C3.1 Algebraic Topology

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Please be aware there are
likely typos in these notes:
comments/corrections are welcome!

Course Book

- Hatcher, Algebraic Topology – Chp. 2 & 3

  This is also freely available from the author's website.
  You are expected to read chapters 2 & 3.

Other references

- Ulrike Tillmann's C3.1 notes – see course page
- The discussion of orientations in my B3.2 Geometry of Surfaces notes may be helpful.

Other books

Massey, A basic course in Algebraic Topology

MORE BASIC but full of ideas:

Fulton, Algebraic Topology : a first course

MORE ADVANCED:

May, A concise course in Algebraic Topology

Davis & Kirk, Lecture notes in Algebraic Topology

Bredon, Topology and Geometry

Bott & Tu, Differential forms in Algebraic Topology

Classics by Spanier, Dold, also see references in May's book

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0. OVERVIEW OF THE COURSE

Motivation, category theory, functors H_* and H^*: some computations
why functors are useful: Invariance of dimension, Brower fixed pt thm

1. ALGEBRAIC PRELIMINARIES : CHAIN COMPLEXES

Graded abelian groups
chain complexes, chain maps, subcomplex, quotient complex
chain map induces map on homology
exact sequence, snake lemma: SES induces LES on H_*, naturality of LES
5-Lemma, SES splits $\Leftrightarrow$ direct sum

2. Δ-COMPLEXES AND SIMPLICIAL HOMOLOGY

Δ^n, n-simplices, Δ-complex (structure), simplicial cx, triangulation
simplicial chain complex, $H_*^\Delta(S^n)$, $H_*^\Delta(T^2)$, remark about orientations
$H_*^\Delta(\sqcup \text{ conn.comp}) \cong \oplus H_*^\Delta(\text{conn.comp.})$, $H_0^\Delta(X) \cong \mathbb{Z}^{\#\text{conn.comp}}$

3. SINGULAR HOMOLOGY

Motivation, singular chain cx
naturality / functoriality, $H_*(\text{point})$

4. CHAIN HOMOTOPIES AND HOMOTOPY INVARIANCE

chain homotopy, prism operator
homotopic maps $f \simeq g$ (relative A), homotopy equivalent spaces $X \simeq Y$
contractible space, homotopic maps induce equal maps on homology
homotopy equivalence induces iso on H_*, $H_*(\mathbb{R}^n) = H_*(\mathbb{D}^n) = H_*(pt)$
pairs of spaces, relative homology $H_*(X,A)$, LES in H_* for pair
reduced homology $\widetilde{H}_*(X)$, LES for $\widetilde{H}_*$, $H_*(\mathbb{D}^n, S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})$
naturality of LES for pairs

5. EXCISION THEOREM AND QUOTIENTS

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good pairs $\Rightarrow H^*(X,A) \cong \widetilde{H}_*(X/A)$, generator of $H_n(S^n) \cong H_n(\mathbb{D}^n, S^{n-1}) \cong \mathbb{Z}$
another remark about orientations

6. MAYER-VIETORIS SEQUENCE

MV LES, $H_*(S^n)$

wedge sum $X \vee Y$, cone CX, suspension ΣX, connected sum $X \# Y$

7. DEGREE OF MAPS OF SPHERES

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local degree, proof of fundamental thm of algebra

8. CELLULAR HOMOLOGY

CW complexes, cellular complex, rank $H_n^{CW} \leqslant \# \text{n-cells}$

$H_*^{CW}(\mathbb{D}^1 \times \mathbb{D}^1)$, $H_*^{CW}(\mathbb{RP}^n)$, $H_*^{CW}(S^n)$, $H_*^{CW}(\Sigma_g)$

Δ-cx $\Rightarrow$ CW cx, $H_*^{CW}(X) \cong H_*^{\Delta}(X) \cong H_*(X)$, Axioms for homology

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functoriality, homotopy invariance, cochain homotopy, dual of a SES

excision, LES, Mayer-Vietoris for H^*, axioms for cohomology

10. CUP PRODUCT

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examples: $H^*(T^2)$, $H^*(\Sigma_2)$, remarks about intersection theory

11. KÜNNETH FORMULA AND PRODUCT SPACES

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algebraic Künneth thm, product spaces $X \times Y$, Euler characteristic χ

