [b_1, b_{01}, e_0])$

geometrically:



so for $\sigma: \Delta^2 \to X$ you take $S(\sigma) = \sigma|_{[b, b_{12}, e_2]} - \sigma|_{[b, b_{12}, e_1]} - \dots$

② S chain hpic to id:

$T: C_n(X) \longrightarrow C_{n+1}(X)$

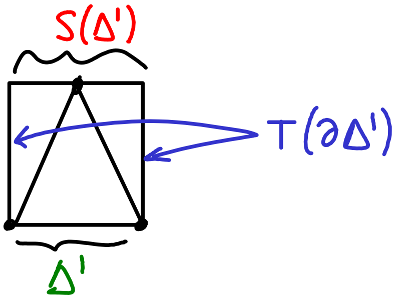
$T(\sigma): \Delta^n \times I \xrightarrow{\text{project}} \Delta^n \xrightarrow{\sigma} X$

exercise: $\partial T + T\partial = S - id$

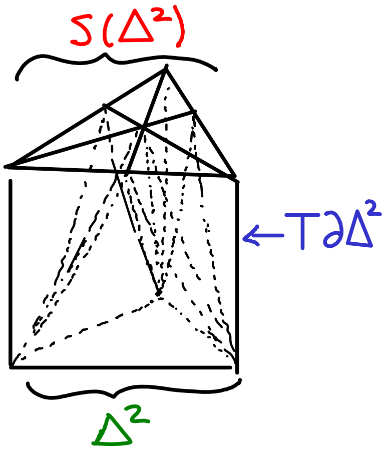
$\Rightarrow \boxed{S_*: H_*(X) \xrightarrow{id} H_*(X)}$

Idea:

$\Delta^1 \times I =$



$\Delta^2 \times I =$



③ $\forall$ n-simplex $\sigma: \Delta^n \to X$, apply $S(\cdot)$ enough times until σ(each n-simplex of subdivision) $\subseteq \mathcal{U}_i$ some i

③ $\Rightarrow$ $\forall$ cycle c, $\exists n$ s.t. $S^n(c) \in C^{\mathcal{U}}_*(X)$ cycle

$\Rightarrow H^{\mathcal{U}}_*(C) \longrightarrow H_*(X)$ surjective

$[S^n(c)] \longmapsto S^n_*[c] = [c]$ by ②

③ $\Rightarrow$ $\forall$ bdry $c = \partial b$, $\exists n$ s.t. $S^n(b) \in C_*^{\mathcal{U}}(X)$

claim: $H^{\mathcal{U}}_*(C) \longrightarrow H_*(X)$ injective

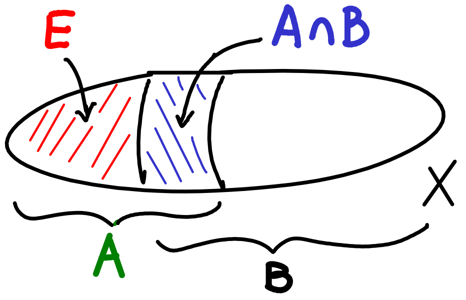
suppose $[c] \longmapsto 0$ then $c = \partial b$ for $b \in C_*(X)$

now $S^n c, S^n b \in C^{\mathcal{U}}_*(X)$ for large n

$\Rightarrow \partial S^n b = S^n \partial b = S^n c$ in $C^{\mathcal{U}}_*(X)$

$\Rightarrow [c] \underset{②}{=} S^n_*[c] = [S^n c] = [\partial S^n b] = 0$ in $H^{\mathcal{U}}_*(X)$ ✓ □

Proof of excision theorem



Let $B = X \setminus E$

use $\mathcal{U} = \{A, B\}$

so $C^{\mathcal{U}}_*(X) = C_*(A) + C_*(B) \subseteq C_*(X)$

$$\Rightarrow \frac{C_*(X \setminus E)}{C_*(A \setminus E)} = \frac{C_*(B)}{C_*(A \cap B)} \cong \frac{C_*(B)}{C_*(A) \cap C_*(B)} \cong C^{\mathcal{U}}_*(X) / C_*(A)$$

2nd isomorphism theorem for groups

$\Rightarrow$ Compare LES's:

$$\begin{tikzcd}
 & & H_*(X \setminus E, A \setminus E) \arrow[d, phantom, "\cong \leftarrow \text{by above isos}"] & & \\
H_*(A) \arrow[r] \arrow[d, equal] & H_*(C^{\mathcal{U}}_* X) \arrow[r] \arrow[d, "\text{locality}"', "\cong"] & H_*(C^{\mathcal{U}}_* X / C_* A) \arrow[r] \arrow[d, "\text{iso by 5-lemma}"] & H_{*-1}(A) \arrow[r] \arrow[d, equal] & H_{*-1}(C^{\mathcal{U}} X) \arrow[d, "\text{locality}"', "\cong"] \\
H_*(A) \arrow[r] & H_*(X) \arrow[r] & H_*(C_* X / C_* A) \arrow[r] \arrow[d, equal] & H_{*-1}(A) \arrow[r] & H_{*-1}(X) \\
 & & H_*(X, A) & &
\end{tikzcd}$$

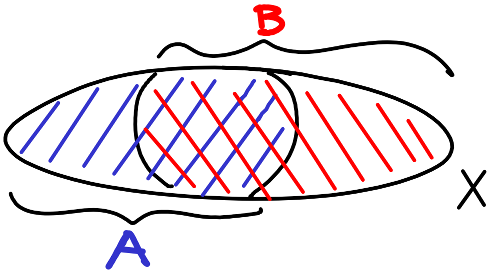
(we are using naturality of LES's induced by SES's)

□

6. MAYER-VIETORIS SEQUENCE ← Key computational tool

$X = A \cup B$ s.t. $X = A^\circ \cup B^\circ$

any subspaces



MV Theorem ∃ LES :

$$\boxed{\cdots \longrightarrow H_*(A\cap B) \xrightarrow{i_*} H_*(A)\oplus H_*(B) \longrightarrow H_*(X) \longrightarrow H_{*-1}(A\cap B) \xrightarrow{i_*[-1]} \cdots}$$

& same holds for $\widetilde{H}_*$ provided $A\cap B \neq \emptyset$.

Pf SES $0 \longrightarrow C_*(A\cap B) \longrightarrow C_*(A)\oplus C_*(B) \longrightarrow C_*^{\mathcal{U}}(X) \longrightarrow 0$ $\quad \mathcal{U}=\{A,B\}$

$$\sigma \longmapsto (\sigma, -\sigma)$$

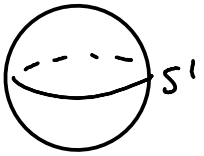
$$(\alpha, \beta) \longmapsto \alpha+\beta$$

⇒ induces the LES (using locality $H_*^{\mathcal{U}}X \cong H_*X$). □

Exercise connecting map is $\delta : H_*(X) \longrightarrow H_{*-1}(A\cap B)$

$$[\alpha+\beta] \longmapsto [\partial\alpha] = -[\partial\beta]$$

Example S^2



$A \simeq pt$

$B \simeq pt$

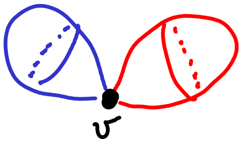
$A\cap B \simeq S^1$

$$\cdots \to \underset{\overset{\shortparallel}{0}}{H_2(pt)\oplus H_2(pt)} \to \underset{\overset{\uparrow}{\text{hence } \mathbb{Z}}}{H_2S^2} \longrightarrow \underset{\overset{\cong}{\mathbb{Z}}}{H_1(S^1)} \longrightarrow \underset{\overset{\shortparallel}{0}}{H_1(pt)\oplus H_1(pt)} \to \cdots$$

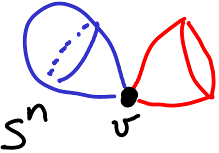
Exercise Compute $H_*S^n = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$ using MV

Example **wedge sum** of X, Y with basepoints $x \in X$, $y \in Y$ $\qquad X \vee Y = \dfrac{X \times Y}{x\sim y}$

$X = S^n \vee S^n =$

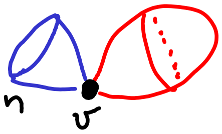


$A =$



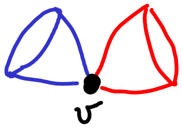
$\simeq S^n$

$B =$



$\simeq S^n$

$A\cap B =$



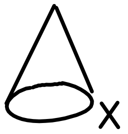
$\simeq pt$

$$0 \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \xrightarrow[\cong]{} H_n(X) \longrightarrow 0 \to \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \underset{\cong \mathbb{Z}}{H_0(X)} \longrightarrow 0$$

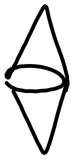
$$1 \longmapsto (1,-1)$$

Similarly $\boxed{H_*(X\vee Y) \cong H_*(X)\oplus H_*(Y) \text{ for } * \neq 0}$ if ∃ contractible nbhds of $x \in X$, of $y \in Y$.

Cones and suspensions



$$CX \underset{\simeq pt}{=} (X \times [0,1]) \big/ (x,s)\sim(y,t) \text{ iff equal or } s=t=1$$

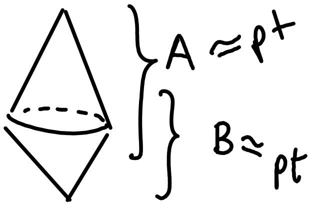


$$\Sigma X = (X \times [0,1]) \big/ (x,s)\sim(y,t) \text{ iff equal or } s=t=0 \text{ or } s=t=1$$

Example $CS^n \cong D^{n+1}$, $\Sigma S^n \cong S^{n+1}$.

Lemma $\boxed{\tilde H_*(\Sigma X) \cong \tilde H_{*-1}(X)}$

Pf



$A \cap B \simeq X$ now apply MV. □

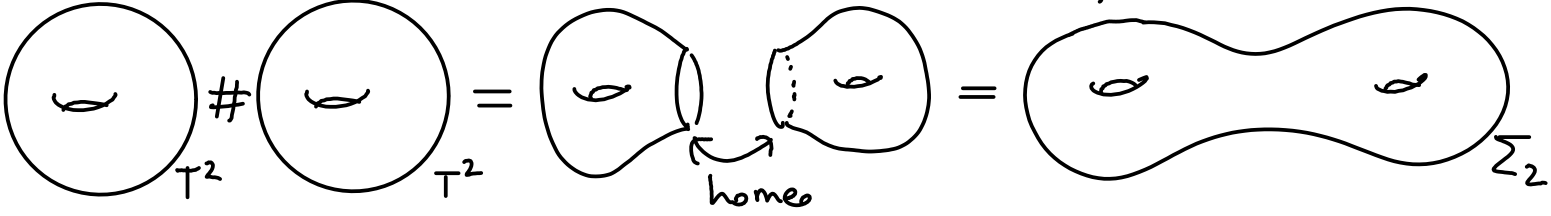
Rmk $\emptyset \neq A \subseteq X \Rightarrow \tilde H_*(X \cup_A CA) \overset{LES}{\cong} H_*(X \cup_A CA, CA) \overset{exc.}{\cong} H_*(X,A)$

↑ identify $a \in A \subseteq X$ with $(a,0) \in CA$

Connected sum

M, N connected n-manifolds $\Rightarrow$ $M \# N = \underline{(M \setminus \text{open } n\text{-ball}) \cup (N \setminus \text{open } n\text{-ball})}$

identify ∂ balls via a homeo



Fact compact connected orientable surfaces are homeo to S^2 or $\underbrace{T^2 \# \dots \# T^2}_{g = \# \text{ copies}}$

and " " non-orientable ones: $\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2$.

genus $g=0$

called Σ_g

Exercise (Homework) For M, N compact connected

By MV, $\boxed{H_*(M\#N) \cong H_*(M) \oplus H_*(N) \text{ for } 1 \leq * \leq n-2}$

If M or N orientable: $* = n-1$ also works

If both non-orientable: $* = n-1$ one of $\mathbb{Z}/2$ summands becomes $\mathbb{Z}$

$H_0(M\#N) \cong \mathbb{Z}$ since connected

fact: $H_n(M\#N)$ is $\mathbb{Z}$ or 0

↑ if M, N both orientable ↖ else

(see later in course)

Cor 1) $\chi(M\#N) = \chi(M) + \chi(N) - \underbrace{\begin{cases} 2 & \text{for } n \text{ even} \\ 0 & \text{" " odd} \end{cases}}_{= \chi(S^n)}$

2) $H_*(\Sigma_g) \leftarrow$ genus g $\cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{2g} & * = 1 \\ \mathbb{Z} & * = 2 \end{cases}$

7. DEGREE OF MAPS OF SPHERES

$f: S^n \longrightarrow S^n \implies$

$$\begin{tikzcd} f_n : H_nS^n \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_nS^n \arrow[d, phantom, sloped, "\cong"] \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \\ 1 \arrow[r, mapsto] & \underline{\deg(f)} \in \mathbb{Z} \end{tikzcd}$$

$\implies f_*: \widetilde{H}_* S^n \to \widetilde{H}_* S^n$ is $\deg(f)\cdot \mathrm{id}$

Properties

1) $\deg(\mathrm{id}) = 1$
2) $\deg(f \circ g) = \deg f \cdot \deg g$
3) $f \simeq g \implies \deg f = \deg g$
4) $f \simeq \mathrm{const} \implies \deg f = 0$
5) f homeomorphism $\implies \deg f = \pm 1$ ← sign depends on whether f is orientation-preserving or reversing

Pf

$\mathrm{id}_* = \mathrm{id}$, $(f\circ g)_* = f_* \circ g_*$, $f \simeq g \Rightarrow f_* = g_*$, $\mathrm{const}_* = 0$, f homeo $\Rightarrow f_n$ iso. ☐

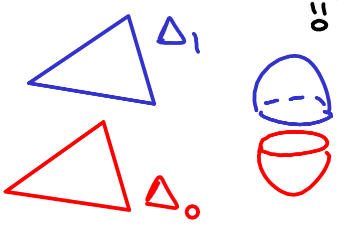
since $S^n \to pt \to S^n$ factors so $H^nS^n \to \underset{0}{\underset{\|}{H^n(pt)}} \to H^nS^n$

Examples

1)
$$S^n = \frac{\Delta^n \times 1 \cup \Delta^n \times 0}{(b,1) \sim (b,0) \text{ if } b \in \partial\Delta}$$

call this Δ_1 ($\Delta^n \times 1$)

call this Δ_0 ($\Delta^n \times 0$)



recall $H_nS^n = \mathbb{Z}\cdot(\Delta_1 - \Delta_0)$

reflection: $r: S^n \longrightarrow S^n$, $r(x,t) = (x, 1-t)$

so $\Delta_0 \leftrightarrow \Delta_1$ swapped by r, so $r_*(\Delta_1 - \Delta_0) = -(\Delta_1 - \Delta_0)$

$\implies \deg(r) = -1$

2) antipodal map $-\mathrm{id}: S^n \longrightarrow S^n$ viewing $S^n \subseteq \mathbb{R}^{n+1}$

$$\implies \boxed{\deg(-\mathrm{id}) = (-1)^{n+1}}$$

Pf
$$-\mathrm{id} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} \circ \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \circ \cdots \circ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$$
composition of $n+1$ reflections each homotopic to r. ☐

3) $A \in O(n) \Rightarrow A: S^{n-1} \to S^{n-1} \Rightarrow \deg A = \det A \in \{\pm 1\}$

Pf fact $SO(n)$ is path-connected so $A \in SO(n)$ is $\simeq \mathrm{id}$ so $\deg A = \det A = +1$

The other path-component of $O(n)$ is $r \circ O(n)$ where r is any reflection. ☐

4) $\boxed{f \text{ not surjective} \implies \deg f = 0}$

Pf If $y \notin \mathrm{Im} f \Rightarrow$
$$\begin{tikzcd} H_n(S^n) \arrow[rr, "f_*"] \arrow[dr, "f_*"'] & & H_n(S^n) \\ & H_n(S^n \setminus y) \cong H_n(\mathbb{R}^n) = 0 \arrow[ur] & \end{tikzcd}$$
☐

Application to vector fields on S^n

$v: S^n \longrightarrow \mathbb{R}^{n+1}$ tangent vector field on S^n

so $v(x) \perp x$

Cor **Hairy ball theorem** $\boxed{\exists \text{ nowhere zero v.f. on } S^n \iff n \text{ odd}}$

(case $n=2$: "you cannot comb a ball of hair without creating a tuft")

Pf Suppose $v(x) \neq 0 \quad \forall x$

$\Rightarrow$ hpy $F: S^n \times [0,1] \longrightarrow S^n$

$$F(x,t) = \cos(\pi t)\, x + \sin(\pi t) \frac{v(x)}{\|v(x)\|}$$

$\Rightarrow F_0 = id, \quad F_1 = -id$

$\Rightarrow 1 = \deg F_0 = \deg F_1 = (-1)^{n+1}$

$\Rightarrow n$ odd

For n odd $\exists\ v(x) = (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1})$ $\square$

Cultural Remark Adams in 1962 proved using alg. topology:

(max # pointwise linearly independent vector fields on S^n) $= 2^b + 8a - 1$

where $n+1 = 2^{4a+b} \cdot$ (odd number), $0 \leq b \leq 3$, $a, b \in \mathbb{N}$, $n \geq 1$.

get 0 if n even $\Rightarrow$ Cor ✓

Local degree

$$f: \begin{tikzcd} S^n \arrow[r] & S^n \\ x \arrow[u, phantom, sloped, "\in"] \arrow[r] & y = f(x) \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

⊛ Suppose points $\neq x$ near x do not map to y:

$\exists$ nbhds $x \in \underset{\substack{\cong \\ D^n}}{U}$, $y \in \underset{\substack{\cong \\ D^n}}{V}$ s.t. $\boxed{(U, U\setminus x) \xrightarrow{f} (V, V\setminus y)}$

call this $f|_x$ local map at x

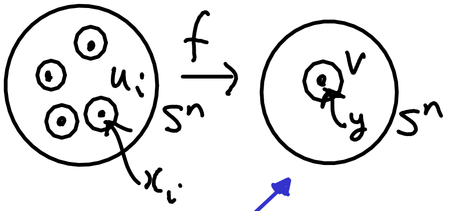
$$\Rightarrow \begin{tikzcd} (f|_x)_* : H_n(U, U\setminus x) \arrow[r, "f_*"] \arrow[d, "\cong", "\text{excise } S^n\setminus U"'] & H_n(V, V\setminus y) \arrow[ddd, equal] \\ H_n(S^n, \underbrace{S^n\setminus x}_{\cong pt}) \arrow[d, "\cong"] & \\ \tilde{H}_n S^n \arrow[d, "\cong"] & \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \\ 1 \arrow[r, mapsto] & \deg_x f \end{tikzcd}$$

Lemma $f: S^n \to S^n$, $f^{-1}(y) = \{x_1, \dots, x_k\}$

$$\Rightarrow \boxed{\deg f = \sum \deg_{x_i} f}$$

Pf

$$\begin{tikzcd}
\tilde{H}_n S^n \arrow[r, "f_*"] \arrow[d, "\text{quotient}"'] & \tilde{H}_n(S^n) \arrow[d, "\text{quotient}"] \\
H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) \arrow[r, "f_*"] \arrow[d, "\text{exc. } S^n \setminus \cup U_i"', "\cong"] & H_n(S^n, S^n \setminus y) \arrow[d, "\text{exc. } S^n \setminus V"] \\
\bigoplus H_n(U_i, U_i \setminus x_i) \arrow[r, "\oplus (f|_{x_i})_*"] & H_n(V, V \setminus y)
\end{tikzcd}$$



Rmk
can use same V
for all i by taking
$\tilde{V} = \cap V_i$
$\tilde{U}_i = f^{-1}(V) \cap U_i$

is:

$$\begin{tikzcd}
1 \arrow[d] & \in & \mathbb{Z} \arrow[r, "\deg f"] \arrow[d] & \mathbb{Z} \arrow[d] \\
(1, \dots, 1) & \in & \bigoplus_i \mathbb{Z} \arrow[r, "\oplus \deg_{x_i} f"'] & \mathbb{Z}
\end{tikzcd} \quad \square$$

map to each summand is exc. of $S^n \setminus U_i$ so iso.

(the 2 squares commute:
1st: quotient is natural
2nd: excision is natural)

Example $p: \mathbb{C} \to \mathbb{C}$ polynomial $a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, $a_i \in \mathbb{C}$, $a_n \neq 0$

$$\Rightarrow f: S^2 = \mathbb{C}P^1 \to \mathbb{C}P^1 = S^2 \qquad \text{(where view } \mathbb{C}P^1 = \mathbb{C} \cup \infty \cong S^2\text{)}$$

$$z \mapsto p(z)$$
$$\infty \mapsto \infty$$

stereographic projection

$$\Rightarrow \text{hpy } F(z,t) = a_n z^n + t(a_{n-1} z^{n-1} + \dots + a_0)$$

$F_0 = a_n z^n$ and $F_1 = f$

hpy is continuous at ∞ since $a_n z^n$ dominates other terms: $F^{-1}(\mathbb{C}P^1 \setminus K) = \mathbb{C}P^1 \setminus$(some compact set) $\forall$ compact K.

this would fail if you tried to homotope $t(a_n z^n) + a_{n-1} z^{n-1} + \dots$

$$\Rightarrow \deg f = \deg(a_n z^n) \overset{\text{Lemma}}{=\!=} \sum_{k=0}^{n-1} \underbrace{\deg_{w_k} \underbrace{a_n z^n}}_{=1} \quad \leftarrow w_k = e^{\frac{2\pi i k}{n}}$$
$$= n$$
$$= \text{degree of the poly } p.$$

orient^n preserving homeo near w_k

holomorphic maps are always orientation preserving

Cor (Fundamental Thm of Algebra) $n \geq 1 \Rightarrow p$ has a root

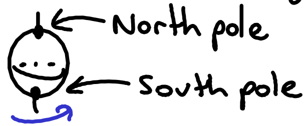
Pf $p^{-1}(0) = \emptyset \Rightarrow f^{-1}(0) = \emptyset \Rightarrow \deg f = 0$ ↯ $\square$

Cultural Rmk For smooth $f: S^n \to S^n$

$\deg f =$ (the number of preimages of a generic point.)

(i.e. almost any point works)

Example $S^2 \to S^2$



rotate by $\frac{2\pi}{d}$ about vertical axis

$\Rightarrow \deg = d = \#$ preimages of a point

except if pick North/South pole

8. CELLULAR HOMOLOGY

Def **CW complex** X is sequence $\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots$

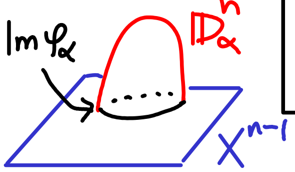
s.t. X^0 is any set

n-discs labelled by some index set I_n

n-skeleton $\boxed{X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I_n} \mathbb{D}^n_\alpha \Big/ x \sim \varphi_\alpha(x)}$

$\varphi_\alpha : \partial\mathbb{D}^n \longrightarrow X^{n-1}$ **attaching map**

(any continuous map, often not injective)



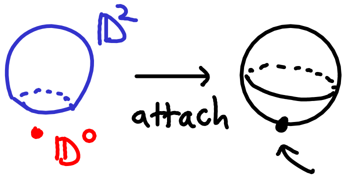
$\Longrightarrow X = \bigcup_{n \geq 0} X^n$ top. space with **weak topology**:

$U \subseteq X$ open $\iff U \cap X^n \subseteq X^n$ open $\forall n$.

$(\iff U \cap \mathbb{D}^n_\alpha \subseteq \mathbb{D}^n_\alpha$ open $\forall n, \alpha)$

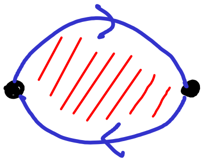
Call X **n-dimensional** if $X = X^n$ and this is the least such n.

Example $S^n = (\mathbb{D}^0 \sqcup \mathbb{D}^n) / (\mathbb{D}^0 \sim \partial\mathbb{D}^n)$

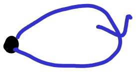


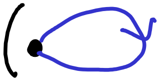
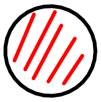
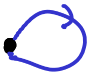
$= S^2$

boundary $S^1 = \partial\mathbb{D}^2$ identified with •

Example $X = \mathbb{RP}^2 =$ 

$X^0 = \bullet = \mathbb{D}^0$

$X^1 =$  $= S^1 = (\mathbb{D}^0 \sqcup \mathbb{D}^1) / (\bullet \sim \varphi_1(x))$, $\partial\mathbb{D}^1 = S^0 = \{0,1\} \xrightarrow[\varphi_1]{} X^0 = \bullet$

$X^2 = ($  $\sqcup$  $) / ($wrap ∂ of  twice around  $)$

$= (X^1 \sqcup \mathbb{D}^2) \Big/ \Big(\overset{\partial\mathbb{D}^2 = S^1}{\overset{\rotatebox{90}{$\in$}}{z}} \sim z^2 \in X^1 = S^1 \Big)$ $\quad \partial\mathbb{D}^2 = S^1 \xrightarrow{\varphi_2} S^1$, $\varphi_2(z) = z^2$

Fact If we homotope φ_α, we get a homotopy equivalent space

Example If use another degree 2 map φ_2 above, get $X \simeq \mathbb{RP}^2$.

X is partitioned as a set by interiors of **n-cells** $\boxed{e^n_\alpha = \mathrm{Image}(\mathbb{D}^n_\alpha \to X)}$

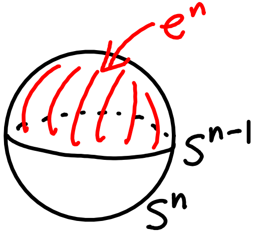
$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I_\alpha} \mathring{e}^n_\alpha$$

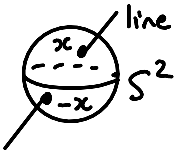
$$= \Big(\bigsqcup_{\alpha \in I_0} e^0_\alpha\Big) \sqcup \Big(\bigsqcup_{\alpha \in I_1} \mathring{e}^1_\alpha\Big) \sqcup \Big(\bigsqcup_{\alpha \in I_2} \mathring{e}^2_\alpha\Big) \sqcup \cdots$$

Rmk interior $\mathbb{D}^0 = \mathbb{D}^0$ so $\mathring{e}^0_\alpha = e^0_\alpha$

Examples **real projective space** $\mathbb{RP}^n = S^n/(\mathbb{Z}/2\text{-action by } \pm id)$

so $x \sim -x$





$X^k = \mathbb{R}P^k$ inductively

$X^n = X^{n-1} \cup e^n$ with $\varphi : S^{n-1} \to X^{n-1} = \mathbb{R}P^{n-1} = S^{n-1}/\pm id$

$x \longmapsto [x] = [-x]$

complex projective space

$\mathbb{C}P^n = (S^{2n+1} \subseteq \mathbb{C}^n)/(S^1\text{-action by } \lambda \cdot Id)$

$x \sim \lambda x$ for $\lambda \in S^1 \subseteq \mathbb{C}^*$

$X^0 = X^1 = pt = \mathbb{C}P^0$

$X^2 = X^3 = X^0 \cup e^2 = \mathbb{C}P^1$, $\varphi : S^1 \to pt$ ← $\mathbb{C}P^1 \cong$

$\cong S^2$

$X^4 = X^5 = X^2 \cup e^4 = \mathbb{C}P^2$, $\varphi : S^3 \to \mathbb{C}P^1 = S^3/(S^1\text{-action})$

$x \longmapsto [x] = [\lambda x], \forall \lambda \in S^1$

$X^{2n} = X^{2n-2} \cup e^{2n} = \mathbb{C}P^n$, $\varphi : S^{2n-1} \to X^{2n-2} = \mathbb{C}P^{n-1}$

$x \longmapsto [x]$

In coordinates: $\mathbb{C}P^n = \{[z_0 : \dots : z_n] : \text{not all } z_j \in \mathbb{C} \text{ are } 0\}$ and $[z] \sim [\lambda z], \forall \lambda \in \mathbb{C}^*$

Can rescale so that $\sum |z_j|^2 = 1$ so $z \in S^{2n-1}$ and left with rescaling by $\lambda \in S^1 \subseteq \mathbb{C}^*$.

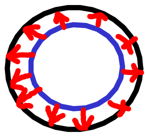
$\mathbb{C}P^{n-1} \cong X^{n-2} = \{[z_0 : \dots : z_{n-1} : 0]\} \subseteq \mathbb{C}P^n = X^n$ and

notice this $=0$ if $w \in S^{2n-1} \cong \partial \mathbb{D}^{2n}$

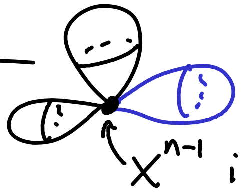
$e^{2n} : \mathbb{D}^{2n} = \{(w_0, \dots, w_{n-1}) : \sum |w_j|^2 \leq 1\} \longrightarrow X^n$ via $[w_0 : \dots : w_{n-1} : \sqrt{1 - \sum |w_j|^2}]$

Observe: For X CW complex, for $n \geq 1$: ← (For $n=0$: $(X^0, X^{-1}) = (X^0, \emptyset)$, $X^0/X^{-1} = X^0$)

- (X^n, X^{n-1}) is a good pair ← (since $\exists$ nbhd of $\partial \mathbb{D}^n$ that deformation retracts to $\partial \mathbb{D}^n$)



- $\boxed{X^n/X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n}$ ←



$S^n_\alpha = \mathbb{D}^n_\alpha / \partial \mathbb{D}^n_\alpha$

X^{n-1} identified to a point

Def **Cellular complex** for X a CW cx,

$$\boxed{\begin{aligned} C^{CW}_n(X) = H_n(X^n, X^{n-1}) &\cong H_n(X^n/X^{n-1}) \\ &= \text{free abelian gp gen. by the } n\text{-cells } e^n_\alpha \end{aligned}}$$

since $\Delta^n \cong \mathbb{D}^n \longrightarrow (e^n \subseteq X^n) \longrightarrow \mathbb{D}^n_\alpha/\partial \mathbb{D}^n_\alpha = S^n_\alpha$ generate

as usual we use the standard orientations of $\Delta^n, \mathbb{D}^n, S^n$.

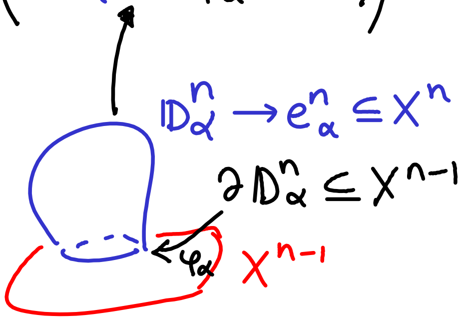
Will build **cellular differential** d, prove $d \circ d = 0$,

$\Rightarrow$ get $\boxed{H^{CW}_*(X) = H_*(C^{CW}_*(X), d)}$

$$\boxed{d\, e^n_\alpha = \sum_{\beta\in I_{n-1}} d^n_{\alpha\beta}\cdot e^{n-1}_\beta}$$

now describe the coefficients $d^n_{\alpha\beta}\in\mathbb{Z}$ and why that is a finite sum.

$$\begin{tikzcd}
d: C_n^{CW} \arrow[rr] \arrow[d, equal] & & C_{n-1}^{CW} \arrow[d, equal] \\
H_n(X^n, X^{n-1}) \arrow[r, "\delta^n_*", "\text{connecting map of LES}"'] & H_{n-1}(X^{n-1}) \arrow[r, "q^{n-1}_*", "\text{quotient map}"'] & H_{n-1}(X^{n-1}, X^{n-2}) \\
e^n_\alpha \arrow[r, mapsto] & \left(\partial \mathbb{D}^n_\alpha \to \varphi_\alpha(S^{m-1})\right) \arrow[r, mapsto] & \sum d_{\alpha\beta}\, e^{n-1}_\beta
\end{tikzcd}$$



Recall LES

$$\begin{tikzcd}
H_*(X^{n-1}) \arrow[rr] & & H_*(X^n) \arrow[dl, "q^n_*"] \\
& H_*(X^n, X^{n-1}) \arrow[ul, "\delta^n_*"', "{[-1]}"] &
\end{tikzcd}$$

here it is important that we chose identifications $\Delta^n \cong \mathbb{D}^n$, $S^n \cong \mathbb{D}^n/\partial\mathbb{D}^n$ compatibly with orientations.

Therefore:

Quotient by $\bigvee_{I_{n-1}\setminus\beta} S^{n-1}$

$$\boxed{d^n_{\alpha\beta} = \deg\Big(\underset{\partial\mathbb{D}^n_\alpha}{\overset{\|}{S^{n-1}}} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \cong \bigvee_{I_{n-1}} S^{n-1} \longrightarrow \underset{\mathbb{D}^{n-1}_\beta/\partial\mathbb{D}^{n-1}_\beta}{\overset{\|}{S^{n-1}}}\Big)}$$

Rmk Only finitely many $d^n_{\alpha\beta}\neq 0$ (for fixed α) because φ_α, q are continuous and S^{n-1} compact, so get a compact image in $\bigvee_\beta S^{n-1}$, therefore cannot surject onto ∞ many S^{n-1}_β.

recall if don't surject then deg=0

Lemma $d\circ d = 0$

Pf $d_n = q^{n-1}_{n-1}\circ\delta^n_n$

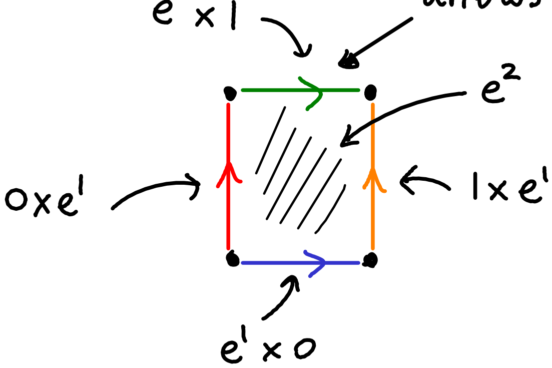
$$d_{n-1}\circ d_n = q^{n-2}_{n-2}\circ \overbrace{\delta^{n-1}_{n-1}\circ q^{n-1}_{n-1}}^{=0 \text{ by LES}}\circ\,\delta^n_n \qquad \square$$

Cor $\boxed{\operatorname{rank} H^{CW}_n(X) \leq \#n\text{-cells}}$

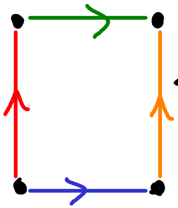
Pf $\#n\text{-cells} = \operatorname{rank} C^{CW}_n(X) \geq \operatorname{rank}\operatorname{Ker} d^{CW}_n \geq \operatorname{rank} H^{CW}_n(X)$ $\square$

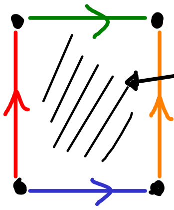
Example $I \times I$ $\quad I = [0,1]$ $\quad \mathbb{D}^1 = [-1,1]$

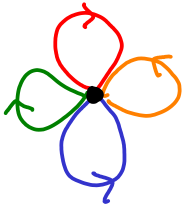
arrows here tell us how we map $[-1,1] \to$ edge (so orientation)

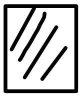


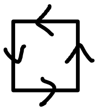
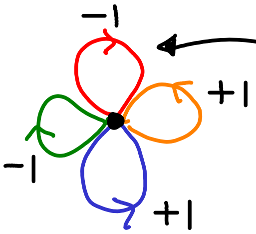
$X^0 = \vdots\vdots = 4$ 0-cells

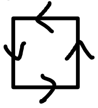
$X^1 =$  ← 4 1-cells $e^1 \times 0$, $1 \times e^1$, $e^1 \times 1$, $0 \times e^1$

$X^2 =$  ← 2-cell e^2

$X^1/X^0 =$  orientations of cells tell us how to orient the circles

$e^2 : \mathbb{D}^2 \cong$  $\longrightarrow X^1$

$\partial e^2 : S^1 \cong$  $\longrightarrow X^1/X^0 =$ 

← degree -1 because top edge of  maps to  by an orientation-reversing homeomorphism.

$$\Rightarrow \partial e^2 = +e^1 \times 0 + 1 \times e^1 - e^1 \times 1 - 0 \times e^1$$

$$\left(= (\partial e^1) \times e^1 - e^1 \times (\partial e^1) \right.$$ ← we come back to this later)

Example $\mathbb{R}P^n$ recall: 1 cell in each dim, $\varphi : S^k \longrightarrow X^k = \mathbb{R}P^k$

$x \longmapsto [\pm x]$

$S^{k-1} =$ 

$$\Delta_1 \xrightarrow{\varphi} X^{k-1}/X^{k-2} = \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \cong S^{k-1} \qquad \deg = +1$$

$$\Delta_2 \longrightarrow -\mathrm{id}(\Delta_1) \qquad \deg = (-1)^k$$

$$\Rightarrow d^k_{\alpha\beta} = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

$$C^{CW}_*(\mathbb{R}P^n) \quad 0 \to \underset{k=n}{\mathbb{Z}} \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow{2} \underset{k=1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \longrightarrow \underset{-1}{0}$$

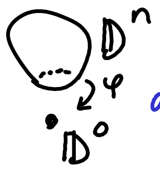
2 if n even, 0 " odd

0, 2 resp.

$$H^{CW}_*(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0, \text{ also } * = n \text{ if odd} \\ \mathbb{Z}/2 & * = 1,3,5,\ldots < n \\ 0 & \text{else} \end{cases}$$

Example S^n: $C_*^{CW}(S^n)$: $n \geq 2$: $0 \longrightarrow \mathbb{Z}\cdot D^n \xrightarrow{0} 0 \to \cdots \to 0 \to \mathbb{Z}\cdot D^0 \to 0$

$n=1$: $0 \longrightarrow \mathbb{Z}\cdot D^1 \xrightarrow{0} \mathbb{Z} D^0 \longrightarrow 0$



$\deg \varphi = 0$ $\Longrightarrow H_*^{CW}(S^n) \cong \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{else} \end{cases}$

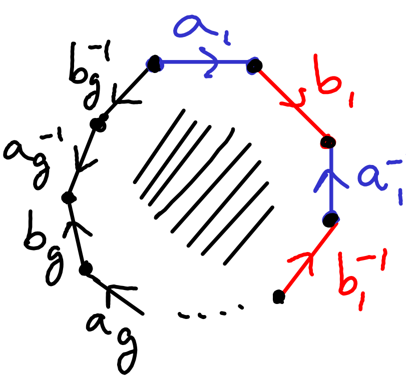
$H_1(S^1, pt) \xrightarrow{\delta} H_0(pt) \xrightarrow{q} H_0(pt, \emptyset)$

$(\Delta^1 \cong [0,1] \to S^1) \longmapsto \partial\sigma$
σ = quotient $0 \sim 1$
$= pt - pt$
$= 0$

if you work with degrees, need to remember orientations:
$\partial D^1 \cong \partial[0,1] = [1] - [0] \longrightarrow$ point
so degree $= +1 - 1 = 0$

Example $\Sigma_g =$

genus g surface



boundary identifications
$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

← Notice all vertices are identified, call vertex v

$\partial a_i = v - v = 0$
$\partial b_i = v - v = 0$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial"] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{Z}^{2g} \arrow[r, "\partial = 0"] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{Z} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\ & \mathbb{Z}\cdot D & \mathbb{Z}\langle a_1, b_1, \ldots, a_g, b_g \rangle & \mathbb{Z}\cdot v & \end{tikzcd}$$

$$D \longmapsto -a_1 - b_1 + a_1 + b_1 + \cdots - a_g - b_g + a_g + b_g = 0$$

signs: compare edge orientation with anticlockwise orientation of ∂D

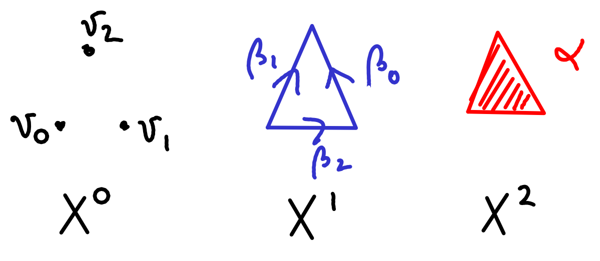
$$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0, 2 \\ \mathbb{Z}^{2g} & * = 1 \\ 0 & \text{else} \end{cases}$$

Lemma X Δ-cx structure $\Longrightarrow$ induces CW-cx structure on X and $(C_*^{CW}(X), d^{CW}) \cong (C_*^{\Delta}(X), d^{\Delta})$

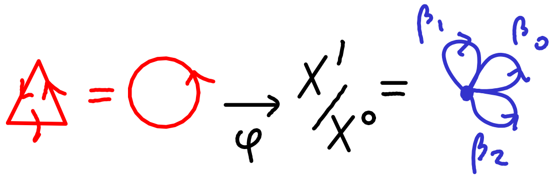
$$\Longrightarrow \boxed{H_*^{CW}(X) \cong H_*^{\Delta}(X)}$$

Pf $X^n = \bigcup$ n-simplices of X and degrees are ± 1 depending on orient^{n} so can identify d^{CW} and d^{Δ}. ☐
$\overset{\cong}{D^n}$

Example X = triangle $= \Delta^2$



$\Rightarrow d^{\Delta}\alpha = \beta_0 - \beta_1 + \beta_2$



$d_{\alpha\beta_2} = d_{\alpha\beta_0} = +1$, $d_{\alpha\beta_1} = -1$

$\Longrightarrow d^{CW}\alpha = d^{\Delta}\alpha$ ✓ ☐

Theorem X CW cx (or Δ-cx) $\Longrightarrow$ $\boxed{H_*^{CW}(X) \cong H_*(X)}$

$\Longrightarrow$ H_*^{Δ}, H_*^{CW} independent of choice of CW-cx/Δ-cx structure.

Pf ① $H_*(X^n, X^{n-1}) \cong \widetilde{H}_*(X^n/X^{n-1}) \cong \widetilde{H}_*(\bigvee_\alpha S^n) \cong \underbrace{\bigoplus_\alpha \widetilde{H}_* S^n}_{\text{lives in degree } n}$

$= 0 \iff * \neq n$

LES for $(X^n, X^{n-1}) \Rightarrow H_*(X^{n-1}) \to H_*(X^n)$ iso for $* < n-1$, $* > n$

② for $*<n$: $H_*(X^n) \cong H_*(X^{n+1}) \cong H_*(X^{n+2}) \cong \dots \cong H_*(X)$

by ①

by compactness each sing. chain lands in X^N, some N

③ for $* > n$: $H_*(X^n) \cong H_*(X^{n-1}) \cong H_*(X^{n-2}) \cong \dots \cong H_*(X^{-1}) = 0$

④ LES: $\dots \to \underset{\substack{\| \\ 0}}{H_n(X^{n-1})} \longrightarrow H_n(X^n) \xrightarrow[q_n^n]{} H_n(X^n, X^{n-1}) \to \dots$ by ③

$\Rightarrow$ q_n^n injective $\forall n$

⑤ LES: $\dots \xrightarrow{q_{n+1}^{n+1}} H_{n+1}(X^{n+1}, X^n) \xrightarrow{\delta_{n+1}^{n+1}} H_n(X^n) \longrightarrow H_n(X^{n+1}) \to 0$ ①

UPSHOT

$$H_n(X) \overset{②}{\cong} H_n(X^{n+1})$$

$$\overset{⑤}{\cong} H_n(X^n) / \operatorname{im} \delta_{n+1}^{n+1}$$

1st iso thm

$$\overset{④}{\cong} \underbrace{(q_n^n H_n(X^n))}_{\substack{\| \\ \operatorname{im} q_n^n \\ \| \\ \operatorname{Ker} \delta_n^n}} / \operatorname{im} \underbrace{q_n^n \circ \delta_{n+1}^{n+1}}_{\substack{\| \\ d_{n+1}^{CW}}} \cong H_n^{CW}(X)$$

exactness LES

$$\operatorname{Ker} \delta_n^n \overset{④}{=} \operatorname{Ker} \underbrace{q_{n-1}^{n-1} \circ \delta_n^n}_{\substack{\| \\ d_n^{CW}}}$$

□

Rmk by ① $\boxed{H_k \text{ } \underline{\text{not}} \text{ affected if attach } (k+2)\text{-cells or higher}}$

by ② $\boxed{\text{Inclusion } X^n \to X \text{ induces iso } H_*(X^n) \to H_*(X) \text{ for } *<n}$

Cor X n-dimensional cell cx $\Longrightarrow$ $H_*(X) = 0$ for $* > n$

Axioms for homology: Eilenberg-Steenrod axioms

It was no accident that H_*^{Δ}, H_*^{CW}, H_* all agreed.