CW-cx for product space, Künneth thm, $H^*(S^n \times S^m)$, $H^*(T^n)$

12. UNIVERSAL COEFFICIENTS THEOREM

Universal coeff. thm, Background on Ext groups and free resolutions

(Co)homology with coefficients in a ring/field/module, $H^*(\mathbb{RP}^2; \mathbb{Z}/2)$

Univ. coeff. thm for PID R, Duality $H^*(X;\mathbb{F}) \simeq H_*(X;\mathbb{F})$ over fields

Structure thm for f.g. mods M over PID R, $\mathrm{Ext}^1_R(M;R)$, torsion shift H_* to H^{*+1}

13. MANIFOLDS : POINCARÉ-LEFSCHETZ DUALITY

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Alexander duality, Knot complements, Jordan curve thm

0. OVERVIEW OF THE COURSE

Motivation

Space $X \xrightarrow{\text{associate}}$ Algebraic object $A(X)$ like numbers, groups, rings, ...

Isomorphism of spaces $X \cong Y$ $\Longrightarrow$ Isomorphism $A(X) \cong A(Y)$

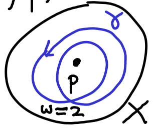
PAY-OFF Can tell spaces apart:
compute $A(X), A(Y) \rightsquigarrow$ if $A(X) \not\cong A(Y)$ then $X \not\cong Y$

Examples

1) Set $X \longrightarrow A(X) = \# X \in \mathbb{N}$
(bijection $X \to Y$) $\Longrightarrow$ same size

2) Vector space $X \longrightarrow A(X) = \dim X \in \mathbb{N}$
(linear iso $X \to Y$) $\Longrightarrow$ same dim

3) Topological Space X
$\longrightarrow \# \pi_0(X) = \#$ **path components**
$\longrightarrow$ **# connected components**
$\longrightarrow \chi(X) =$ **Euler characteristic**
$\Big\} \in \mathbb{N}$
$\longrightarrow$ Function $X \times \underbrace{\mathcal{L}X}_{\text{loops} = C^0(S^1, X)} \longrightarrow \mathbb{Z}$
$(p, \gamma) \longmapsto w(\gamma; p)$
Winding number of γ around p.



(Homeomorphism $X \to Y$) $\longrightarrow A(X) = A(Y)$

CONVENTION for this course:

Space = topological space

Map of spaces = continuous map = cts map

For spaces, "$\cong$" means homeomorphism

"id" = identity map

All diagrams commute unless we say otherwise, e.g.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

means $\beta \circ \alpha = \delta \circ \gamma$

Category theory is the best language to phrase all this

Algebraic topology builds functors such as

$$\begin{tikzcd} & \left\{\begin{array}{l}\text{Groups}\\ \text{and group homs}\end{array}\right\} \\ F : \left\{\begin{array}{l}\text{Topological spaces}\\ \text{and continuous maps}\end{array}\right\} \arrow[ur] \arrow[r] \arrow[dr] & \left\{\begin{array}{l}\text{Abelian groups}\\ \text{and group homs}\end{array}\right\} \\ & \left\{\text{Rings and ring homs}\right\} \end{tikzcd}$$

We will not use much category theory, just basic terminology:

Def A **category** C consists of the data:

$Ob(C)$ = a collection of **objects**

$Hom(A,B)$ = a set of **morphisms** between any $A, B \in Ob C$ ("**arrows**")

- with **composition** rule $Hom(B,C) \times Hom(A,B) \xrightarrow{\circ} Hom(A,C)$ which is associative.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

- with **identity** morphs $id_A \in Hom(A,A)$ s.t. $f \circ id_A = id_B \circ f = f$ $\forall (f: A \to B) \in Hom(A,B)$

Example
Sets = { sets with all maps between sets }
Top = { topological spaces with continuous maps }
Gps = { groups with group homs }

Def A (covariant) **functor** $F : C_1 \longrightarrow C_2$ is the data:

- an assignment $(A \in Ob\, C_1) \longmapsto (F(A) \in Ob\, C_2)$
- an assignment $(\underset{\overset{\cap}{Hom_{C_1}(A,B)}}{A \xrightarrow{f} B}) \longmapsto (\underset{\overset{\cap}{Hom_{C_2}(F(A),F(B))}}{F(A) \xrightarrow{F(f)} F(B)})$

compatible with identities and compositions.