Def A **generalised homology theory (GHT)**

is a **functor** $F : \text{TopPairs} = \left(\begin{array}{l}\text{Category of pairs}\\ \text{of spaces, and}\\ \text{maps of pairs}\end{array}\right) \longrightarrow \text{Graded Abelian Gps}$

with a **natural transformation** $\delta : F_*(X,A) \longrightarrow \underbrace{F_{*-1}(X,\emptyset)}_{\leftarrow \text{abbreviated: } F_{*-1}(X)}$ satisfying:

1) **homotopy invariance**: $f \simeq g \Rightarrow F(f) = F(g)$

2) **exactness**: $\exists$ LES

$$\begin{tikzcd} \cdots \arrow[r] & F_*(A) \arrow[r, "F(\text{incl}:A\to X)"'] & F_*(X) \arrow[r, "F(\text{incl}:(X,\emptyset)\to(X,A))"'] & F_*(X,A) \arrow[r, "\delta"] & F_{*-1}(A) \arrow[r] & \cdots \end{tikzcd}$$

3) **additivity**: $(X,A) = \bigsqcup (X_i, A_i)$, $\text{incl}_i : (X_i,A_i) \longrightarrow (X,A)$

then $\boxed{\sum F(\text{incl}) : \bigoplus F(X_i,A_i) \xrightarrow{\cong} F(X,A)}$

4) **excision**: $\boxed{\overline{E} \subseteq A^\circ \subseteq X \quad \Longrightarrow \quad F(X\setminus E, A \setminus E) \xrightarrow[\uparrow F(\text{incl})]{\cong} F(X,A)}$

Remark (4) $\Longleftrightarrow$ $X = A^\circ \cup B^\circ$, $\text{incl} : (B, A\cap B) \longrightarrow (X,A)$

then $\boxed{F(\text{incl}) : F(B, A\cap B) \xrightarrow{\cong} F(X,A)}$

Pf $B = X \setminus E$, $E = X \setminus B$ noticing that $(X\setminus E)^\circ \cup A^\circ = X$

$E = A \setminus B$ noticing that $\overline{E} \subseteq \overline{A} \setminus B^\circ \subseteq A^\circ \setminus B^\circ \subseteq A^\circ$. $\square$ $\leftarrow$ $X = A^\circ \cup B^\circ$ So $\partial B \subseteq A^\circ$

Rmk In (3), the topology on the disjoint union $\bigsqcup (X_i, A_i)$ is defined by: $U \subseteq \bigsqcup (X_i,A_i)$ open $\Longleftrightarrow$ $U \cap X_i \subseteq X_i$ open $\forall i$

FACT **Theorem**

a) $(F,\delta_F),(G,\delta_G)$ GHTs, $\alpha : F \to G$ a natural transformation commuting with δ_F, δ_G such that $\alpha_{\text{point}} : F(\text{point}) \to G(\text{point})$ is an iso, then α is an iso.

b) If (F,δ_F) GHT satisfies (5) **dimension**: $\boxed{F_*(\text{point}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}}$

Then $\exists$ natural iso $F \cong H_*$ (such an F is called a **homology theory**)

Rmk In (b) if require $F_0(\text{point}) = G$ an abelian group (instead of $\mathbb{Z}$)

$\Longrightarrow F(X,A) \cong H_*(X,A;G)$ = (homology with coefficients in G) $\leftarrow$ later in course

9. COHOMOLOGY

$C_* \cong \bigoplus_\alpha \mathbb{Z}$

(C_*, ∂_*) chain cx s.t. C_* free $\mathbb{Z}$-module

Def **n-cochains** $\boxed{C^n = \mathrm{Hom}(C_n, \mathbb{Z})}$

coboundary map $\partial^n : C^n \longrightarrow C^{n+1}$

(this is the dual of ∂) $\partial^n(\phi) = \phi \circ \partial_{n+1}$

$$\begin{tikzcd}
C_n \arrow[d, "\phi"'] & C_{n+1} \arrow[l, "\partial_{n+1}"'] \arrow[dl, "\partial^*\phi = \phi\circ\partial"] \\
\mathbb{Z} &
\end{tikzcd}$$

Notice ∂^* is degree $+1$ map (not -1)

$$\boxed{H^m(C_*, \partial_*) = \mathrm{Ker}\,\partial^m / \mathrm{Im}\,\partial^{m-1}}$$

cocycles ($\mathrm{Ker}\,\partial^m$)

coboundaries ($\mathrm{Im}\,\partial^{m-1}$)

(Note $\partial^* \circ \partial^* = 0$: $\partial^*\partial^*\phi = \phi \circ \underbrace{\partial\circ\partial}_{=0} = 0$)

Rmk If use negative grading, (C^{-*}, ∂^{-*}) is a chain complex with homology so many results from H_* carry over to H^*. So abusively write

$$\boxed{H^*(C_*, \partial_*) = H_*(\mathrm{Hom}(C_*, \mathbb{Z}), \partial^*)}$$

Remarks about dualisation

$\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$ generated by projection maps $\pi_i(x_1, \dots, x_n) = x_i$

this is the dual of the standard basis: $\pi_i = e_i^* : e_i \to 1$, $e_k \to 0, k \neq i$

$\alpha : \mathbb{Z}^n \to \mathbb{Z}^m$, $x \longmapsto A \cdot x$ (mxn matrix) $\Longrightarrow$

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z}) \arrow[l, "\text{dual}"', "\alpha^*"] \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{Z}^n & \mathbb{Z}^m \arrow[l, "\text{transpose}(A)"']
\end{tikzcd}$$

$\alpha^*\phi = \phi \circ \alpha$

Def X space $\Longrightarrow$ **singular cohomology** $\boxed{H^*(X) = H^*(C^*(X), \partial^*)}$

dualise $C_* = C_*(X)$

similarly define H^*_Δ, H^*_{CW}

Example $\mathbb{RP}^3$: $C^{CW}_*(\mathbb{RP}^3)$: $0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$

dualise: $C^*_{CW}(\mathbb{RP}^3)$: $0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longrightarrow 0$

$$H^*(\mathbb{RP}^3) \cong H^*_{CW}(\mathbb{RP}^3) = \begin{cases} \mathbb{Z} & * = 0, 3 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

$\leftarrow$ notice $H_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$ has moved to grading 2.

Functoriality

$f: X \longrightarrow Y \quad \Rightarrow \quad f_* : C_* X \longrightarrow C_* Y$ $\swarrow$ called **pull-back**

$\Rightarrow \quad C^* X \xleftarrow{f^*} C^* Y$ is dual : $\boxed{f^* \phi = \phi \circ f_*}$

Lemma f^* is a **cochain map** (meaning $\partial^* \circ f^* = f^* \circ \partial^*$)

$\Rightarrow \quad \boxed{f^* : H^* Y \longrightarrow H^* X}$

Pf
$$\begin{aligned}
(\partial^* \circ f^*)(\phi) &= \partial^*(\phi \circ f_*) \\
&= (\phi \circ f_*) \circ \partial \\
&= (\phi \circ \partial) \circ f_* \qquad \text{as } f_* \text{ chain map} \\
&= f^* \circ (\phi \circ \partial) \\
&= f^* \circ (\partial^* \phi) \\
&= (f^* \circ \partial^*)(\phi)
\end{aligned}$$

Properties
- $id^* = id$
- $(f \circ g)^* = g^* \circ f^*$ notice order!

$\Longrightarrow \quad \boxed{H^* : \text{Top} \longrightarrow \text{GradedAbGps}}$ contravariant functor

Exercise $H^0(X) = \prod_{\pi_0 X} \mathbb{Z}$ where $\pi_0 X = \{$path-components of $X\}$

Homotopy invariance

Lemma $f_*, g_* : \underbrace{C_* \longrightarrow \tilde{C}_*}_{\text{free}}$ chain hpic $\Rightarrow f^* = g^* : H^* \tilde{C} \to H^* C$

Pf $f_* - g_* = \tilde{\partial} \circ h + h \circ \partial$ some $h : C_* \longrightarrow \tilde{C}_*[1]$

$f^* - g^* = h^* \circ \tilde{\partial}^* + \partial^* \circ h^*$ for dual $h^* : \tilde{C}^* \longrightarrow C^*[-1]$.

(notice degree -1, not $+1$) □

Def h^* called **cochain homotopy**

Cor $f \simeq g : X \longrightarrow Y \Rightarrow f^* = g^* : H^* Y \longrightarrow H^* X$ □

Algebra : dual of SES

Lemma $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ exact, A, B, C free

$\Longrightarrow 0 \leftarrow A^* \xleftarrow{i^*} B^* \xleftarrow{j^*} C^* \leftarrow 0$ exact ←

Cultural Remark

$\left(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}\right)^* = \prod_{n\in\mathbb{N}} \mathbb{Z}^*$

is not free.

(Baer 1937)

so A^*, B^*, C^* are not free unless A, B, C have finite ranks

Pf C free $\Rightarrow$ $\exists$ splitting

$$\begin{tikzcd} B \arrow[r, "j"] & C \arrow[l, bend left, "s"] \end{tikzcd}$$

$j \circ s = id$

↑ Pick preimages b_i for basis e_i of C, then $s(e_i) = b_i$

$\Rightarrow A \oplus C \xrightarrow[i\oplus s]{\cong} B$

Rmk inverse is

$B \cong A \oplus C$

$b \longmapsto i^{-1}(b - s(b)) \oplus j(b)$

dual

$\Rightarrow A^* \oplus C^* \xleftarrow[i^*\oplus s^*]{\cong} B^*$ and $s^* \circ j^* = id$

→ so i^* surj → so j^* inj

$$\begin{tikzcd} \Rightarrow \quad 0 & A^* \arrow[l] & B^* \arrow[l, "i^*"'] \arrow[r, bend right, "s^*"'] & C^* \arrow[l, "j^*"'] & 0 \arrow[l] \end{tikzcd}$$

where $0 = (j \circ i)^* = i^* \circ j^*$ so $\mathrm{Im}\, j^* \subseteq \mathrm{Ker}\, i^*$

prove $\supseteq$: $i^* b = 0 \Rightarrow b - j^* s^* b \in \mathrm{Ker}\, i^* \cap \mathrm{Ker}\, s^* = \{0\}$

$\Rightarrow b = j^* s^* b \in \mathrm{Im}\, j^*$

since $s^* j^* = id$

since $i^* \oplus s^*$ is iso.

$\Rightarrow \mathrm{Ker}\, i^* = \mathrm{Im}\, j^*$ □

Excision, LES, Mayer-Vietoris

By previous Lemma get dual results:

Excision $\overline{E} \subseteq A^\circ \subseteq X \Rightarrow H^*(X \setminus E, A \setminus E) \xleftarrow[\cong]{i^*} H^*(X, A)$

LES for pair (X,A) $\cdots \xleftarrow{q^*[+1]} H^{*+1}(X,A) \xleftarrow{\delta} H^*(A) \xleftarrow{i^*} H^*(X) \xleftarrow{q^*} H^*(X,A) \leftarrow \cdots$

M.V. $X = A^\circ \cup B^\circ \Rightarrow \cdots \leftarrow H^{*+1}(X) \leftarrow H^*(A\cap B) \xleftarrow[i_A^* \oplus -i_B^*]{} H^*(A) \oplus H^*(B) \xleftarrow[j_A^* \oplus j_B^*]{} H^*(X) \leftarrow \cdots$

where

$$\begin{tikzcd} & A \arrow[dr, "j_A"] & \\ A\cap B \arrow[ur, "i_A"] \arrow[dr, "i_B"'] & & X \\ & B \arrow[ur, "j_B"'] & \end{tikzcd}$$

are the obvious maps

Axioms for cohomology These are analogous to the axioms for homology except we reverse all arrows, and we change axiom (3): $\prod$ instead of $\oplus$

additivity: $(X, A) = \bigsqcup (X_i, A_i)$, $incl_i : (X_i, A_i) \longrightarrow (X, A)$

then $\boxed{\prod F(incl_i) : \prod F(X_i, A_i) \xleftarrow{\cong} F(X, A)}$

10. CUP PRODUCT

Theorem $H^*(\overset{\text{space}}{X})$ is $\overset{①}{\underline{\text{unital}}}$ $\overset{②}{\underline{\text{graded-commutative}}}$ ring via

$\cup : H^k(X) \times H^\ell(X) \longrightarrow H^{k+\ell}(X)$ determined by

$$\boxed{\begin{aligned} \cup : \; & C^k(X) \times C^\ell(X) \longrightarrow C^{k+\ell}(X) \\ & (\phi \cup \psi)(\sigma) = \phi\left(\sigma|_{[e_0,\dots,\underline{\underline{e_k}}]}\right) \cdot \psi\left(\sigma|_{[\underline{\underline{e_k}},\dots,e_{k+\ell}]}\right) \end{aligned}}$$

① $1 \in C^0(X)$ constant function $\Rightarrow$ $1 \cup \phi = \phi \cup 1 = \phi$

② $\phi \cup \psi = (-1)^{\deg\phi \cdot \deg\psi}\, \psi \cup \phi$

Useful trick

$$\phi\circ\text{incl} \longleftarrow\!\shortmid\; \phi$$

If X is CW-cx, then $C_*^{CW}(X) \xrightarrow[\simeq]{\text{inclusion}} C_*(X)$, so $C^*_{cw}(X) \xleftarrow[\simeq]{\text{restriction}} C^*(X)$. So to define/determine a class in $H^*(X)$ it is enough to define its values on CW chains (provided it is a CW-cocycle). So doing:

$$\begin{tikzcd} H^k_{cw} \times H^\ell_{cw} \arrow[r, "\cong"] \arrow[d] & H^k \times H^\ell \arrow[d, "\cup"] \\ H^{k+\ell}_{cw} \arrow[r, "\cong"] & H^{k+\ell} \end{tikzcd}$$

we get an induced product

Proof of Theorem

$$\begin{aligned} \partial^*(\phi\cup\psi)(\sigma) &= (\phi\cup\psi)(\partial\sigma) \\ &= (\phi\cup\psi) \; \sum (-1)^i \, \sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_n]} \qquad n = k+\ell \\ &= \sum_{i\leq k} (-1)^i \, \phi\left(\sigma|_{[e_0,\dots,\hat{e}_i,\dots,\underline{\underline{e_{k+1}}}]}\right) \cdot \psi\left(\sigma|_{[\underline{\underline{e_{k+1}}},\dots,e_n]}\right) \\ &+ \sum_{i> k} (-1)^i \, \phi\left(\sigma|_{[e_0,\dots,e_k]}\right) \cdot \psi\left(\sigma|_{[e_k,\dots,\hat{e}_i,\dots,e_n]}\right) \cdot \underbrace{(-1)^{i-k}(-1)^{k-i}}_{1} \\ &= \left((\partial^*\phi) \cup \psi(\sigma)\right) + (-1)^k \, \phi \cup \partial^*\psi \end{aligned}$$

<u>induces $[\phi]\cup[\psi] = [\phi\cup\psi]$:</u>

Well-defined:
- cycles → cycle: $\partial(\phi\cup\psi) = \overbrace{(\partial\phi)}^{=0}\cup\psi \pm \phi\cup\overbrace{(\partial\psi)}^{=0} = 0$ ✓
- $[\phi] = [\phi + \partial\alpha]$ so need $[(\partial\alpha)\cup\psi] = 0$

  $(\partial\alpha)\cup\psi \underset{\substack{\uparrow \\ \partial\psi = 0}}{=} \partial(\alpha\cup\psi)$ ✓
- Similarly $[\phi]\cup[\partial\beta] = 0$

bilinear, associative, distributive: true at chain level unital: $(\partial 1)(\sigma) = 1(\sigma|_{[e_0]}) - 1(\sigma|_{[e_0]}) = 1-1=0$

$(1\smile\psi)(\sigma) = 1(\sigma|_{[e_0]})\cdot\psi(\sigma|_{[e_0,\dots,e_n]}) = \psi(\sigma)$ ($\phi\smile 1=\phi$ similar)

graded-comm. sketch proof: ⟵ non-examinable

Let $r: C_n(X)\longrightarrow C_n(X)$, $r(\sigma)=\varepsilon_n\,\overline{\sigma}$ where: $\varepsilon_n=(-1)^{\frac{n(n+1)}{2}}$

and $\overline{\sigma}|_{[v_0,\dots,v_n]} = \sigma|_{[v_n,\dots,v_0]}$ ⟵ reverse order of vertices: is product of $\underbrace{n+(n-1)+\dots+1}_{n(n+1)/2}$ transpositions

(idea: each transposition is a reflection in hyperplane, so reverses orientation of simplex, so insert ε_n to compensate)

one checks:

- r chain map
- $\underbrace{r^*\varphi\smile r^*\psi}_{\varepsilon_k\varepsilon_\ell} = \underbrace{r^*(\psi\smile\varphi)}_{\varepsilon_{k+\ell}}$

  differ by $(-1)^{k\ell}$
- $r\simeq id$ so can drop $r^*=id$ on cohomology

  ($r-id = P\partial+\partial P$ with

  $P\sigma = \sum(-1)^i\,\varepsilon_{n-i}\,(\sigma\circ\pi)|_{[v_0,\dots,v_i,\underline{\underline{w_n}},\dots,w_i]}$)

  v_i, w_i like for prism operator

  projection $\Delta^n\times I\xrightarrow{\pi}\Delta^n$ □

Naturality of cup product

Lemma $f:X\longrightarrow Y \implies f^*:H^*Y\longrightarrow H^*X$ hom of unital rings

Pf
$$\begin{aligned} f^*(\varphi\smile\psi)(\sigma) &= (\varphi\smile\psi)(f_*\sigma)\\ &= \varphi(f_*\sigma|_{[e_0,\dots,e_k]})\cdot\psi(f_*\sigma|_{[e_k,\dots,e_n]})\\ &= ((\varphi\circ f_*)\smile(\psi\circ f_*))(\sigma)\\ &= (f^*\varphi\smile f^*\psi)(\sigma)\end{aligned}$$

unital: $f^*(1) = 1\circ f_* = 1$ □

UPSHOT

H^*: Top $\longrightarrow$ {Graded-commutative unital rings with graded unital ring homs}
contravariant functor.

Warning An (iso)morphism $H^*(Y)\longrightarrow H^*(X)$ of groups will also preserve the ring structure if f^* is induced by a map of spaces $X\longrightarrow Y$ (by above Lemma).

$\implies$ Cor The excision theorem iso on cohomology is an iso of rings.

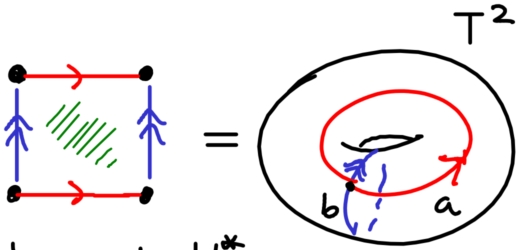
However the connecting hom in M.V. or LES cannot possibly be a ring hom since it drops gradings by 1 ($\Rightarrow \delta(a\smile b)$ and $\delta(a)\smile\delta(b)$ have different grading!)

Example $H^1(T^2)\times H^1(T^2)\xrightarrow{\cup}H^2(T^2)$ bilinear form $\mathbb{Z}^2\times\mathbb{Z}^2\to\mathbb{Z}$ with matrix $\begin{pmatrix}0&+1\\-1&0\end{pmatrix}$

Pf recall:

$*$		$H_*(T^2)$	$H^*(T^2)$
0	$\mathbb{Z}$	$\mathbb{Z}\cdot pt$	$\mathbb{Z}\cdot 1$
1	$\mathbb{Z}^2$	$\mathbb{Z}a\oplus\mathbb{Z}b$	$\mathbb{Z}a^*+\mathbb{Z}b^*$
2	$\mathbb{Z}$	$\mathbb{Z}\cdot D$	$\mathbb{Z}\cdot D^*$

where $D:\Delta^2\cong$ square $\longrightarrow$



$1, a^*, b^*, D^*$ are dual basis in H^*

Identify $H^*(T^2)\cong H^*_\Delta(T^2)$ so at chain level: $\leftarrow X=T^2$

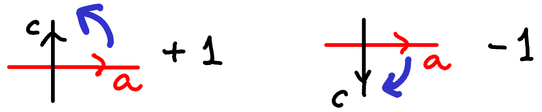
$a^*: C_1^{CW}(X)\longrightarrow\mathbb{Z}$
$a\longmapsto 1$
$b\longmapsto 0$

$b^*: C_1^{CW}(X)\longrightarrow\mathbb{Z}$
$a\longmapsto 0$
$b\longmapsto 1$

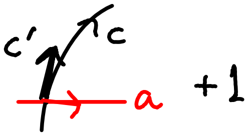
$D^*: C_2^{CW}(X)\longrightarrow\mathbb{Z}$
$D\longmapsto 1$

$\Rightarrow$ $b^*(\underset{C_1^{CW}}{\underset{\cap}{c}})$ = # a intersects c counted with orientation signs

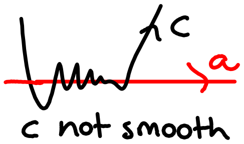
$a^*(\underset{C_1^{CW}}{\underset{\cap}{c}}) = -$ # b intersects c counted with signs.



Fact Same holds for smooth singular 1-chains $c:\Delta^1\cong I\longrightarrow T^2$ which intersect a transversely: velocity vectors a', c' span $\mathbb{R}^2$



Otherwise ill-defined:

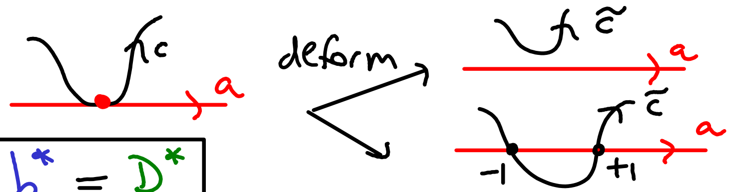


and

are bad.

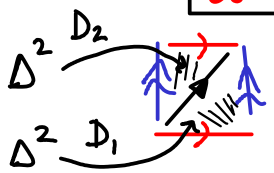
trick need first pick homologous representative which is smooth & transverse, by continuously deforming the chain (continuous map ≃ id so id on H_*)

Example



both cases: $a^*(\tilde c)=0$

claim $\boxed{a^*\cup b^* = D^*}$



$$(a^*\cup b^*)(\underbrace{D_1+D_2}_{\text{homologous to }D}) = a^*(\underbrace{D_1|_{[e_0,e_1]}}_{\|\ \rightarrow})\cdot b^*(\underbrace{D_1|_{[e_1,e_2]}}_{\|\ \uparrow\uparrow}) + \text{same for } D_2$$

$$= a^*(a)\,b^*(b)+a^*(b)\,b^*(a)$$

$$= 1$$

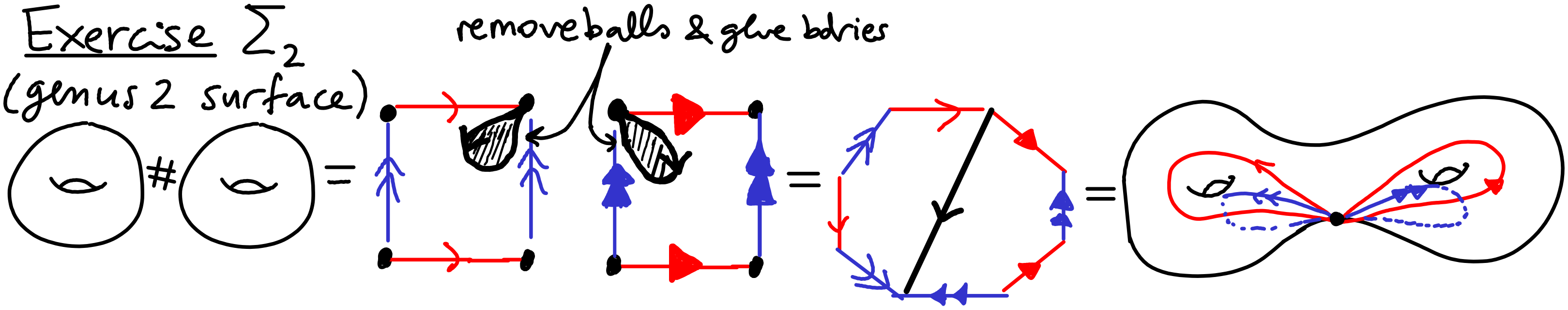
Notice we are using the "Useful Trick" (start of Sec. 10)
We view D as the singular cycle D_1+D_2.

Graded-comm. $\Rightarrow b^*\cup a^* = -D^*$, $a^*\cup a^* = (-1)^{1\cdot 1}a^*\cup a^*$ so $=0$, sim'ly $b^*\cup b^*=0$. □

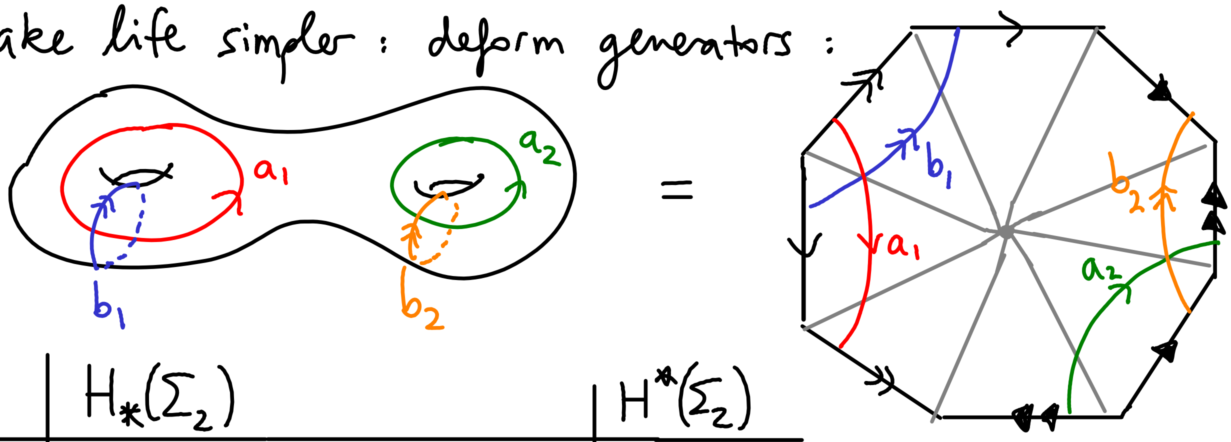
Idea $\cup$ just counts (signed) geometric intersection # of corresponding curves.
Why "$a\cap a=0$"? Can deform a to make it disjoint from a:

Exercise Σ_2

(genus 2 surface)



Make life simpler : deform generators :



		$H_*(\Sigma_2)$	$H^*(\Sigma_2)$
0	$\mathbb{Z}$	$\mathbb{Z}\cdot pt$	$\mathbb{Z}\cdot 1$
1	$\mathbb{Z}^4$	$\mathbb{Z}a_1+\mathbb{Z}b_1+\mathbb{Z}a_2+\mathbb{Z}b_2$	$\mathbb{Z}\langle a_1^*, b_1^*, a_2^*, b_2^*\rangle$ ← dual basis
2	$\mathbb{Z}$	$\mathbb{Z}\cdot D$	$\mathbb{Z}\cdot D^*$

Notice on $C_1^{CW}(\Sigma_2)$: $a_i^*(c) = -\overset{\text{signed count}}{\#}(b_i \text{ intersects } c)$

$b_i^*(c) = \#(a_i$ " " $)$

Exercise $a_i^* \cup b_j^* = \delta_{ij} \cdot D^* = - b_j^* \cup a_i^*$ $\quad \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$a_i^* \cup a_i^* = b_i^* \cup b_i^* = 0$

} so same as geometric intersection numbers of corresponding curves.

hint: D is homologous to the sum of ± triangles in last picture (orientation signs)

Cultural Rmk on general theory (Intersection Theory/Differential Topology)

M^m oriented m-mfd
$N^n \subseteq M^m$ oriented compact n-dim submfd $\Rightarrow$ $H_n(N) \xrightarrow{incl_*} H_n(M)$ ← see later in course

$\exists\,[N] \longmapsto [N]$

N, M also smooth (see Differential Geometry course) $\Rightarrow$ $\omega_N \in H^{m-n}(M)$ counts $\overset{\text{with signs}}{\#}$ intersecns with N

$N_1^{n_1}, N_2^{n_2} \subseteq M$ compact oriented smooth submfds
$n_1 + n_2 = m$ (so complementary dimensions) $\Rightarrow$ $\omega_{N_1} \cup \omega_{N_2} = \underbrace{\#(N_1 \cap N_2)}_{\text{geometric intersection }\#} \cdot [M]^*$

may require ← deforming unless N_1, N_2 are already transverse

Fact (Thom 1954)
Not all $a \in H^j(M)$ arise as ω_N for connected compact oriented codim$=j$ smooth submfd N
But $\exists\, N \in \mathbb{N}$ s.t. $N\cdot a$ does arise. They do arise for $H^*(M;\mathbb{Q})$, $H^*(M,\mathbb{R})$, $H^*(M;\mathbb{Z}/2)$

11. KÜNNETH FORMULA AND PRODUCT SPACES

Algebra : tensor products

e.g. abelian groups $= \mathbb{Z}$-mods
vector spaces$/\mathbb{F} = \mathbb{F}$-mods

R ring (comm. with 1)

Def A, B R-modules ⇒ **Tensor product** is R-module

$$\boxed{A \underset{R}{\otimes} B = \langle (a\, ,\, b) : \ a\in A,\ b\in B \rangle / \text{relations of } \underline{\text{bilinearity}} \ \& \ \underline{\text{rescaling}}}$$

(or $A\otimes B$) R-mod generated write $a\otimes b$ for its class

bilinearity : $(a_1 + a_2)\otimes b = a_1\otimes b + a_2\otimes b$

$a\otimes (b_1 + b_2) = a\otimes b_1 + a\otimes b_2$

rescaling : $r\cdot (a\otimes b) = (ra)\otimes b = a\otimes (rb)$ ∀$r\in R$

("can move $r\in R$ across the $\otimes$ symbol")

- So general element looks like $\sum a_k\otimes b_k$ (finite sum) ← NOT UNIQUELY!
- Don't confuse with $A\times B$: e.g. $0\otimes b = 0$ ∀b

Rmk Can define $A\underset{R}{\otimes}B$ also by a universal property : for all R-mods C,

$$\boxed{\mathrm{Hom}_R(A\underset{R}{\otimes}B\, ; C) \xrightarrow[\text{natural}]{\cong} \{R\text{-bilinear maps } A\times B\to C)}$$

Using above description of $A\otimes B$: $\varphi \longmapsto (A\times B\to C,\ (a,b)\mapsto \varphi(a\otimes b))$

Example $(R=\mathbb{F})$ V, W v.s.$/\mathbb{F}$ ⇒ $V\otimes W$ v.s.$/\mathbb{F}$ basis $v_i\otimes w_j$

basis v_i basis w_j

$\dim_{\mathbb{F}} V\otimes W = \dim V\cdot \dim W$

Exercise V, W finite dim$/\mathbb{F}$ ⇒ $V^*\otimes W \cong \mathrm{Hom}_{\mathbb{F}}(V,W)$

Hint $f: V\to\mathbb{F}$, $w\in W$, $f\otimes w\longmapsto (V\to W,\ v\longmapsto f(v)\cdot w)$

e.g. $(\mathbb{R}^n)^*\otimes\mathbb{R}^m \cong \mathrm{Mat}_{m\times n}(\mathbb{R}) \equiv \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)$
$e_i^*\otimes e_j \longleftrightarrow$ matrix A with $A_{ji}=1$, 0 else.

Examples $(R=\mathbb{Z})$

- $\mathbb{Z}^n\otimes\mathbb{Z}^m \cong \mathbb{Z}^{n\cdot m}$
- $\mathbb{Z}/n\otimes\mathbb{Z}\cong\mathbb{Z}/n \longleftarrow \sum a_i\otimes b_i = (\sum a_ib_i)\otimes 1$
- $\mathbb{Z}/2\otimes\mathbb{Z}/3 = 0 \longleftarrow 1\otimes x = 3\otimes x = 1\otimes 3x = 0$
- $\mathbb{Z}/2\otimes\mathbb{Z}/4\cong\mathbb{Z}/2 \longleftarrow \begin{cases} 1\otimes 1 = 3\otimes 1 = 1\otimes 3 \\ 1\otimes 2 = 2\otimes 1 = 0\end{cases}$

Examples

- $A\otimes B\cong B\otimes A$
- $(\bigoplus_i A_i)\otimes(\bigoplus_j B_j) \cong \bigoplus_i\bigoplus_j (A_i\otimes B_j)$
- $A\otimes R\cong A$ (so "$\cdot\underset{R}{\otimes}R$ does nothing")
- $A\otimes R/d \cong A/d\cdot A$

} hence now know $A\otimes B$ for any f.g. R-mods A, B.

for example $\mathbb{Z}/2\otimes\mathbb{Z}/4 \cong (\mathbb{Z}/4)/2\cdot\mathbb{Z}/4 \cong \mathbb{Z}/2$ ← (Rmk $(\mathbb{Z}/n)/m\cdot\mathbb{Z}/n \cong \mathbb{Z}/\gcd(m,n)$)

More generally: $\begin{cases} R/I \underset{R}{\otimes} R/J \cong R/(I+J) & \text{for ideals } I,J\subseteq R. \\ A\otimes_R R/J \cong A/J\cdot A \end{cases}$

Warning $\cdot\otimes A$ often not an exact functor, i.e. does not preserve exact sequences

indeed it can ruin injectivity: $0\to\mathbb{Z}\xrightarrow{\cdot 2}\mathbb{Z}$ now take $\cdot\otimes\mathbb{Z}/2$ get $0\to\mathbb{Z}/2\xrightarrow{0}\mathbb{Z}/2$.

Fact $\boxed{\cdot\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \text{ and } \cdot\underset{\mathbb{Z}}{\otimes}\mathbb{R} \text{ are exact functors on } \mathbb{Z}\text{-mods}}$ ← More generally $\cdot\underset{R}{\otimes}\mathrm{Frac}(R)$ is exact on R-mods where Frac R is fraction field, and R is an integral domain "localisation is an exact functor"

example A f.g. $\mathbb{Z}$-mod $\Rightarrow A\cong\mathbb{Z}^r\oplus\mathbb{Z}/_{d_1}\oplus\cdots\oplus\mathbb{Z}/_{d_k}$ some $d_i\neq 0$

$$\Rightarrow A\otimes\mathbb{Q}\cong\mathbb{Q}^r \Rightarrow \boxed{r=\operatorname{rank}A=\dim_{\mathbb{Q}}(A\otimes\mathbb{Q})}$$

Tensor product of chain cxes

$C_*,\tilde{C}_*$ chain cxes of R-mods

$$(C_*\otimes\tilde{C}_*)_n=\bigoplus_{i+j=n}C_i\otimes\tilde{C}_j$$

$$\boxed{\partial(x\otimes y)=(\partial x)\otimes y+(-1)^{\deg x}x\otimes\tilde{\partial}y}$$ "Leibniz rule"

Think of ∂ as an operator of deg $=-1$ acting from left

since ∂ "jumps over x" get $(-1)^{\deg\partial\cdot\deg x}$

Exercise $\partial\circ\partial=0$ ← would fail without sign

recall $Z_*=\ker\partial=$ cycles
$B_*=\operatorname{im}\partial=$ boundaries

$$Z_i\otimes\tilde{Z}_j\subseteq Z_{i+j}(C_*\otimes\tilde{C}_*) \text{ and } \left.\begin{matrix}Z_i\otimes\tilde{B}_j\\ B_i\otimes\tilde{Z}_j\end{matrix}\right\}\subseteq B_{i+j}(C_*\otimes\tilde{C}_*)$$

Cor $\exists$ natural maps

$$\boxed{\begin{aligned}H_i(C_*)\otimes H_j(\tilde{C}_*)&\longrightarrow H_{i+j}(C_*\otimes\tilde{C}_*)\\ \textstyle\sum[c_k]\otimes[\tilde{c}_k]&\longmapsto\textstyle\sum[c_k\otimes\tilde{c}_k]\end{aligned}}$$

FACT:

Algebraic Künneth Thm

$C_*,H_*(C_*)$ f.g. free R-mods (no assumption on $\tilde{C}_*$) ← PID (principal ideal domain)

$$\Rightarrow\boxed{\bigoplus_{i+j=n}H_i(C_*)\otimes H_j(\tilde{C}_*)\cong H_n(C_*\otimes\tilde{C}_*)} \text{ via}$$

Algebra: Euler characteristic

C finitely generated graded abelian gp (so $\mathbb{Z}$-mod)
(more generally: R-mod for PID R)

Def **Euler characteristic** $\chi(C)=\sum(-1)^i\operatorname{rank}C_i$

Example/Motivation X finite CW-cx then take $C=C_*^{CW}(X)$ to get

$$\boxed{\chi(X)=\#(0\text{-cells})-\#(1\text{-cells})+\#(2\text{-cells})-\cdots}$$

Lemma If C_* f.g. chain cx $\Longrightarrow\boxed{\chi(C_*)=\chi(H_*(C_*))}$ $\left(=\sum(-1)^i\operatorname{rank}H_i(C)\right)$

Pf Observation: rank $C_i = \dim_{\mathbb{Q}}(C_i \otimes_{\mathbb{Z}} \mathbb{Q})$ ← for R-mods, do $\dim_{\mathbb{F}}(C_i \otimes_R \mathbb{F})$ with $\mathbb{F} = \mathrm{Frac}(R)$

$\Rightarrow$ WLOG assume C_i are vector spaces/field $\mathbb{F}$.

Abbreviate $|C_i| = \dim_{\mathbb{F}} C_i$.

Rank-nullity thm

$$0 \to Z_i \to C_i \xrightarrow{\partial} B_{i+1} \to 0 \Rightarrow |C_i| = |Z_i| + |B_{i+1}|$$
$$0 \to B_i \to Z_i \to H_i \to 0 \Rightarrow |H_i| = |Z_i| - |B_i|$$
$$\Rightarrow |C_i| - |H_i| = |B_{i+1}| - |B_i|$$

$$\Rightarrow \chi(C_*) - \chi(H_*) = \sum (-1)^i |B_{i+1}| - \sum (-1)^i |B_i| = \sum (-1)^i (-|B_i| + |B_i|) = 0. \quad \square$$

Cor X space $\Rightarrow$
$$\boxed{\begin{aligned}\chi(X) &= \sum (-1)^i \operatorname{rank} H_i(X) \\ &= \sum (-1)^i \operatorname{rank} C_i(X)\end{aligned}}$$
← if finite rank $H_*(X)$
← if finite rank $C_*(X)$

So $\chi(X)$ is invariant up to hpy equivalence! Example $\chi\left(\substack{\text{platonic}\\\text{solid}}\right) = \chi(S^2) = 2$

Product spaces

X, Y CW-cxes $\Rightarrow$ $X \times Y$ CW-cx with cells $e_\alpha \times e_\beta$ attaching maps

e_α ↗ (at X), e_β ↖ (at Y)

$$\begin{tikzcd}
\partial(\mathbb{D}^i_\alpha \times \mathbb{D}^j_\beta) = (\partial \mathbb{D}^i_\alpha) \times \mathbb{D}^j_\beta \arrow[d, "\varphi_\alpha \times id"'] & \cup & \mathbb{D}^i_\alpha \times (\partial \mathbb{D}^j_\beta) \arrow[d, "id \times \varphi_\beta"] \\
X^{i-1} \times Y^j \arrow[dr] & & X^i \times Y^{j-1} \arrow[dl] \\
& (X \times Y)^{i+j-1} &
\end{tikzcd}$$

Cor $\boxed{\chi(X \times Y) = \chi(X) \cdot \chi(Y)}$

$\forall$ finite CW-cxes X, Y

Pf $\sum (-1)^k \operatorname{rank} H_k^{CW}(X \times Y)$

$$= \sum (-1)^k \operatorname{rank} C_k^{CW}(X \times Y) = \sum_k \sum_{i+j=k} (-1)^{i+j} \operatorname{rank} C_i^{CW}(X) \cdot \operatorname{rank} C_j^{CW}(Y)$$
$\square$

Lemma $d(e^i_\alpha \times e^j_\beta) = (d e^i_\alpha) \times e^j_\beta + (-1)^i e^i_\alpha \times (d e^j_\beta)$

(proof later)

hence $\boxed{C_*^{CW}(X \times Y) \cong C_*^{CW}(X) \otimes C_*^{CW}(Y)}$

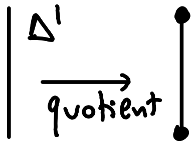
Hence if $H_*(Y)$ free then by Künneth $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$.