$F(id_A) \overset{\nearrow}{=} id_{F(A)}$ $F(g \circ f) \overset{\nearrow}{=} F(g) \circ F(f)$

A **contravariant functor** is defined similarly except it reverses the direction of arrows: $(F(A) \xleftarrow{F(f)} F(B)) \in Hom(F(\underline{B}), F(\underline{A}))$
(so $F(g \circ f) = F(f) \circ F(g)$ reverses order of compositions)

Examples

1) $F : Top \longrightarrow Sets$, $A \longmapsto A$, $f \longmapsto f$ "forget the topology and continuity"

2) $F : Sets \longrightarrow Gps$, $A \longmapsto$ free abelian group generated by A

$$\mathbb{Z}\langle A\rangle = \left\{ \sum_{\text{finite}} n_i \cdot a_i \; : \; \begin{array}{l} a_i \in A \\ n_i \in \mathbb{Z} \end{array} \right\}$$

$$(A \xrightarrow{f} B) \longmapsto \left(\begin{array}{l} F(A) : \mathbb{Z}\langle A\rangle \longrightarrow \mathbb{Z}\langle B\rangle \\ \quad \sum n_i \cdot a_i \longmapsto \sum n_i \cdot f(a_i) \end{array} \right)$$

When we say a construction is **natural** we mean functorial:

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[d, "f"] \arrow[dd, bend right, "g\circ f"'] & A(X) \arrow[d, "A(f)"] \arrow[dd, bend left, "A(g\circ f) = A(g)\circ A(f)"] \\ Y \arrow[r, "A"] \arrow[d, "g"] & A(Y) \arrow[d, "A(g)"] \\ Z \arrow[r, "A"] & A(Z) \end{tikzcd}$$

A : (a category of spaces) → (a cat. of algebraic objects)
The algebraic objects we assigned are assigned compatibly with maps of spaces, and the compatibility maps $A(f)$ are also compatible w.r.t. composition.
So we made compatible choices in constructing A.

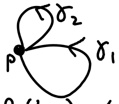
Not to be confused with natural transformations of functors (later) which is about relating two such constructions A_1, A_2 in a compatible way

Example of a functor in algebraic topology (see B3.5 Topology and Groups course)

$\underset{\substack{\uparrow \\ \text{topological space}}}{\pi_1(X}, p) = $ **Fundamental group** $= \left\{ \begin{array}{l} \gamma \in \mathcal{L}X \\ \gamma(1) = p \end{array} \right\} \Big/$ continuous deformations of loops based at p

$p \in X$

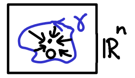
Group multiplication: concatenate loops $\gamma_1 * \gamma_2$



(each travelling twice as fast)

Examples

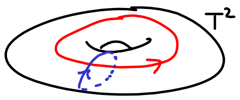
$\pi_1(\mathbb{R}^n) = 0$ ← deform: $h : S^1 \times [0,1] \longrightarrow \mathbb{R}^n$, $h(t,s) = (1-s)\gamma(t)$



$\pi_1(S^1) \cong \mathbb{Z}$ ← total # times wind around circle

$\pi_1(S^n) \cong 0$ $n \geq 2$ (not obvious)

$\pi_1(\text{torus}) \cong \mathbb{Z}^2$

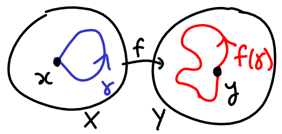


those loops generate π_1

FUNCTOR

$$\text{Based Top} = \left\{ \begin{array}{l} \text{Topological spaces with choice of basepoint,} \\ \text{and continuous basepoint-preserving maps} \end{array} \right\} \xrightarrow{\pi_1} Gps$$

$$(X, p) \longmapsto \pi_1(X, p)$$



$$((X,x) \xrightarrow[f]{\text{cts}} (Y,y)) \longmapsto \left(\begin{array}{l} \pi_1(X,x) \xrightarrow{\text{gp. hom.}} \pi_1(Y,y) \\ \quad \gamma \longmapsto f \circ \gamma \end{array} \right)$$

Lemma Functors map isomorphisms to isomorphisms (iso. means ∃ inverse w.r.t. composition)