Example

$*$	$H_*(S^1)$
0	$A \cong \mathbb{Z}$
1	$B \cong \mathbb{Z}$
2	0

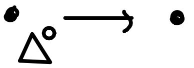
$*$	$H_*(S^1) \otimes H_*(S^1) \cong H_*(S^1 \times S^1)$ ← torus	
0	$A \otimes A$	$\cong \mathbb{Z}$
1	$(A \otimes B) \oplus (B \otimes A)$	$\cong \mathbb{Z}^2$
2	$B \otimes B$	$\cong \mathbb{Z}$
3	0	

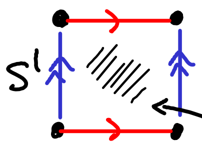
B generated by



$S^1 = \Delta^1 / \text{endpts}$

A generated by

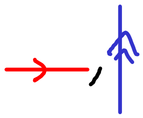


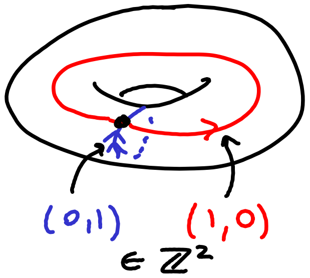


Ho generator is •

square $\cong \Delta^2$ generator of H_2

H_1 generators:





Pf $(\partial \mathbb{D}^i_\alpha)\times \mathbb{D}^j_\gamma \xrightarrow[\varphi_\alpha\times id]{} X^{i-1}\times Y^j \longrightarrow X^{i-1}\times Y^j \Big/ \underbrace{(X\times Y)^{i+j-2}\cap(X^{i-1}\times Y^j)}_{\substack{\| \leftarrow \text{easy check} \\ X^{i-2}\times Y^j \,\cup\, X^{i-1}\times Y^{j-1}}}$

↑
This proof is Non-examinable

$\circledast :=$

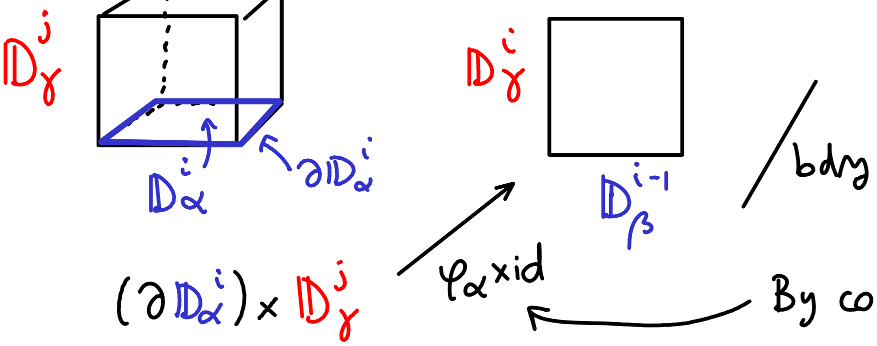
$$X^{i-1} = X^{i-2}\cup\left(\mathbb{D}^{i-1}_\beta \cup \ldots\right)\big/\sim$$
$$Y^j = Y^{j-1}\cup\left(\mathbb{D}^j_\gamma \cup \ldots\right)\big/\sim$$

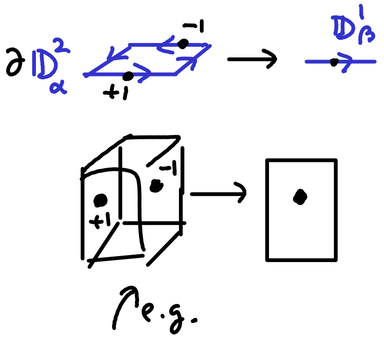
} get ~ from attaching maps

$$X^{i-1}\times Y^j = X^{i-2}\times Y^j \cup X^{i-1}\times Y^{j-1} \cup \left(\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma \cup \ldots\right)\big/\sim$$

$$\Rightarrow \circledast = \left(\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma \cup \ldots\right)\big/\text{boundaries}$$
$$= \mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma\big/\partial(\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma) \quad \vee \quad \ldots$$

$$(\partial\mathbb{D}^i_\alpha)\times\mathbb{D}^j_\gamma \xrightarrow[\varphi_\alpha\times id]{} \mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma\big/\text{bdry} \quad \vee \ldots$$





By considering local degrees now we see we get degree $= d_{\alpha\beta}$ for this.
$\Rightarrow$ get contribution $(de^i_\alpha)\times e^j_\beta$ ✓

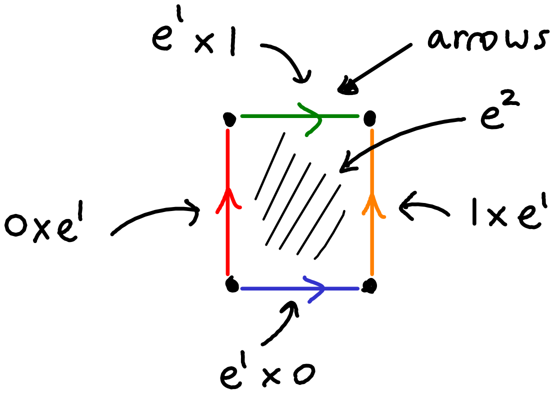
similarly

$$\mathbb{D}^i_\alpha\times\partial\mathbb{D}^j_\gamma \xrightarrow[id\times\varphi_\gamma]{} \mathbb{D}^i_\alpha\times\mathbb{D}^{j-1}_\delta\big/\text{bdry}$$

$\Rightarrow$ degree $(-1)^i d_{\gamma\delta}$
so get $(-1)^i e^i_\alpha \times de^j_\beta$

$(-1)^i$ caused by orientations:
Could reorder factors: $\mathbb{D}^i_\alpha\times\mathbb{D}^j_\gamma \cong \mathbb{D}^j_\gamma\times\mathbb{D}^i_\alpha$ by $\begin{pmatrix} 0 & Id_j \\ Id_i & 0\end{pmatrix}$ whose det $= (-1)^{ij}$. Then $\partial\mathbb{D}^j_\gamma\times\mathbb{D}^i_\alpha \to \mathbb{D}^{j-1}_\delta\times\mathbb{D}^i_\alpha/\text{bdry}$ gives degree $d_{\gamma\delta}$.
Swap factors $\mathbb{D}^{j-1}_\delta\times\mathbb{D}^i_\alpha/\text{bdry}$ by $\begin{pmatrix} 0 & Id_i \\ Id_{j-1} & 0\end{pmatrix}$, det $= (-1)^{i(j-1)}$. Total sign $= (-1)^i$.

Example Recall after definition of H_*^{cw} we had example $I\times I$:



arrows here tell us how we map $[-1,1]\to$ edge (so orientation)

$$\partial e^2 = +e^1\times 0 + 1\times e^1 - e^1\times 1 - 0\times e^1$$
$$= (\partial e^1)\times e^1 \underset{\substack{\uparrow\\ (-1)^{\dim e^1}}}{-} e^1\times(\partial e^1) \qquad ✓$$

A further comment on orientation sign $(-1)^i$

$$\mathbb{D}^i\times\mathbb{D}^j \cong \underset{\substack{\parallel\\ [v_0,\dots,v_i]}}{\Delta^i}\times\underset{\substack{\parallel\\ [w_0,\dots,w_j]}}{\Delta^j}$$

$\leftarrow$ viewed in $\mathbb{R}^i,\mathbb{R}^j$
project $\mathbb{R}^{i+1}\to\mathbb{R}^i$
$(t_0,\dots,t_i)\mapsto\underline{\underline{(t_1,\dots,t_i)}}$

$$\partial(\mathbb{D}^i\times\mathbb{D}^j)\cong \underbrace{\partial\Delta^i}\times\Delta^j \cup \Delta^i\times\underbrace{\partial\Delta^j}$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$\sum_k(-1)^k[v_0,\dots,\hat{v}_k,\dots,v_i] \qquad\qquad \sum\underline{\underline{(-1)^k}}[w_0,\dots,\hat{w}_k,\dots,w_j]$$

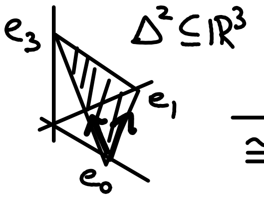
would be correct orientation sign for basis $w_1-w_0,\dots,\widehat{w_k-w_0},\dots,w_j-w_0$ but actually we have $[v_0,\dots,v_i]\times[w_0,\dots,\hat{w}_k,\dots,w_j]\subseteq\mathbb{R}^i\times\mathbb{R}^j$ and $(-1)^{i+k}$ is the orientation sign for the basis

$$v_1-v_0,\dots,v_i-v_0,\ w_1-w_0,\dots,\widehat{w_k-w_0},\dots,w_j-w_0$$

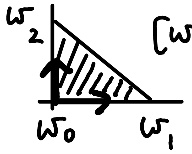
for the hyperplane in $\mathbb{R}^{i+j+1}$ containing

$\Longrightarrow$ need $(-1)^i$ to fix orientation sign.

Example $\Delta^1\times\Delta^2$

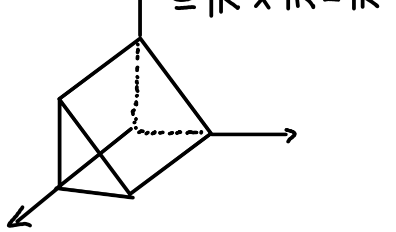


$\xrightarrow{\ \cong\ }$

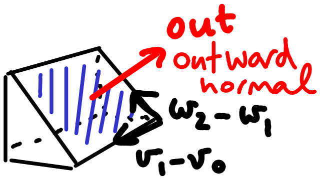


$[w_0,w_1,w_2]\subseteq\mathbb{R}^2$

$\Delta^1\times\Delta^2\cong$



$$[v_0,v_1]\times\underbrace{[\hat{w}_0,w_1,w_2]}$$



out, v_1-v_0, w_2-w_1 is negative $\mathbb{R}^3$-basis

out, w_2-w_1 is positive $\mathbb{R}^2$-basis $\longleftarrow$ differ due to $(-1)^i$, $i=1$.

Projections
$$\begin{tikzcd} & X \\ X\times Y \arrow[ur, "p_X"] \arrow[dr, "p_Y"'] & \\ & Y \end{tikzcd}$$

FACT:

e.g. $Y\simeq$ finite CW cx

no conditions on X

automatic if use field coefficients

Künneth Theorem If $H_n(Y)$ finitely generated, free $\forall n$

$$H_n(X\times Y)\cong \bigoplus_{i+j=n} H_i(X)\otimes H_j(Y)$$

$$H^n(X\times Y)\cong \bigoplus_{i+j=n} H^i(X)\otimes H^j(Y)$$

$$p_X^*a\cup p_Y^*b \longleftarrow a\otimes b \qquad \circledast$$

Recall for cellular homology this on generators is:
$$e^i_\alpha\times e^j_\beta \mapsfrom e^i_\alpha\otimes e^j_\beta$$

This is hom of rings if use following product
$$(a\otimes b)\cdot(\tilde a\otimes\tilde b)=(-1)^{|b|\cdot|\tilde a|}(a\cup\tilde a)\otimes(b\cup\tilde b)$$
gradings

think of it as "exchanging order of $b,\tilde a$"

Rmk

An indirect proof the Thm is to write down two generalised cohomology theories $F(X,A)=H^*(X,A)\otimes H^*(Y)$ and $G(X,A)=H^*(X\times Y, A\times Y)$, and consider the natural transformation $\alpha: F\to G$ given by $\circledast$, notice for $X=pt$, $A=\emptyset$ both F,G give $H^*(Y)$.

Example $X=S^n$, $Y=S^m$ $n\neq m$

$$H_*(S^n\times S^m)\cong\begin{cases}\mathbb{Z} & *=0,n,m,n+m\\ 0 & \text{else}\end{cases}\cong H^*(S^n\times S^m)$$
$$a_n\cup a_m=a_{n+m}$$
where $a_i=dual(e^i)$

$$H_*(S^n\times S^n)\cong\begin{cases}\mathbb{Z} & *=0,2n\\ \mathbb{Z}^2 & *=n\\ 0 & \text{else}\end{cases}\cong H^*(S^n\times S^n)$$
$$a_n\cup a_n=a_{2n}$$

n-torus $S^1\times\dots\times S^1$

Cor $H^k(T^n)\cong\Lambda^k[x_1,\dots,x_n]$

exterior algebra $=$ free abelian gp. on gens. $\{x_{i_1}\wedge\dots\wedge x_{i_k} : i_1<\dots<i_k\}$ so rank $=\binom{n}{k}$

where $x_i=p_i^*(\text{gen. of } H^1(S^1))$

$p_i: T^n\to S^1$ projections to factors.

Pf idea Künneth & induction $(T^n=T^{n-1}\times S^1)$ □

FACT cup product equals composition

$$\cup: H^i(X)\otimes H^j(X)\longrightarrow H^{i+j}(X\times X)\xrightarrow{\Delta^*}H^{i+j}(X)$$

$$(\Delta^i\xrightarrow[\sigma_1]{}X)\otimes(\Delta^j\xrightarrow[\sigma_2]{}X)\longmapsto(\Delta^i\times\Delta^j\xrightarrow[\sigma_1\times\sigma_2]{}X\times X)$$
$$\overset{\sim}{\|}$$
$$\Delta^{i+j}$$

$\Delta=$ diagonal map
$$X\longrightarrow X\times X$$
$$x\longmapsto(x,x)$$

12. UNIVERSAL COEFFICIENTS THEOREM

This proof is Non-examinable

(C_*, ∂_*) chain cx

$$\begin{tikzcd}
\Rightarrow\ 0 \arrow[r] & Z_* = \operatorname{Ker}\partial_* \arrow[r, "\text{incl}"] \arrow[loop below, "\partial=0"] & C_* \arrow[r, "\partial_*"] & B_{*-1} = \operatorname{Im}\partial_{*-1} \arrow[r] \arrow[loop below, "\partial=0"] & 0 \quad \text{is SES}
\end{tikzcd}$$

FACT: Submodules of a free $\mathbb{Z}$-module are free

Rmk The same holds for R-mods if R is PID

($\mathbb{Z}$-module ≡ abelian gp
free means: $\bigoplus_{\text{indexing set}} \mathbb{Z}$)

(PID = principal ideal domain
= integral domain R s.t.
every ideal $= R \cdot a$ some a)

Assume C_* free $\mathbb{Z}$-mod

FACT $\Rightarrow$ Z_*, B_* free (as $\operatorname{Ker}\partial^*, \operatorname{Im}\partial^*$ are submods of C_*)

$\Rightarrow$ SES splits, choose splitting $\begin{tikzcd} C_* \arrow[r, bend left, "\partial_*"] & B_{*-1} \arrow[l, bend left, "s"] \end{tikzcd}$ so $\partial_* \circ s = id$

dual SES $\Rightarrow$

$$\begin{tikzcd}
0 & Z^* \arrow[l] & C^* \arrow[l, "\text{incl}^*"'] & B^{*-1} \arrow[l, "\partial^*"'] & 0 \arrow[l]
\end{tikzcd}$$

note: $\text{incl}^* =$ restrict to Z_*
since $\text{incl}^* \circ \phi : Z_* \xrightarrow{\text{incl}} B_* \xrightarrow{\phi} \mathbb{Z}$

$$\begin{tikzcd}
0 & Z^n \arrow[l] & C^n \arrow[l] & B^{n-1} \arrow[l, "\partial^*"'] & 0 \arrow[l] \\
0 & Z^{n-1} \arrow[l] \arrow[u, "\partial=0"] & C^{n-1} \arrow[l] \arrow[u, "\partial"] & B^{n-2} \arrow[l, "\partial^*"'] \arrow[u, "\partial=0"] & 0 \arrow[l]
\end{tikzcd}$$

(Rmk Although $\partial^* = 0 : B^* \to B^{*+1}$
the map $\partial^* : B^{n-1} \to C^n$ need not $=0$
$\psi : B_{n-1} \to \mathbb{Z}$
$\Rightarrow \partial^*\psi = \psi \circ \partial : C_n \xrightarrow{\partial} B_{n-1} \xrightarrow{\psi} \mathbb{Z}$)

Connecting map
$\delta : Z^{n-1} \to B^{n-1}$
of LES:

$$\begin{tikzcd}
0 & \partial^*\varphi & \varphi|_{B_*} = \phi|_{B_*} \arrow[l, "\partial^*"'] \\
\varphi|_{Z_*} = \phi \arrow[u, mapsto] & \exists\varphi \arrow[l, mapsto] \arrow[u, mapsto] &
\end{tikzcd}$$

$B_* \subseteq Z_*$

$\Rightarrow \boxed{\delta(\phi) = \phi|_{B_*}}$

LES $\Rightarrow$

$$\begin{tikzcd}
\cdots & Z^n \arrow[l] & H^nC \arrow[l] & B^{n-1} \arrow[l, "\partial^*"'] & Z^{n-1} \arrow[l, "\delta^{n-1}"'] \\
 & & & \phi|_{B_{n-1}} & \phi \arrow[l, mapsto]
\end{tikzcd}$$

($H^*B = B^*, H^*C = C^*$ since $\partial^* = 0$)

$$\Rightarrow \quad 0 \leftarrow \operatorname{Ker}\delta^n \leftarrow H^nC \leftarrow B^{n-1}/\operatorname{Im}\delta^{n-1} \leftarrow 0$$

$\operatorname{Ker}\delta^n = \{\phi \in Z^n : \phi(B_n) = 0\} \Rightarrow$ so:

$$\begin{tikzcd}
\phi : Z_n \arrow[rr] \arrow[dr] & & \mathbb{Z} \\
 & Z_n/B_n = H_n(C_*) \arrow[ur] &
\end{tikzcd}$$

$= \operatorname{Hom}(H_n(C_*), \mathbb{Z})$

✪ Universal Coefficients Thm:

$$\begin{tikzcd}
0 \arrow[r] & B^{n-1}/\operatorname{Im}\delta^{n-1} \arrow[r] \arrow[d, equal] & H^n(C) \arrow[r] & \operatorname{Hom}(H_n(C_*), \mathbb{Z}) \arrow[r] & 0 \\
 & \mathbf{Ext}^1(H_{n-1}(C); \mathbb{Z}) & {[\varphi]} \arrow[r, mapsto] & (\varphi : H_n(C_*) \to \mathbb{Z}) &
\end{tikzcd}$$

is SES

see next Lemma

and SES splits (but not naturally): $\begin{tikzcd} B^{n-1}/\operatorname{Im}\delta^{n-1} \arrow[r, bend left, "\partial^*"] & H^n(C) \arrow[l, bend left, "s^*"] \end{tikzcd}$

$s^* \circ \partial^* = id$
(since $\partial \circ s = id$
$\Rightarrow id = (\partial \circ s)^* = s^* \circ \partial^*$)

$\Rightarrow H^n(C) \cong \operatorname{Hom}(H_n(C), \mathbb{Z}) \oplus \operatorname{Ext}^1(H_{n-1}(C); \mathbb{Z})$

Lemma $\mathrm{Ext}^1(H_{n-1}(C);\mathbb{Z}) \cong B^{n-1}/\mathrm{Im}\,\delta^{n-1}$ canonically

Algebra background on Extension groups $\mathrm{Ext}^i(M;\mathbb{Z})$

general case

M R-module, R ring (comm. with 1)

$\Rightarrow \exists$ free resolution :

$$\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \to 0$$

exact, P_i free R-mods

(pick gens x_α for M $\Rightarrow P_0 = \bigoplus_\alpha R \xrightarrow[\varphi_0]{} M$, $e_\alpha \mapsto x_\alpha$

" " y_β for $\ker\varphi_0 \Rightarrow P_1 = \bigoplus_\beta R \xrightarrow[\varphi_1]{} \ker\varphi_0$, $e_\beta \mapsto y_\beta$

continue inductively)

our case

$H_{n-1}(C_*)$ $\mathbb{Z}$-mod

$$\begin{tikzcd} 0 \arrow[r] & B_{n-1} \arrow[r, hook] \arrow[d, equal] & Z_{n-1} \arrow[r] \arrow[d, equal] & H_{n-1}(C) \arrow[r] \arrow[d, equal] & 0 \\ & P_1 & P_0 & M & \end{tikzcd}$$

Take $\mathrm{Hom}(\cdot\,;\mathbb{Z})$ and drop $\mathrm{Hom}(M;\mathbb{Z})$

$$0 \to \mathrm{Hom}(P_0;\mathbb{Z}) \xrightarrow{\varphi_1^*} \mathrm{Hom}(P_1;\mathbb{Z}) \xrightarrow{\varphi_2^*} \cdots$$

Is cochain complex but not exact

$\Rightarrow$ take cohomology groups :

Def $\mathrm{Ext}^0(M;\mathbb{Z}) = \ker\varphi_1^*$

$\mathrm{Ext}^1(M;\mathbb{Z}) = \ker\varphi_2^* / \mathrm{Im}\,\varphi_2^*$

$\cdots$

Fact independent of choices P_i, φ_i

$$0 \to B^{n-1} \to Z^{n-1} \to 0$$

Example 1 $\mathrm{Ext}^0(M;\mathbb{Z}) \cong \mathrm{Hom}(M,\mathbb{Z})$

$$\begin{tikzcd} P_1 \arrow[r, "\varphi_1"] \arrow[dr, "0"'] & P_0 \arrow[r, two heads, "\varphi_0"] \arrow[d, "\phi"] & M \arrow[dl, dotted] \\ & \mathbb{Z} & \end{tikzcd}$$

descends : $m \longmapsto \phi(\varphi_0^{-1} m)$

well defined since $\phi(\underset{\substack{\| \\ \mathrm{Im}\,\varphi_1}}{\ker\varphi_0}) = 0$

Example 2 $\mathrm{Ext}^1(M;\mathbb{Z}) =$

$$\left\{ \phi : \begin{tikzcd} P_2 \arrow[r] \arrow[dr, "0"'] & P_1 \arrow[r] \arrow[d, "\phi"] & P_0 \\ & \mathbb{Z} & \end{tikzcd} \right\} \Big/ \left\{ \phi = \varphi\circ\varphi_1 : \begin{tikzcd} P_1 \arrow[r, "\varphi_1"] \arrow[dr, "\phi"'] & P_0 \arrow[d, "\varphi"] \\ & \mathbb{Z} \end{tikzcd} \right\}$$

Proof of Lemma

By **Example 2**,

$\mathrm{Ext}^1(H_{n-1}(C_*);\mathbb{Z}) =$

$$= \left\{ \begin{tikzcd} 0 \arrow[r] & B_{n-1} \arrow[r] \arrow[d, "\phi"'] & Z_{n-1} \\ & \mathbb{Z} & \end{tikzcd} \right\} \text{modulo}$$

those arising from restriction

$$\left\{ \begin{tikzcd} 0 \arrow[r] & B_{n-1} \arrow[r] \arrow[dr, "\phi|_{B_{n-1}}"'] & Z_{n-1} \arrow[d, "\phi"] \\ & & \mathbb{Z} \end{tikzcd} \right\}$$

Thus $B^{n-1}/\mathrm{Im}\,\delta^{n-1}$. □

Rmk If R PID, then $\ker(P_0 \to M)$ is free (since submod of free mod P_0)

$\Rightarrow$ can pick $P_1 = \ker(P_0 \to M)$, $P_k = 0$ for $k \geq 2 \Rightarrow \mathrm{Ext}^k(M;\mathbb{Z}) = 0$ $k \geq 2$

(Co)homology with coefficients in a ring/field/module

Motivation

So far we had (C_*, ∂_*) chain cx of abelian groups $\}$ in graded sense

group hom

$\Rightarrow H_*(C_*) = \operatorname{Ker}\partial_* / \operatorname{Im}\partial_*$ abelian group (since $\operatorname{Ker}\partial$, $\operatorname{Im}\partial$ are)

We cannot use a chain cx of (non-abelian) groups, because $\operatorname{Im}\partial_*$ need not be a normal subgroup of $\operatorname{Ker}\partial_*$.