Pf $\underbrace{A\xrightarrow{f}B\xrightarrow{g}A}_{id} \Rightarrow \underbrace{FA\xrightarrow{Ff}FB\xrightarrow{Fg}FA}_{F(id)=id}$, similarly for $\underbrace{B\xrightarrow{g}A\xrightarrow{f}B}_{id}$. □

Def **Natural transformation** $\alpha: F\longrightarrow G$ between functors $\mathcal{C}_1 \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{C}_2$ is an association $(A\in \mathrm{Ob}\,\mathcal{C}_1)\longmapsto(\alpha_A : F(A)\longrightarrow G(A))$ $\in \mathrm{Hom}_{\mathcal{C}_2}(F(A),G(A))$

such that $(A\xrightarrow{f}B)$, $\in \mathrm{Hom}_{\mathcal{C}_1}(A,B)$ $\Rightarrow$

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\alpha_B"] & G(B)
\end{tikzcd}$$

(commutes)

It is called a **natural isomomorphism** if each α_A is an isomorphism in $\mathcal{C}_1$

Example of a natural transformation in algebraic topology

Let $H_1(X,p)$ = abelianisation of $\pi_1(X,p)$ (want to identify $ab=ba$ so quotient by $\langle aba^{-1}b^{-1}\rangle$) ← **commutators**

$\Rightarrow$ natural trans. $(\text{Based Top}\xrightarrow{\pi_1}\text{Gps})\xrightarrow{\alpha}(\text{Based Top}\xrightarrow{H_1}\text{Gps})$

which associates $\underset{\in\text{Based Top}}{(X,p)}\longmapsto(\alpha_{(X,p)} : \pi_1(X,p)\xrightarrow{\text{quotient}}H_1(X,p))$

Cultural Rmk **higher homotopy groups** $\pi_n(X,p)=\left\{\begin{matrix}S^n\xrightarrow{\text{cts}}X\\ \text{basept}\mapsto p\end{matrix}\right\}\Big/\text{cts deform}^n$

FACT abelian for $n\geq 2$, but hard: e.g. $\pi_k(S^n)$ not all known.

We will not study these in this course.
We will study simpler invariants called HOMOLOGY groups $H_n(X)$

FACT (Hurewicz) ∃ natural transformation $\pi_n\longrightarrow H_n$
which will make sense at the end of course:
$f: S^n\xrightarrow{\text{cts}}X$ gives rise to a class $f_*[S^n]\in H_n(X)$.

Exercise to practice these notions from category theory:
Summarise your undergraduate linear algebra as follows:

1) ∃ functor F: $\underbrace{\left\{\begin{matrix}\mathbb{R}^n \text{ for } n\in\mathbb{N}\\ \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)=\left\{\begin{matrix}m\times n\\ \text{matrices}\end{matrix}\right\}\end{matrix}\right\}}_{\text{Mat}} \to \underbrace{\left\{\begin{matrix}\text{finite dimensional real vector spaces}\\ \text{and linear maps}\end{matrix}\right\}}_{\text{Vect}}$

2) A choice of basis for each vector space V determines a functor $G:\text{Vect}\to\text{Mat}$

3) Construct natural isomorphisms $G\circ F\xrightarrow{\alpha}\mathrm{Id}_{\text{Mat}}$, $F\circ G\xrightarrow{\beta}\mathrm{Id}_{\text{Vect}}$

When functors satisfying such natural isos exist, the categories are called **equivalent** (not isomorphic). So Mat, Vect are equivalent categories.

Aim of the course: build a functor

HOMOLOGY H_* : Top $\longrightarrow$ **Graded abelian groups**

$X \longmapsto H_*(X)$ ← grading $*\in\mathbb{Z}$

$(X\to Y)\longmapsto \left(\begin{matrix}H_*(X)\longrightarrow H_*(Y)\\ \text{grading preserving hom}\end{matrix}\right)$

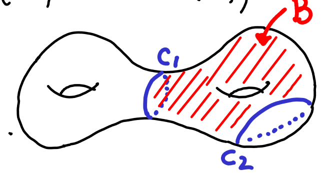
and a contravariant functor

COHOMOLOGY H^* : Top $\longrightarrow$ **Graded rings**

$X\longmapsto H^*(X)$

$(X\to Y)\longmapsto (H^*(X)\longleftarrow H^*(Y))$

Rough idea:
H_*X is generated by "nice" subspaces $C\subseteq X$ which have no boundary: $\partial C=\emptyset$, modulo identify C_1,C_2 if $C_1\cup C_2$ arises as a boundary ∂B.
Call such C_1,C_2 **homologous**.