However, abelian groups can be thought of as $\mathbb{Z}$-modules, then given any **abelian group G**, define **homology with coeffs in G**

$$\boxed{H_*(C_*;G) = H_*(C_* \otimes_{\mathbb{Z}} G)}$$

with differential $\partial_* \otimes \mathrm{id}$

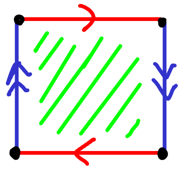
Def X space $\Rightarrow$ $\boxed{H_*(X;G) = H_*(C_*(X) \otimes_{\mathbb{Z}} G)}$

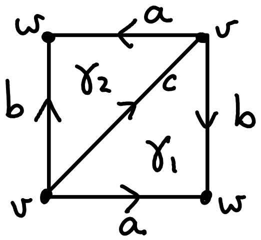
Explanation:

$C_k(X)$ free $\mathbb{Z}$-mod $\cong \bigoplus_{I_k} \mathbb{Z} \Rightarrow C_k(X) \otimes_{\mathbb{Z}} G \cong \bigoplus_{I_k} G$: just replace $\mathbb{Z}$ by G (as $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot \cong \cdot$)

Why care? We hope to get more/new invariants of spaces

Example $X = \mathbb{RP}^2 =$





$*$	$C^{\Delta}_*(\mathbb{RP}^2; G)$
0	$Gv \oplus Gw$
1	$Ga \oplus Gb \oplus Gc$
2	$G\gamma_1 \oplus G\gamma_2$

for $G = \mathbb{Z}/2$:

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow[\begin{pmatrix}1&1\\1&1\\1&1\end{pmatrix}]{\partial_2} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow[\begin{pmatrix}1&1&0\\1&1&0\end{pmatrix}]{\partial_1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0$$

$$\Rightarrow H_*(\mathbb{RP}^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

compare: $H_*(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ 0 & \text{else} \end{cases}$ ($G = \mathbb{Z}$ case)

Form cochain complex using $\operatorname{Hom}_{\mathbb{Z}}(\cdot, G)$ (= group homs to G) in place of $\operatorname{Hom}(\cdot, \mathbb{Z})$

$$\boxed{\begin{aligned} H^*(C_*;G) &= H_*(\operatorname{Hom}_{\mathbb{Z}}(C_*, G)) \\ H^*(X;G) &= H^*(\operatorname{Hom}_{\mathbb{Z}}(C_*(X);G)) \end{aligned}}$$

X space

with differential ∂^*: $\partial^*\phi = \phi \circ \partial_*$

so: $H^*(C_*(X);G)$

Universal coefficients thm (same proof using $\operatorname{Hom}_{\mathbb{Z}}(\cdot, G)$)

$$\boxed{\begin{aligned} 0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(C_*);G) \to H^n(C_*;G) &\longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_n(C_*), G) \to 0 \\ [\varphi] &\longmapsto (\varphi : H_n(C_*) \to G) \end{aligned}}$$

Example $X = \mathbb{R}P^2$, $G = \mathbb{Z}/2$, apply $\mathrm{Hom}_{\mathbb{Z}}(\cdot, G)$ to the complex in previous example

$$0 \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow[\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}]{\partial^1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow[\begin{pmatrix}1&1\\1&1\\0&0\end{pmatrix}]{\partial^0} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \leftarrow 0$$

$$\Rightarrow \quad H^*(\mathbb{R}P^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

compare: $H^*(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$

($G = \mathbb{Z}$ case)

Can generalise further:

$C_* =$ chain cx of ...	coefficients in:	
abelian gps ($\mathbb{Z}$-mods)	abelian gp G ($\mathbb{Z}$-mod)	$H_*(C_*; G) = H_*(C_* \otimes_{\mathbb{Z}} G)$
R-modules ↖ ring (comm. with 1)	R-module M	$H_*(C_*; M) = H_*(C_* \otimes_R M)$

Rmk $H_*(C; M)$ will be an R-module since $\ker \partial, \mathrm{Im}\, \partial$ are (∂_* is R-linear hom by assumption)

X space $\Rightarrow$ $C_k(X; R) = C_k(X) \otimes_{\mathbb{Z}} R \cong \bigoplus_{I_k} R$: just replace $\mathbb{Z}$ by R (as $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot \cong \cdot$)

$$\Rightarrow \boxed{H_*(X; M) = H_*(C_*(X; R); M) = H_*(C_*(X;R) \otimes_R M)}$$

so just replace each $\mathbb{Z}$ by M in $C_*(X)$

Form cochain complex using $\mathrm{Hom}_R(\cdot, M)$ (= R-linear homs to M) in place of $\mathrm{Hom}(\cdot, \mathbb{Z})$

with differential ∂^*: $\partial^* \phi = \phi \circ \partial_*$

$$\boxed{\begin{aligned} H^*(C_*; M) &= H_*(\mathrm{Hom}_R(C_*, M)) \\ H^*(X, M) &= H^*(\mathrm{Hom}_R(C_*(X, R); M)) \end{aligned}}$$

X space

so: $H^*(C_*(X; R); M)$

Rmk These are R-mods. If we use $M = R$, then they are also rings via cup product

Universal Coefficients Thm For R any PID, C_* chain cx of R-mods,

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*); M) \to H^n(C_*; M) \to \mathrm{Hom}_R(H_n(C_*), M) \to 0 \text{ is SES}$$

$\mathrm{Ext}^1_R \cong B^{n-1}/\mathrm{im}\, \delta^{n-1}$ working over R using homs to M

$[\varphi] \longmapsto (\varphi : H_n(C_*) \to M)$

and the SES splits but the splitting is not natural.

Same proof using $\mathrm{Hom}_R(\cdot, M)$

Example $R = \mathbb{F}$ field $\Rightarrow$ C_*, H_*, H^* are vector spaces $/\mathbb{F}$.

Rmk all $\mathbb{F}$-mods (i.e. vector spaces$/\mathbb{F}$) are free $\mathbb{F}$-mods $\cong \bigoplus_{\text{basis } b_i} \mathbb{F} b_i$

up to iso they are determined by $\dim_{\mathbb{F}}$ = cardinality of basis.

<u>Cor</u> C_* = chain cx of $\mathbb{F}$-vector spaces $\Rightarrow \boxed{H^n(C_*;\mathbb{F}) \cong \left(H_n(C_*)\right)^*}$ ← dual v.s.: $\mathrm{Hom}_{\mathbb{F}}(H_n(C_*);\mathbb{F})$

<u>Pf</u> Pick any basis v_i for $\mathbb{F}$-v.s. B_{n-1}, extend it to a basis v_i, w_j of Z_{n-1} (also works in ∞ dim case).

$\Rightarrow$ can extend any $\mathbb{F}$-linear map $\psi : B_{n-1} \to \mathbb{F}$ to $\phi : Z_{n-1} \to \mathbb{F}$ just pick any values $\phi(w_j) \in \mathbb{F}$ e.g. $\phi(w_j) = 0$.

$\Rightarrow B^{n-1}/\mathrm{im}\,\delta^{n-1} = 0$ so $H^n(C_*;\mathbb{F}) \longrightarrow \mathrm{Hom}(H_n(C_*);\mathbb{F})$ iso □

<u>Cor</u> $\boxed{H^n(X;\mathbb{F}) \cong H_n(X;\mathbb{F})^*}$ ← dual v.s. for any field $\mathbb{F}$.

<u>Cor</u>
$$\boxed{H^n(X;M) \underset{\substack{\uparrow \\ \text{if } X \simeq \text{CW-cx}}}{\cong} H^n_{CW}(X;M) \underset{\substack{\uparrow \\ \text{if } X \simeq \Delta\text{-cx}}}{\cong} H^n_{\Delta}(X;M)}$$

<u>Pf</u> Cor holds for <u>homology</u> and the isos are <u>natural</u>. ← i.e. functorial w.r.t. maps

The universal coeff. thm SES is <u>natural</u>. So result holds by 5-Lemma. □

<u>Algebra: structure thm for f.g. abelian groups and R-mods</u>

<u>Fact 1</u> A f.g. abelian gp $\Longrightarrow A \cong \underbrace{\mathbb{Z}^r}_{\text{free part } F} \oplus \underbrace{\mathbb{Z}/p_1^{n_1} \oplus \cdots \oplus \mathbb{Z}/p_k^{n_k}}_{\text{torsion part } T}$

where $p_i \in \mathbb{Z}$ prime (need not be distinct)
Also r, k, p_i, n_i are unique (up to reordering)

<u>Example</u> $\mathbb{Z}/4 = \mathbb{Z}/2^2 \not\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$ (Chinese Remainder Thm)

<u>Fact 2</u> $T \cong \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_k$ with $d_1 | d_2 | \cdots | d_k$ ($d_i \in \mathbb{N}$ unique)

<u>Example</u> $\mathbb{Z}/2 \oplus \mathbb{Z}/2^2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3 \cdot 2^2$ $\quad d_1 = 2$, $d_2 = 12$

<u>Fact 3</u> M f.g. R-mod, R <u>PID</u>, then:

$$\boxed{\begin{aligned} M &\cong F \oplus T \\ F &\cong R^r \\ T &\cong R/d_1 \oplus R/d_2 \oplus \cdots \oplus R/d_k \end{aligned}}$$

$r \in \mathbb{N}$ unique, called <u>rank</u> of M

$d_1 | \cdots | d_k$ non-zero, not invertible
d_i called <u>invariant factors</u>
unique up to mult^n by invertible elements ← e.g. ± 1 if $R = \mathbb{Z}$

<u>Rmk</u> $T = \{m \in M : rm = 0 \text{ some } r \neq 0 \in R\}$ = <u>torsion elements</u>
$F \cong M/T$

Torsion shift

Easy Exercise ↓

$$\boxed{\mathrm{Ext}^*_R\Big(\bigoplus_i M_i\,;\,\prod_j N_j\Big)\cong \prod_i\prod_j \mathrm{Ext}^*_R(M_i\,;N_j)}$$ ← any R-mods M_i, N_j

Upshot To compute $\mathrm{Ext}^1_R(M;R)$ for $M = R^r \oplus R/d_1 \oplus \cdots$ just need:

$$\boxed{\begin{aligned}\mathrm{Ext}^1_R(R\,;R) &= 0\\ \mathrm{Ext}^1_R(R/d\,;R) &\cong R/d\end{aligned}}$$

$\mathrm{Ext}^1_R(R;R)=0$ ← since
$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & R \arrow[r, "id"] \arrow[d, equal] & R \arrow[r] & 0 \\ P_1 & P_0 & & \end{tikzcd}$$

$\mathrm{Ext}^1_R(R/d;R)\cong R/d$ ← since
$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "d"] \arrow[d, "\phi"] & R \arrow[r, "1"] & R/d \arrow[r] & 0 \\ & R & & & \end{tikzcd}$$
so choice of $\phi(1)\in R$ modulo ϕ coming from
$$\begin{tikzcd} R \arrow[r, "d"] \arrow[d, "\phi"'] & R \arrow[dl, "\varphi"] \\ R & \end{tikzcd}$$
so $\phi(1)=d\cdot\varphi(1)$
$\in d\cdot R$

$$\Longrightarrow\ \boxed{\mathrm{Ext}^1_R(M\,;R)\cong \mathrm{Torsion}(M)}$$

Exercises

- $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\,;\mathbb{Z}/m)\cong \mathbb{Z}/\gcd(m,n)$
- G abelian gp $\Rightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z};G)=0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/d\,;G)\cong G/d\cdot G$ ← $d\neq 0$
- R any ring (comm. with 1)
  $x\in R$ not zero divisor $\Longrightarrow$ $\mathrm{Ext}^*_R(R/(x)\,;N)\cong\begin{cases}\{n\in N: x\cdot n=0\} & *=0\\ N/x\cdot N & *=1\\ 0 & \text{else}\end{cases}$ ← N R-mod

Cor If $H_n(X;R)$ f.g. R-mod $\forall n$, R PID,

$\Longrightarrow\ H_n(X;R) = R^{r_n}\oplus T_n$ (free & torsion parts)

$$\Longrightarrow\ \boxed{H^n(X;R)\cong R^{r_n}\oplus T_{n-1}}$$ ← torsion moves up!

↑ not natural (the $\cong$)

Pf $0\longrightarrow T_{n-1}\longrightarrow H^n(X;R)\longrightarrow \mathrm{Hom}(R^{r_n}\oplus T_{n-1}\,;R)\to 0$

$$\mathrm{Hom}(R^{r_n}\oplus T_{n-1}\,;R)\cong \underbrace{(\mathrm{Hom}(R\,;R))^{r_n}}_{\cong R^{r_n}}\oplus\underbrace{\mathrm{Hom}(T_{n-1};R)}_{0}$$

$R\to R$, $1\mapsto x$ ⟶ $\cong R^{r_n}$

x determines the hom

0 since $T_{n-1}\to R$, $1\longmapsto 0$
(R is integral domain, so no torsion elts $\neq 0$)

$\Longrightarrow\ 0\to T_{n-1}\to H^n(X;R)\to \underbrace{R^{r_n}}\to 0$

free, so can split the SES (pick lifts of basis). □ ← so not canonical

Example

$*$	$H_*(\mathbb{R}P^3)$	$H^*(\mathbb{R}P^3)$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/2$	0
2	0	$\mathbb{Z}/2$
3	$\mathbb{Z}$	$\mathbb{Z}$

torsion moves up

Universal coefficients Theorem in homology

FACT Theorem C_* chain cx of free $\overset{\leftarrow \text{PID}}{R}$-mods, M R-module

$$\Rightarrow \boxed{\begin{array}{l} \text{SES } \; 0 \to H_*(C_*)\otimes_R M \to H_*(C_*\otimes_R M) \to \mathrm{Tor}_1^R(C_{*-1}; M) \to 0 \\ \qquad\qquad [c]\otimes m \mapsto [c\otimes m] \qquad\qquad \nwarrow \text{defined below.} \\ \text{The SES splits, but the splitting is not natural.} \end{array}}$$

Torsion groups: A, B R-mods (R comm. ring with 1)

exact sequence, P_i free R-mods

pick $\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} A \to 0$ free resolution

take $\cdot\otimes B$
omit $A\otimes B$

$$\Rightarrow \cdots \to P_2\otimes B \xrightarrow[\varphi_2\otimes id]{} P_1\otimes B \xrightarrow[\varphi_1\otimes id]{} P_0\otimes B \to 0$$

not exact
but is chain cx

$$\mathrm{Tor}_k^R(A,B) = H_k(\text{this complex}) \leftarrow$$

fact independent of choices of P_i, φ_i

Rmk R PID $\Rightarrow \ker\varphi_0$ free $\Rightarrow$ can pick $P_1 = \ker\varphi_0$, $P_k = 0$ for $k \geq 2$

$\Rightarrow$ only $\mathrm{Tor}_0^R, \mathrm{Tor}_1^R$ can be non-zero

Example $\mathrm{Tor}_*^{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}/b) = ?$

$$0 \to \mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z} \xrightarrow[\text{quotient}]{\varphi_0} \mathbb{Z}/a \to 0 \quad \text{free resolution}$$

take $\cdot\otimes\mathbb{Z}/b$
drop $\mathbb{Z}/a\otimes\mathbb{Z}/b$

$$\Rightarrow 0 \to \mathbb{Z}/b \xrightarrow{\cdot a} \mathbb{Z}/b \to 0 \qquad (\text{since } \mathbb{Z}\otimes_{\mathbb{Z}} G \cong G \text{ any } G)$$

$$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/a; \mathbb{Z}/b) = (\mathbb{Z}/b)/a\cdot\mathbb{Z}/b \cong \mathbb{Z}/\gcd(a,b) \cong \mathbb{Z}/a\otimes\mathbb{Z}/b$$

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/a; \mathbb{Z}/b) = \{x \in \mathbb{Z}/b : a\cdot x = 0\} \cong \mathbb{Z}/\gcd(a,b)$$

Facts $\mathrm{Tor}_0^R(A,B) = P_0\otimes B / \mathrm{Im}(\varphi_1\otimes id) \cong A\otimes B$

$\mathrm{Tor}_*^R(A,B) \cong \mathrm{Tor}_*^R(B,A)$

Exercise $\mathrm{Tor}_*^R(\oplus A_i, \oplus B_j) \cong \bigoplus_i\bigoplus_j \mathrm{Tor}_*^R(A_i, B_j)$

$\mathrm{Tor}_*^R(A,B) = 0$ for $* \geq 1$ if A or B is free (use $M\otimes_R R \cong M$)

$\Downarrow$

deduce $\mathrm{Tor}_1^R(A,M)$
for f.g. R-mods A
$\nwarrow$ PID

$$\mathrm{Tor}_*^R(R/u; M) \underset{\substack{\uparrow \\ R \text{ PID}}}{\cong} \begin{cases} M/u\cdot M & * = 0 \\ \underline{u\text{-torsion}}(M) = \{x \in M : u\cdot x = 0\} & * = 1 \\ 0 & \text{else} \end{cases}$$

Example $H_*(\mathbb{R}P^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \end{cases}$ $\quad H_*(\mathbb{R}P^2)\otimes\mathbb{Z}/2 \cong \begin{cases} \mathbb{Z}\otimes\mathbb{Z}/2 \\ \mathbb{Z}/2\otimes\mathbb{Z}/2 \\ 0 \end{cases} \cong \begin{cases} \mathbb{Z}/2 \\ \mathbb{Z}/2 \\ 0 \end{cases}$

$\nwarrow$ caused by $\mathrm{Tor}_1^{\mathbb{Z}}(H_1(\mathbb{R}P^2); \mathbb{Z}/2) = \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$

Künneth Thm

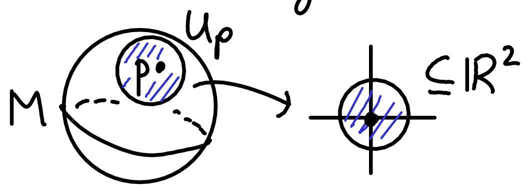
R PID $\Rightarrow$ natural SES:
(C_* free ch. cx. R-mods
D_* any ch. cx. R-mods) $\to$

$$\boxed{0 \to \bigoplus_{i+j=n} H_i(C_*)\otimes H_j(D_*) \to H_n(C_*\otimes D_*) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(H_i(C), H_j(D)) \to 0}$$

and the SES splits but the splitting is not natural. Example R = field, then this $= 0$.

13. MANIFOLDS : POINCARÉ-LEFSCHETZ DUALITY

- M **n-mfd** is Hausdorff topological space s.t. $\forall p \in M$ $\exists$ open neighbourhood $U_p \subseteq M$ homeomorphic to $\mathbb{R}^n$ (equivalently: to an open ball, or any open set in $\mathbb{R}^n$)



One also requires M **second countable** i.e. $\exists$ countable basis of open sets
$\Longleftrightarrow$ M is covered by countably many such U_p.
← exercise

A **submanifold** $N \subseteq M$ is a mfd s.t. inclusion $N \to M$ is an embedding (i.e. a homeomorphism onto its image)

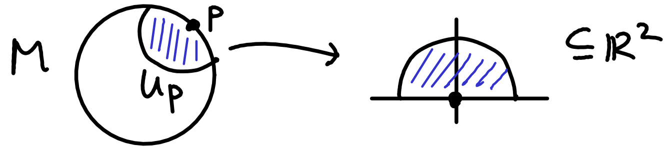
- M **n-mfd with boundary** if also allow $U_p \cong$ **upper half space $\mathbb{H}^n$**

$$\{x \in \mathbb{R}^n : x_n \geq 0\} = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0} = \mathbb{H}^n$$

$p \longmapsto 0$

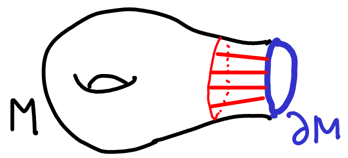
equivalently: any open nbhd of $0 \in \mathbb{H}^n$

Such p are called **boundary points**
they form the **boundary ∂M** which is an (n-1)-mfd without boundary.



FACT (collar nbhd thm) $\partial M \subseteq M$ has an open neighbourhood $\cong \partial M \times (0,1]$
$\partial M \longrightarrow \partial M \times 1$

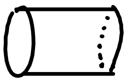
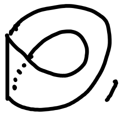
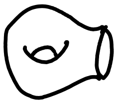
M is **closed** if compact without boundary.



Examples

closed mfds : S^n, $\mathbb{RP}^n$, $T^n = S^1 \times \cdots \times S^1$ (**n-torus**), $\mathbb{CP}^n$, $O(n)$, $SU(n)$

non-compact mfds: $\mathbb{R}^n$, $\mathrm{Mat}_{m \times n} \cong \mathbb{R}^{mn}$, $GL(n, \mathbb{R})$

mfds with bdry: $\mathbb{D}^n$, $\mathbb{D}^1 \times S^1 =$ , Möbius band = , $T^2 \setminus$ open disc = 

FACT (Milnor 1959) Any mfd is homotopy equivalent to a CW-cx

fact If M is a compact manifold then $H_*(M)$ are finitely generated

Rmk M **triangulable** if $M \simeq$ simplicial cx.
Not all mfds are triangulable, but most of those we encounter are.

Compact manifolds have f.g. homology

$\swarrow$ Non-examinable proof

① X space is a **Euclidean neighbourhood retract** if $\exists$ **embedding** $j : X \longrightarrow \mathbb{R}^N$ some N, s.t. $i(X)$ is a retract of a nbhd $V \subseteq \mathbb{R}^n$.
$\nwarrow$(homeo onto image)

② X is **weakly locally contractible** if $\forall$ nbhd $x \in U \subseteq X$, $\exists$ nbhd $x \in V \subseteq U$ s.t. V is contractible inside U.

FACT compact $X \subseteq \mathbb{R}^n$ is ① $\Longleftrightarrow$ X is ②

Rmk If we find nbhd V as in ① with retraction $V \xrightarrow{f} X$ then any smaller nbhd V' also retracts using $f|_{V'} : V' \to X$. Similarly in ② $V' \subseteq V$ is contractible: restrict the hpy.

Lemma A X compact & ① $\Longrightarrow$ X is the retract of a finite simplicial cx

pf $i(X) \subseteq \mathbb{R}^n$ compact $\Rightarrow$ lies inside some large n-simplex $\Delta^n \to \mathbb{R}^n$

Apply barycentric subdivision until simplices have diameter $<$ dist$(X, \partial V)$.

Simpl.cx. $= \bigcup \{$subsimplices which intersect $X\}$ using the restriction of retraction $V \to X$. □

Rmk Also deduce X has f.g. homology since retractions are surjective on H_*.
($\bigoplus \mathbb{Z} \twoheadrightarrow H_*$(finite simpl. cx) $\xrightarrow{\text{retract}}\!\!\!\!\!\rightarrow H_*(X)$ so get surjection from free $\mathbb{Z}$-mod, so f.g.)

Lemma B M compact mfd $\Longrightarrow$ M embeds into $\mathbb{R}^N$, some N.

Pf "Just do it proof":

$\forall p \in M$, $\exists$ homeo $\mathbb{D}^n \xrightarrow{\varphi_p}$ nbhd$(p \in M)$

Pick finite subcover of φ_{p_i} of $M = \bigcup_{p \in M} \varphi_p(\mathbb{D}^n)$. Say $i = 1, \dots, k$

$\psi_{p_i} : M \xrightarrow{\varphi_{p_i}^{-1}} \mathbb{D}^n \to \mathbb{D}^n/\partial\mathbb{D}^n \cong S^n \subseteq \mathbb{R}^{n+1}$ define embedding $(\psi_{p_1}, \dots, \psi_{p_k}) : M \to \mathbb{R}^{k\cdot(n+1)}$

Finally use: a continuous bijection from a compact space to a Hausdorff space is $\cong$ □

Rmk Same works if M has boundary, just consider its **double** $\underbrace{M \cup M}_{\substack{\text{identify}\\ \text{along } \partial M}}$ and apply the Lemma to the double.

Cor M compact mfd (possibly with bdry) $\Longrightarrow$ M has f.g. homology

Pf Mfds satisfy ② since locally ball $\simeq$ pt. M embeds in $\mathbb{R}^N$ by Lemma B.
① holds by FACT. Done by Lemma A. □

Def A **local orientation** of M at $x \in M$ is a choice of generator

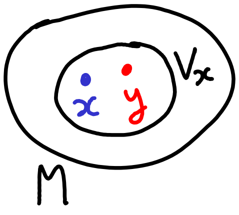
excise complement of nbhd $V_x \cong \mathbb{D}^n$

choice of homeo is not canonical!

$$\begin{aligned}\mu_x \in H_n(M, M \setminus x) &\cong H_n(\mathbb{D}^n, \mathbb{D}^n \setminus 0)\\ &\cong \tilde{H}_n(S^n)\\ &\cong \mathbb{Z}\end{aligned}$$

Def An **orientation** of M is a **locally consistent** choice $x \mapsto \mu_x$

meaning:



$V_x \cong \mathbb{D}^n \simeq pt$

$$\begin{tikzcd} & H_n(M, M \setminus V_x) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ H_n(M, M \setminus x) \arrow[rr, "\cong"] & & H_n(M, M \setminus y) \end{tikzcd}$$

$$\begin{tikzcd} & \exists\, a \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ \mu_x \arrow[rr, mapsto] & & \mu_y \end{tikzcd}$$

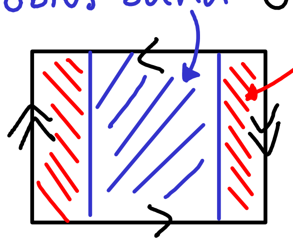
$$\left(H_n(\mathbb{D}^n, \mathbb{D}^n \setminus x) \cong H_n(\mathbb{D}^n, \mathbb{D}^n \setminus y) \atop \uparrow \; \mathbb{D}^n \setminus x \simeq \mathbb{D}^n \setminus y \right)$$

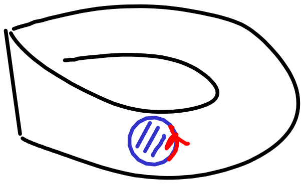
Def M **orientable** if $\exists$ orientation on M

oriented if we chose an orientation

Examples S^n, $\mathbb{R}^n$, $\mathbb{C}P^n$, orientable surfaces Σ_g, $\mathbb{R}P^n \leftarrow$ odd n

Non-example $\mathbb{R}P^2 =$ Möbius band $\cup$ D^2

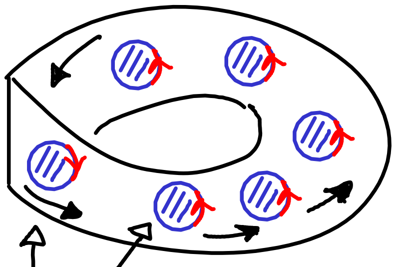




Choice of μ_x is choice of orientation of boundary circle of small disc containing x

$\Longrightarrow \mathbb{R}P^2$ not orientable

by local consistency can move disc continuously and preserves orientation



discs are differently oriented

$\Longrightarrow$ contradicts local consistency.

The fundamental class [M]

FACT

Theorem For M closed n-mfd:

M orientble connected $\Longrightarrow$ $H_n(M) \underset{\text{natural}}{\cong} H_n(M, M\setminus x) \underset{\text{choice}}{\cong} \mathbb{Z}$

$\Longrightarrow$ $\exists\ [M] \longleftrightarrow \mu_x$

↑ once we choose an orientation $(\mu_x)_{x\in M}$

↑ called **fundamental class**

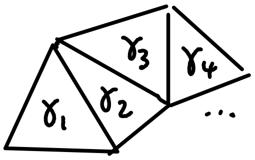
(if swap orientation: for $-\mu_x$ get $-[M]$)

M not orientable connected $\Longrightarrow$ $H_n(M) = 0$

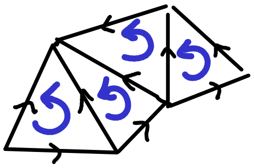
$H_n(M; \mathbb{F}_2) = \mathbb{F}_2 \cdot [M] \cong \mathbb{F}_2$ ← (or any field of characteristic 2)

Construction of [M] if M has Δ-complex structure

M compact $\Longrightarrow$ finite # n-simplices $\gamma_1, \dots, \gamma_N$



M oriented $\Longrightarrow$ pick orientations of $\gamma_1, \dots, \gamma_N$ to agree with given orientation of M: ↙ for $x \in \mathrm{Int}(\gamma_i)$



$$\mathbb{Z}\cdot\mu_x = H_n(M, M\setminus x) \underset{\text{exc.}}{\xrightarrow{\cong}} H_n(\gamma_i, \gamma_i \setminus x) = \mathbb{Z}\cdot\gamma_i$$

$$\mu_x \longmapsto \gamma_i$$

$\Longrightarrow$ $\boxed{[M] := \sum \gamma_i}$ satisfies $\partial [M] = 0$ ✓

$$\begin{tikzcd} H_n(M) \arrow[r] & H_n(M, M\setminus x) \arrow[r, "\cong"] & H_n(\gamma_i, \gamma_i\setminus x) \\ {[M]} \arrow[rr, bend right, mapsto] & \mu_x \arrow[r, mapsto] & \gamma_i \end{tikzcd}$$

Not difficult to see that $H_n^{CW}(X) = \mathbb{Z}\cdot[M]$, so $\Longrightarrow H_n(M) \cong H_n(M, M\setminus x)$

$$[M] \longmapsto \mu_x$$

Also $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}\cdot[M]$ since $C_{n+1}(M) = 0$ ($\nexists$ (n+1)-simplices since $\dim M = n$)

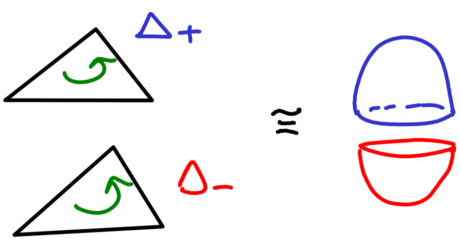
M non-orientble $\Longrightarrow$ each facet of γ_i appears twice in $\partial \sum \gamma_i$

$\Longrightarrow$ $\partial \sum \gamma_i = 0$ over $\mathbb{F}_2$ independently of choices of orientations of γ_i. ✓

Examples

1) $S^n = \dfrac{\overset{\Delta_+}{\Delta^n} \cup \overset{\Delta_-}{\Delta^n}}{\text{glue bdries}}$

S^2:

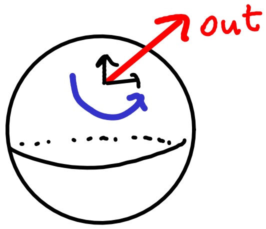


$[S^n] = \Delta_+ - \Delta_-$ if use canonical orientation we discussed

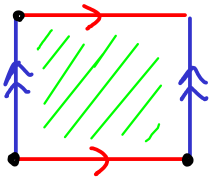
$\uparrow$ hence $\partial[S^n] = \partial\Delta - \partial\Delta = 0$

$\mathbb{D}^n \subseteq \mathbb{R}^n$ canonical orientation

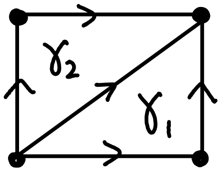
$\Rightarrow S^{n-1} = \partial \mathbb{D}^n$ " using outward normal first rule



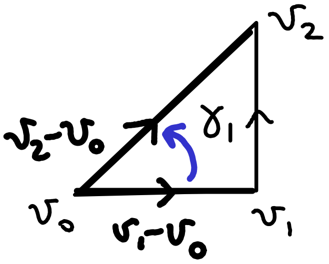
2) $T^2 =$



Δ-complex structure (compatibly with side identifications!)



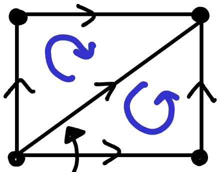
Want orientation induced by square $\subseteq \mathbb{R}^2$



$v_1 - v_0,\ v_2 - v_0$ positive $\mathbb{R}^2$-basis

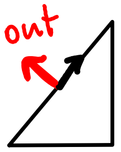
$\Rightarrow \gamma_1$ agrees with orientation

$\Longrightarrow$



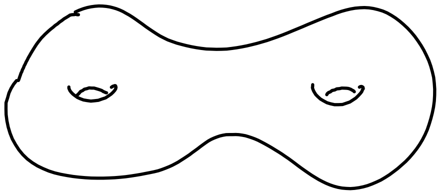
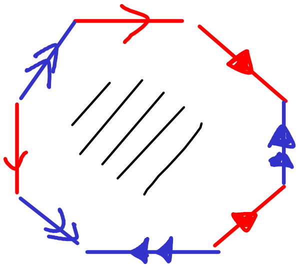
$[T^2] = +\gamma_1 - \gamma_2$

$\uparrow$ γ_2 orientation disagrees

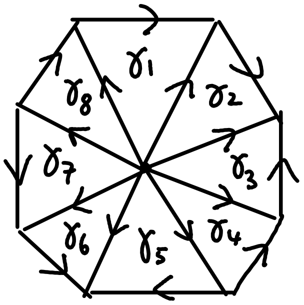


Rmk general feature: if two simplices are identified along a facet then the 2 respective outward normals are related by reflection.

So consistency $\Rightarrow$
- either simplices are compatibly oriented and the two induced orientations on facet are opposite
- or not compatibly oriented but facet orientⁿ is same, then need sign like in example when build $[T^2]$

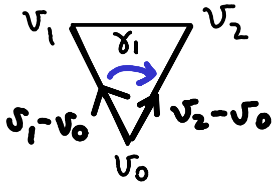
3) Recall $\Sigma_2 =$  $=$ 

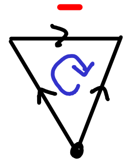
Δ-cx structure (compatible with side identifications!):



Use the orientation induced by polygon $\subseteq \mathbb{R}^2$

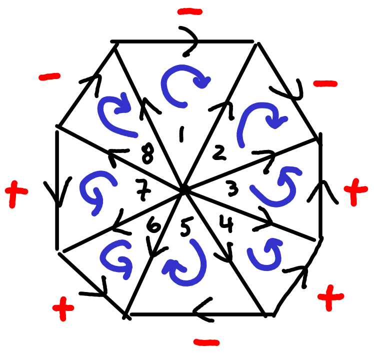
$$\Rightarrow [\Sigma_2] = -\gamma_1 - \gamma_2 + \gamma_3 + \gamma_4 - \gamma_5 + \gamma_6 + \gamma_7 - \gamma_8$$

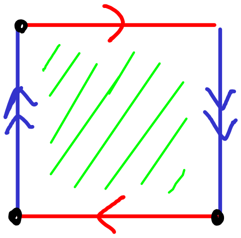
Explanation: 

$v_1 - v_0, v_2 - v_0$ is negative $\mathbb{R}^2$-basis $\Rightarrow$ 

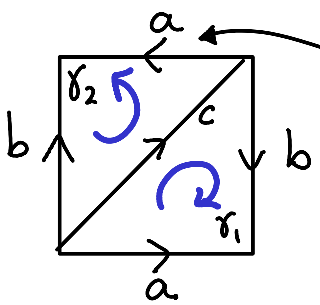
All simplices γ_i have v_0 = centre of polygon

$\Rightarrow$ sign $\begin{cases} - & \text{if outer edge clockwise} \\ + & \text{——anti——} \end{cases}$



3) $\mathbb{RP}^2 =$ 

(non-orientable) example



won't get Δ-cx structure if you try  since get issue here

Use the orientation induced by square $\subseteq \mathbb{R}^2$

$$\Rightarrow [\mathbb{RP}^2] = -\gamma_1 + \gamma_2$$

$$\partial[\mathbb{RP}^2] = -(b - a + c) + (a - b + c)$$

$$= -2b + 2a$$

$\neq 0$ so not cycle in $C_*^{CW}(\mathbb{RP}^2)$

However, working modulo 2:

$\partial[\mathbb{RP}^2] = 0 \in C_*^{CW}(\mathbb{RP}^2; \mathbb{F}_2)$ since $2=0$ in $\mathbb{F}_2$

$$\Rightarrow [\mathbb{RP}^2] \in H^2(\mathbb{RP}^2; \mathbb{F}_2)$$

Degree

Def M, N oriented closed connected n-mfds, $f : M \to N$

$$f_* : H_n(M) \longrightarrow H_n(N)$$
$$[M] \longmapsto \underbrace{\deg(f)}_{\in \mathbb{Z}} \cdot [N]$$

local degree

local map like in chapter 7

Lemma If $f^{-1}(y)$ finite,

then $\deg(f) = \sum_{x \in f^{-1}(y)} \deg(f_y)_*$

pf

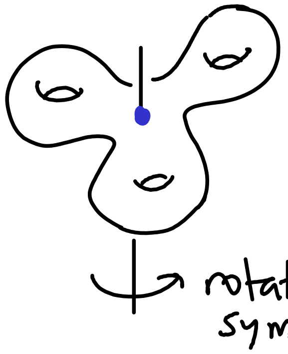
$$\begin{tikzcd}
{[M]} \arrow[dd, mapsto] & H_n(M) \arrow[r, "f_*"] & H_n(N) & {[N]} \\
 & \bigoplus_{x \in f^{-1}(y)} H_n(M, M\setminus x) \arrow[u, phantom, "\cong"] \arrow[r, "(f_x)_*"] & H_n(N, N\setminus y) \arrow[u, phantom, "\cong"] & \mu_y^N \arrow[u, mapsto] \\
\sum \mu_x^M \arrow[rr, mapsto] & & \left(\sum \deg(f_x)_*\right) \cdot \mu_y^N &
\end{tikzcd}$$

$\square$

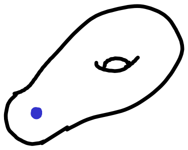
Examples

1) $S^1 \longrightarrow S^1,\ z \longmapsto z^n,\quad [S^1] \longmapsto n \cdot [S^1]$ so $\deg = n$

2) $\Sigma_3 =$



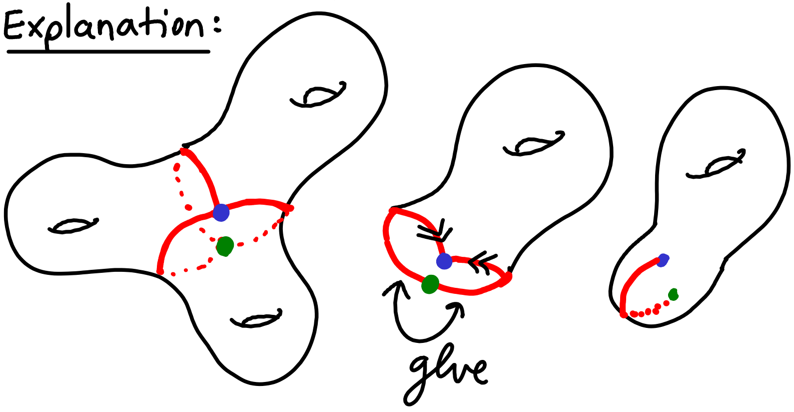
$\xrightarrow[q]{\text{quotient}} \Sigma_3 \big/ \mathbb{Z}/_3$ -rotation action $=$



$= \Sigma_1$

torus

Explanation:



Easy check: $\deg(q) = 3$
(e.g. use local degrees)

Cultural Rmk

For M, N, f smooth, the $\deg f = \#$(preimages of a generic point of N)

Idea: $\deg f$ tells you how many times you cover N. (almost all points work)

Poincaré duality

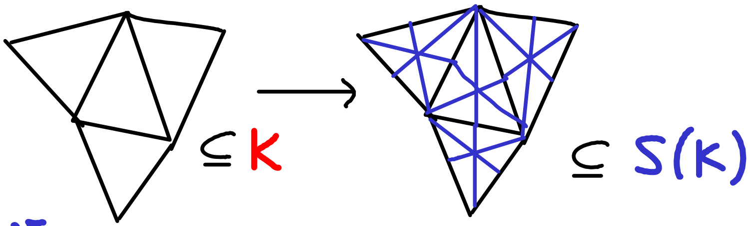
FACT Theorem For M closed n-mfd

M oriented $\Longrightarrow$ $\boxed{H^k(M) \cong H_{n-k}(M)}$ s.t. $\begin{matrix} 1 & \leftrightarrow & [M] \\ \pitchfork & & \pitchfork \\ H^0(M) & \cong & H_n(M) \end{matrix}$

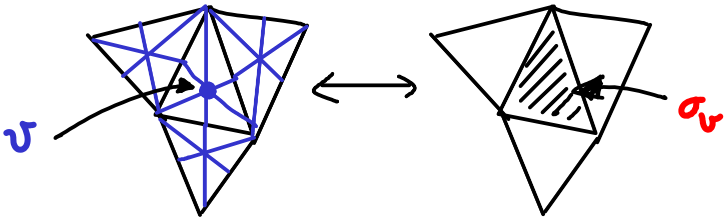
M non-oriented $\Longrightarrow$ same holds with $\mathbb{F}_2$ coefficients

Sketch proof when M is a simplicial complex K (Non-examinable)

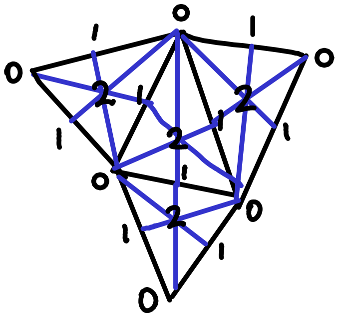
$S(K)$ = barycentric subdivision



1) simplex $\sigma = \sigma_v$ of K with barycentre v $\longleftrightarrow$ $v = v_\sigma$ vertex of $S(K)$

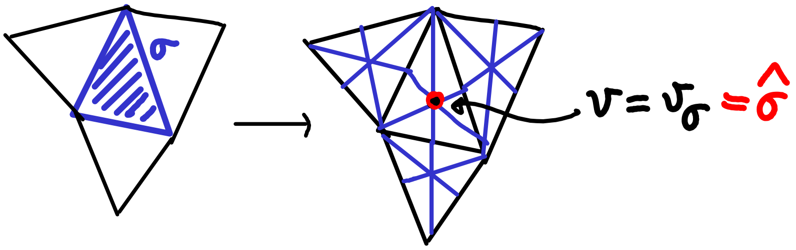


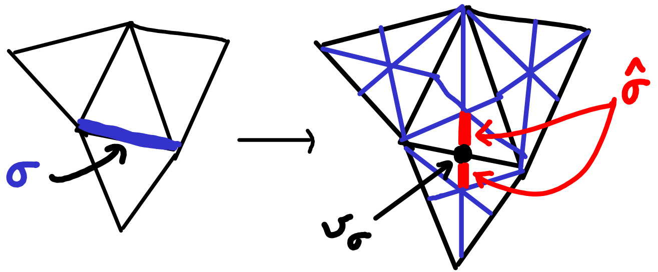
2) $ht(v) = (\text{height of } v) = \dim \sigma_v$