Facts

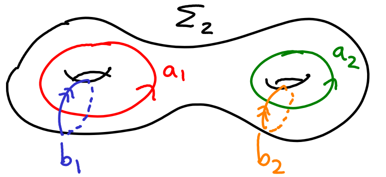
- $H_0(X)\cong\bigoplus_{\pi_0X}\mathbb{Z}$ ← π_0X = {path-connected components} ← generated by a point in each path-comp.
- $X=\bigsqcup X_i$ path-components $\Rightarrow H_*(X)\cong\bigoplus H_*(X_i)$
- $\chi(X) = \sum_{d\geq0}(-1)^d \operatorname{rank} H_d(X)$

↑ **Euler characteristic** ↑ max # $\mathbb{Z}$-linearly independent elements

Example: compact surfaces

$H_*(\Sigma_g) \cong \begin{cases}\mathbb{Z} & *=0\\ \mathbb{Z}^{2g} & *=1\\ \mathbb{Z} & *=2\\ 0 & \text{else}\end{cases}$

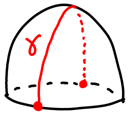
↑ orientable surface genus g
$\chi=2-2g$



We will show that those 4 loops generate $H_1(\Sigma_2)$

$H_*(N_h) \cong \begin{cases}\mathbb{Z} & *=0\\ \mathbb{Z}/_2\oplus\mathbb{Z}^{h-1} & *=1\\ 0 & \text{else}\end{cases}$

↑ non-orientable surface S^2 with h Möbius bands attached
$\chi=2-h$



$N_1=\mathbb{RP}^2 = S^2/\pm\mathrm{Id}$

Notice γ is a loop.
It generates $H_1(N_1)$

Examples of homology calculations

$$H_*(\mathbb{R}^n) \cong H_*(\mathbb{D}^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ 0 & \text{else} \end{cases}$$

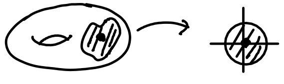
n-dimensional ball

$\mathbb{D}^n = \{x : \|x\| \leq 1\} \subseteq \mathbb{R}^n$

$$H_*(S^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$$

$\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ n-dim sphere

Hausdorff top. space s.t. each pt has an open neighbourhood homeo to an open ball in $\mathbb{R}^n$



$$H_*(X) = \begin{cases} 0 & \text{for } * < 0 \\ 0 & \text{for } * > n \\ \mathbb{Z} & \text{for } * = n \\ 0 & \text{for } * = n \end{cases}$$

for **n-dimensional manifolds** X

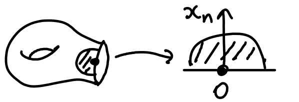
for connected orientable compact manifold

for non-orientable "

non-compact "

connected **manifolds with boundary** $\neq \emptyset$

boundary point has an open nbhd homeo to open nbhd of $0 \in$ half-space: $\{x \in \mathbb{R}^n : x_n \geq 0\}$

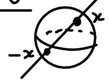


Rmk M compact connected n-mfd

$$\Rightarrow H_{n-1}(M) \cong \begin{cases} \mathbb{Z}^k & \text{some } k \geq 0 \text{ if orientable} \\ \mathbb{Z}^k \oplus \mathbb{Z}/2 & \text{" " non-orientable} \end{cases}$$

$$H_*(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/2 & \text{odd } * = 1,3,5,\dots < n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

$S^n/\pm id$ real projective space



$\mathbb{R}P^n$ orientable $\Leftrightarrow$ n odd (e.g. $\mathbb{R}P^1 \cong S^1$)

$$H_*(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & \text{even } * = 0,2,4,\dots,2n \\ 0 & \text{else} \end{cases}$$

e.g. $\mathbb{C}P^1 \cong S^2$ stereographic projection

space of complex lines through $0 \in \mathbb{C}^{n+1}$

Complex projective space

$$\cong (\mathbb{C}^n \setminus 0)/\mathbb{C}^*\text{-rescaling}$$
$$= \{[z_0, z_1, \dots, z_n] : z_j \in \mathbb{C} \text{ not all } 0\} / [z] = [\lambda z] \text{ for } \lambda \in \mathbb{C}^*$$