3) σ k-simplex of K
$\downarrow$
dual simplex

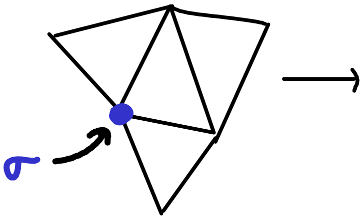
$$\hat{\sigma} = \bigcup_{\substack{\tau \in S(K) \\ ht(v_\sigma) \text{ is min} \\ \text{of heights of} \\ \text{vertices of } \tau}} \tau$$

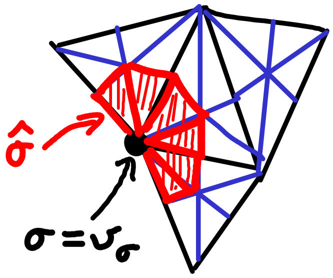




Rmk: $\bigcup \tau$ with $ht(v_\sigma)$ max will give back σ.
Thus $\hat{\sigma}, \sigma$ intersect transversely at v_σ.
One can also describe $\hat{\sigma}$ as

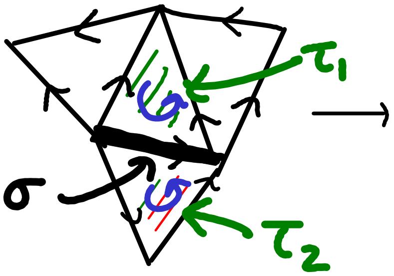
$$\hat{\sigma} = \bigcap_{\substack{\text{vertices} \\ v \in \sigma}} \overline{\text{Star}}_{S(K)}(v)$$ $\leftarrow$ (closed star is the union of simplices of $S(K)$ having v as a face)

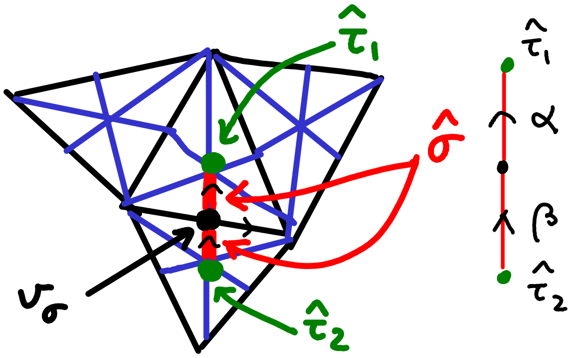


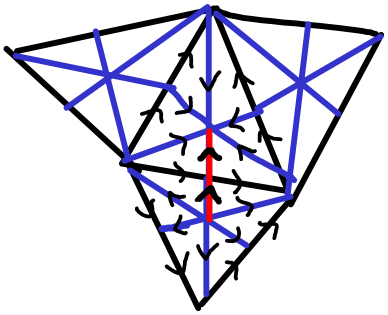


FACTS
- $\dim \hat{\sigma} = n - \dim \sigma$
- dual cells $\hat{\sigma}$ give a cell decomposition of M ("polygonal" complex rather than Δ-cx)

⊛
- $\partial \hat{\sigma} = \sum_{\substack{\text{facet} \\ \sigma \subsetneq \tau \\ \tau \in K}} \pm \hat{\tau}$ need compare orientations of σ, τ (+ if σ as a facet of τ has boundary orientation)

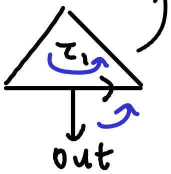


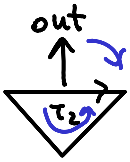




$\hat{\sigma} = \alpha + \beta$

$\partial \hat{\sigma} = + \hat{\tau}_1 - \hat{\tau}_2$





4) dual chain complex

D_{n-k} = free abelian group on dual chains $\hat{\sigma}$

$H_*(M) \cong H_*(D_*, \partial_*)$ (since $\hat{\sigma}$ give a cell decomp. of M)

5) $\varphi : D_{n-k} \longrightarrow C^k(M)$

$\hat{\sigma} \longmapsto \sigma^*$ where $\sigma^*(\alpha) = \begin{cases} 0 \text{ for } k\text{-cells } \alpha \neq \sigma \\ 1 \text{ for } \alpha = \sigma \end{cases}$

- φ linear bijection ✓
- chain map:

$$\varphi(\partial \hat{\sigma}) = \varphi \overset{\text{see ⊛}}{\sum} \pm \hat{\tau} = \sum \pm \tau^*$$

$$\partial^* \varphi(\hat{\sigma}) = \partial^* \sigma^* = \left(\sigma^* \circ \partial : \tau \xrightarrow{\partial} \sum_{\substack{\text{facets } \sigma_i \\ \text{of } \tau}} \pm \sigma_i \xrightarrow{\sigma^*} \begin{cases} \pm 1 & \text{if one of } \sigma_i = \sigma \\ 0 & \text{else} \end{cases} \right)$$

$$= \sum \pm \tau^* = \varphi(\partial \hat{\sigma}) \checkmark$$

UPSHOT φ is chain iso so get iso:

$$H_*(M) \underset{④}{\cong} H_*(D_*, \partial_*) \xrightarrow[\varphi]{\cong} H^{n-*}(M)$$

Cor χ(odd dimensional closed orientable mfd) $= 0$

Pf $b_i = \operatorname{rank} H_i(M)$ (**Betti numbers**)

$$\chi(M) = b_0 - b_1 + \dots + b_{\dim M - 1} - b_{\dim M}$$

equal by Poincaré duality □

(Poincaré–)Lefschetz duality

Theorem

M compact oriented ~~n-mfd~~ n-mfd with boundary

$$\boxed{H^k(M) \cong H_{n-k}(M, \partial M)}$$

$1 \in H^0(M) \longmapsfrom [M, \partial M] \in H_n(M, \partial M)$

relative fundamental class

$$\boxed{H_k(M) \cong H^{n-k}(M, \partial M)}$$

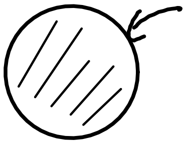
Non-oriented $\Rightarrow$ same holds with $\mathbb{F}_2$ coefficients.

Pf basically same as Poincaré duality. □

Cor M compact, connected, $\partial M \neq \emptyset \Rightarrow H^n(M) = H_0(M, \partial M) = 0$

Examples

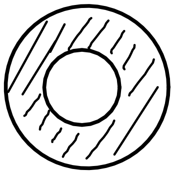
1) $\mathbb{D}^n$



$\partial \mathbb{D}^n = S^{n-1}$

$$\mathbb{Z} \cong H^0 \mathbb{D}^n \cong H_n(\mathbb{D}^n, S^{n-1})$$

2)

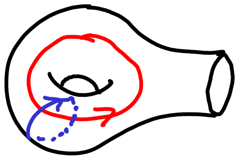


$A = \text{annulus} \subseteq \mathbb{R}^2$

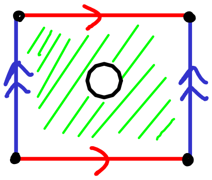
$\simeq S^1$

$$\mathbb{Z} \cong H^0 A \cong H_2(A, \partial A)$$
$$\mathbb{Z} \cong H^1 A \cong H_1(A, \partial A)$$
$$0 \cong H^2 A \cong H_0(A, \partial A)$$

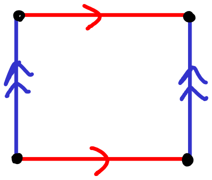
3) $M = T^2 \setminus$ open ball $=$



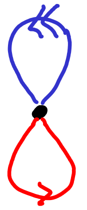
$=$



$\simeq$ def. retract



$\simeq$



$\simeq S^1 \vee S^1$

$$\Rightarrow H_*(M, \partial M) \cong H^{2-*}(S^1 \vee S^1) = \begin{cases} \mathbb{Z}^2 & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$$

$\leftarrow$ gen. by 2 loops

$\leftarrow$ gen. by $[M, \partial M]$

What happens in the non-compact case?

Locally finite homology (Borel-Moore)

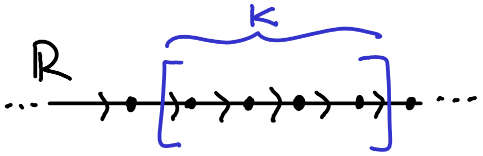
$C_*^{lf}(X)$ allow infinite sums $\sum \overset{\in\mathbb{Z}}{n_i}\, \sigma_i$ ← generators of $C_*(X)$

s.t. given any compact subset $K \subseteq X$,

$$\#\{n_i \neq 0 \,:\, K \cap \operatorname{Im} \sigma_i \neq \emptyset\} < \infty .$$

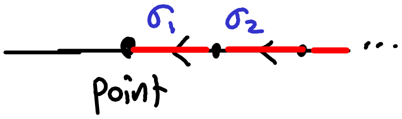
Examples

- $C_1^{lf}(\mathbb{R}) \ni \sum_{m\in\mathbb{Z}} \sigma_m$



$\Rightarrow$ get cycle $[\mathbb{R}] \in H_1^{lf}(\mathbb{R})$ $\qquad$ $\sigma_m : I \cong [m, m+1] \subseteq \mathbb{R}$

- $C_0^{lf}(\mathbb{R}) \ni [\text{point}] \ni \partial(\sigma_1 + \sigma_2 + \cdots)$ is a boundary:



exercise $H_*^{lf}(\mathbb{R}) = \begin{cases} \mathbb{Z} & *=1 \\ 0 & \text{else} \end{cases}$ $\quad (\cong H^{1-*}(\mathbb{R}))$

FACT Theorem $\;$ M orientable n-mfd (possibly not compact) $\Rightarrow$ $\boxed{H^*(M) \cong H_{n-*}^{lf}(M)}$

cohomology with compact supports $H_c^*(X)$

$C_c^*(X)$: only allow cochains $\phi : C_*X \to \mathbb{Z}$ s.t. $\exists$ compact $K \subseteq X$ with ← depends on ϕ

$$\phi(C_*(X \setminus K)) = 0 \qquad \text{(vanish on chains in } X\setminus K)$$

Example $\;$ $\underset{\text{cycle}}{c} \in C_*(X)$ $\Rightarrow$ $\phi(\alpha) =$ signed # intersections of c with α
(geometric intersection #)

$\Rightarrow$ $\phi \in C_c^*(X)$ since $\phi(\alpha) = 0$ if $\alpha \subseteq X \setminus \operatorname{Im}(c)$

Thm $\;$ M orientable n-mfd (possibly not compact) $\Rightarrow$ $\boxed{H_*(M) \cong H_c^{n-*}(M)}$

Warning H_*^{lf}, H_c^* are not homotopy invariant (indeed non-trivial for $\mathbb{R}^n$)

Caused because they are not functorial. They are however functorial for **proper maps**

Mayer-Vietoris holds for H_c^* but not for H_*^{lf}. $\qquad$ (preimages of compact sets are compact

Fact $H_c^*(X) \cong \varinjlim H^*(X, X\setminus K)$ where compacts $K_1 \subseteq K_2$ give $H^*(M, M\setminus K_1) \to H^*(M, M\setminus K_2)$

Direct limit $\varinjlim G_i$ via maps $G_i \to G_j$ means $\bigsqcup G_i \big/$ identify $g \in G_i$ with its images under those maps

(The indices are partially ordered & directed: $\forall i,j, \exists k \geq i,j$ so can compare G_i, G_j inside G_k (via $G_i \to G_k$, $G_j \to G_k$)

Fact $\varinjlim$ is an exact functor.

Cap product and Poincaré duality revisited

X space, $k \geq \ell$

$$\cap : C_k(X) \times C^\ell(X) \longrightarrow C_{k-\ell}(X)$$ cap product

$$(\sigma : \Delta^k \to X) \cap (\phi : C_\ell(X) \to \mathbb{Z}) = \underbrace{\phi(\sigma|_{\underbrace{[e_0,\dots,e_\ell]}_{\text{"bottom face"}}})}_{\in \mathbb{Z}} \cdot \underbrace{\sigma|_{\underbrace{[e_\ell,\dots,e_k]}_{\text{"top face"} \cong \Delta^{k-\ell}}}}_{\in C_{k-\ell}(X)}$$

(easy) Properties

- $\cap$ bilinear
- $\partial(\sigma \cap \phi) = (-1)^\ell (\partial\sigma \cap \phi - \sigma \cap \partial^* \phi)$
- cycle $\cap$ cycle is cycle
- boundary $\cap$ cycle / cycle $\cap$ boundary are boundaries

$$\Longrightarrow \boxed{\cap : H_k(X) \times H^\ell(X) \longrightarrow H_{k-\ell}(X)} \quad \text{bilinear}$$

Theorem (Poincaré duality) The map $\phi \longmapsto [M] \cap \phi$ gives following isos

① For M closed oriented n-mfd

$$[M]\cap \cdot \; : H^*(M) \xrightarrow{\cong} H_{n-*}(M)$$

② For M non-compact oriented n-mfd,

$$[M]\cap \cdot \; : H^*_c(M) \xrightarrow{\cong} H_{n-*}(M) \qquad \circledast$$

$$[M]\cap \cdot \; : H^*(M) \xrightarrow{\cong} H^{\ell f}_{n-*}(M)$$

Sketch Pf of ① for smooth mfds (Non-examinable)

If M smooth $\Rightarrow \exists$ "good cover" U_i of M meaning open cover s.t.

FACT from Riemannian geometry ("convex neighbourhoods")

$U_i \cong \mathbb{R}^n$

$U_{i_1} \cap \dots \cap U_{i_k} \cong \mathbb{R}^n$ or $\emptyset$

Then compute $H^*_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$ and $\circledast$ holds for $\mathbb{R}^n$.

$\Rightarrow \star$ holds $\forall U_i$

$\Rightarrow$ by naturality of $\circledast$ and of Mayer-Vietoris get $\circledast$ for $\underset{\text{finite}}{\cup} U_i$

use 5-lemma

$\Rightarrow \circledast$ for M, which is ①. $\square$

General Pf of Poincaré duality ← Non-examinable

Step 1 : holds for $\mathbb{R}^n$

Pf $H^k_c(\mathbb{R}^n) \underset{\uparrow}{\cong} H^k(D^n, S^{n-1}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z} & k = n \end{cases} \cong H_{n-k}(\mathbb{R}^n)$

can make K larger by picking K = large ball
then $H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong H^k(K, \partial K)$

Pick Δ-cx structure for $\mathbb{R}^n$. So $[\mathbb{R}^n] = \sum \pm \sigma_i$ ← sum over n-simplices.
Say ∃ simplex $\sigma_0 : \Delta^n \to \mathbb{R}^n$. Define $\phi : C^{cw}_c(\mathbb{R}^n) \to \mathbb{Z}$, $\phi(\sigma_0) = \pm 1$ ⊛
$\phi(\text{other simplices}) = 0$
$\Rightarrow \delta\phi = 0$ for dim reasons

$[\mathbb{R}^n] \cap \phi = \sum \phi(\pm\sigma_i) = \phi(\pm\sigma_0) = 1$ (pick sign in ⊛)

Step 2 holds for $A, B, A \cap C \Longrightarrow$ holds for $A \cup B$

Pf Mayer-Vietoris for H^*_c, naturality, 5-lemma ✓

Step 3 holds for A_i, and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \Rightarrow$ holds for $\bigcup A_i$

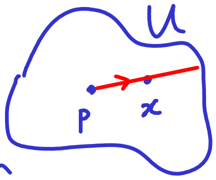
Pf By applying $\varinjlim$: both sides of P.D. iso commute with limits ✓

Step 4 holds for open subsets in $\mathbb{R}^n$

Pf Every such set is a union of convex open sets (e.g. balls)

By Step 3 enough to consider case of finite union.

By induction on # convex open sets :



fix $p \in U$
$U \longrightarrow \mathbb{R}^n$
$x \longmapsto \dfrac{x - p}{\text{dist } p \text{ to } \partial U \text{ along ray to } x}$

1 convex set $U \cong \mathbb{R}^n$ via a proper homeomorphism,
now use Step 1 ✓

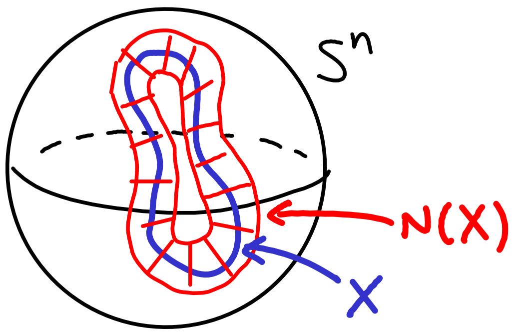
2 convex sets : KEY TRICK convex set ∩ convex set is convex in $\mathbb{R}^n$!
⇒ use Step 2 & previous case

k+1 convex sets : $A = \bigcup\{\text{first } k \text{ convex sets}\}$, $B = $ last convex set
$\Rightarrow A \cap B \subseteq B$ is a union of k convex sets
} ⇒ use Step 2 & Inductive hypothesis ✓

Step 5 holds for mfd M

Consider open sets in M for which it holds.
By a Zorn's Lemma argument we get a maximal open subset U where holds.
If $U \neq M$ pick $p \in M \setminus U$ and nbhd $V \cong \mathbb{R}^n$ of p. Then holds for $U, V, U \cap V$
(note $U \cap V \subseteq V \cong \mathbb{R}^n$ open, so Step 4 applies) so by Step 2 holds for $U \cup V$
Contradicts maximality. ✓ □

Alexander duality



(in fact, enough to assume X is locally contractible)

$\emptyset \neq X \subsetneq S^n$ compact subset s.t.

$\exists$ open neighbourhood $N(X)$ which deformation retracts to X such that $\overline{N(X)} \subseteq S^n$ is an n-mfd with boundary.

Theorem

$$\boxed{\tilde{H}_*(X) \cong \tilde{H}^{n-*-1}(S^n \setminus X)}$$

Pf later

Example $X \subseteq S^3$ **knot** (i.e. $X = \text{image}(S^1 \xrightarrow[\text{homeomorphism onto the image}]{} S^3)$)

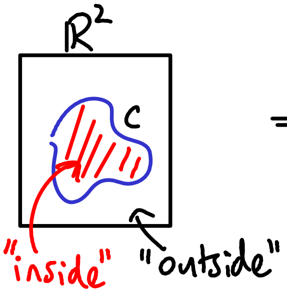
↖ **embedding**

$\Longrightarrow N(X) \cong$ solid torus $\simeq S^1$

$$\Longrightarrow \begin{array}{lllll} \tilde{H}_0(X) & = 0 & = & \tilde{H}^2(S^3 \setminus X) \\ \tilde{H}_1(X) & = \mathbb{Z} & = & \tilde{H}^1 & \text{''} \\ \tilde{H}_2(X) & = 0 & = & \tilde{H}^0 & \text{''} \end{array}$$

so the homology of a **knot complement** does not tell knots apart (always same)

Theorem (**Jordan curve theorem**)



$C \simeq S^1$ closed curve in $\mathbb{R}^2 \subseteq S^2$ ↙ e.g. by stereographic projection $S^2 \cong \mathbb{C} \cup \infty$

$\Longrightarrow \mathbb{R}^2 \setminus C$ has 2 path-components (= connected components)

Similarly for $C \simeq S^n \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}$.

Pf $C \simeq S^n \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}$ $\Rightarrow \tilde{H}_n(S^n) \cong \mathbb{Z} \cong \tilde{H}^0(S^{n+1} \setminus C)$

$\Rightarrow H^0(S^{n+1} \setminus C) \cong \mathbb{Z}^2$

$\Rightarrow S^{n+1} \setminus C$ has 2 path components. $\square$

Proof Alexander duality

$$Y := S^n \setminus N(X) \qquad (\simeq S^n \setminus X)$$

for $* < n-1$

$$\begin{aligned} \tilde{H}^{n-*-1}(Y) &= H^{n-*-1}(Y) \\ &\underset{\text{Lefschetz}}{\cong} H_{*+1}(Y, \partial Y) \\ &\underset{\text{exc.}}{\cong} H_{*+1}(S^n, \overline{N(X)}) \\ &\underset{\substack{\text{LES} \\ \text{using } * < n-1}}{\cong} \tilde{H}_*(\underbrace{\overline{N(X)}}_{\simeq X}) \end{aligned}$$

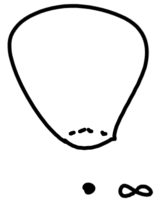
for $* = n-1$

$$\begin{aligned} \tilde{H}^0(Y) \oplus \mathbb{Z} &\cong H^0(Y) \\ &\underset{\text{Lef.}}{\cong} H_n(Y, \partial Y) \\ &\underset{\text{exc.}}{\cong} H_n(S^n, \overline{N(X)}) \\ &\cong \tilde{H}_{n-1}(\underbrace{\overline{N(X)}}_{\simeq X}) \oplus \mathbb{Z} \end{aligned}$$

$$\begin{tikzcd} & & \cong & & \\ 0 \arrow[r] & \tilde{H}_n(S^n) \arrow[r] \arrow[ur] \arrow[dr, "\cong"'] & H_n(S^n, \overline{N(X)}) \arrow[r] \arrow[d] & \tilde{H}_{n-1}(\overline{N(X)}) \arrow[r] & 0 \\ & & H_n(S^n, S^n \setminus \infty) \cong \mathbb{Z} & & \end{tikzcd}$$

$\uparrow$

$S^n = \mathbb{R}^n \cup \infty$



for $* = n$

$$H^{n-*-1}(Y) = H^{-1}(Y) = 0$$

$$H_n(X) \cong H_n(N(X)) \underset{\text{Lef.}}{\cong} 0 \quad \text{n-mfd with bdry} \neq \emptyset.$$

$\square$