Examples of cohomology calculations

$H^0(X) = \prod_{\pi_0 X} \mathbb{Z} \leftarrow$ if $\pi_0 X$ finite, then $\cong \bigoplus_{\pi_0 X} \mathbb{Z} \cong H_0 X$ but if infinite then not: here allow only finite sums

$H^*(X) \cong \prod H^*(X_i) \leftarrow X_i$ path-components of X

FACT

If $H_n(X)$ finitely generated abelian gp, so
$H_n(X) \cong \mathbb{Z}^{r_n} \oplus T_n \leftarrow T_n$ = torsion elements = elements of finite order
Then $H^n(X;R) \cong \mathbb{Z}^{r_n} \oplus \underline{\underline{T_{n-1}}}$ as abelian groups

$\Rightarrow H^*(\Sigma_g), H^*(\mathbb{R}^n), H^*(\mathbb{D}^n), H^*(S^n), H^*(\mathbb{C}P^n)$ same as for H_*, but:

$$H^*(N_h) = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}^{h-1} & *=1 \\ \mathbb{Z}/2 & *=2 \\ 0 & \text{else} \end{cases} \qquad H^*(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & *=0 \\ 0 & *=1 \\ \mathbb{Z}/2 & *=2 \\ 0 & \text{else} \end{cases} (h=1) \qquad H_*(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/2 & * \text{ even} = 2,4,\dots \leq n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

and $H^n(\text{non-orientable compact n-mfd}) \cong \mathbb{Z}/2$.

$\Rightarrow$ The interesting feature is the ring structure:

$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/x^{n+1}$ grading: $|x| = 2$

$\mathbb{Z}[x]$ = polynomials in x with $\mathbb{Z}$-coefficients

$H^*(S^n) \cong \mathbb{Z}[x]/x^2$ $|x| = n$

$H^*(T^n) \cong \Lambda[x_1, \dots, x_n]$ $|x_i| = 1$

$T^n = S^1 \times \dots \times S^1$ **n-torus**

e.g. n=2 (torus) $\begin{cases} \mathbb{Z} \cdot 1 & *=0 \\ \mathbb{Z}x_1 \oplus \mathbb{Z}x_2 & *=1 \\ \mathbb{Z}\, x_1 \wedge x_2 & *=2 \\ 0 & \text{else} \end{cases}$

exterior algebra generated$/\mathbb{Z}$ by symbols $x_{i_1} \wedge \dots \wedge x_{i_k}$ with $i_1 < \dots < i_k$ product given by $\wedge$ using relations $x_i \wedge x_j = -x_j \wedge x_i$.

$H^*(\mathbb{R}P^{2n}) \cong \mathbb{Z}/2[x]/x^{n+1}$ $|x| = 2$

$H^*(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}/2[x]/x^{n+1} \oplus \underbrace{\mathbb{Z}[-2n-1]}$ means: a copy of $\mathbb{Z}$ in degree $2n+1$

$H^*(\Sigma_g) \cong \Lambda[a_1, \dots, a_g, b_1, \dots, b_g] / \langle a_i b_j \text{ for } i \neq j,\ a_i b_i - a_j b_j,\ a_i a_j,\ b_i b_j \rangle$

$|a_i| = |b_i| = 1$ exterior alg. instead of poly. alg since $a_i b_i = -b_i a_i$

Why more information?

connected sum: remove a ball in each, glue along ∂ball

$S^2 \times S^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ have same $H_* = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} \oplus \mathbb{Z} & *=2 \\ \mathbb{Z} & *=4 \end{cases}$

but the rings H^* are not iso, hence $S^2 \# S^2 \ncong \mathbb{C}P^2 \# \mathbb{C}P^2$.

Example of why such functors are useful

Theorem **Invariance of dimension**

Suppose $F_* : \text{Top} \to \text{Gps}$ functors s.t.
$F_*(\mathbb{D}^n) = 0$ for all $*$
$F_*(S^n) \neq 0$ only for $* = n$

Then: $S^n \cong S^m \iff n=m$
$\mathbb{R}^n \cong \mathbb{R}^m \iff n=m$

"homeomorphisms preserve dimension"
Non-trivial result because there are space-filling curves.
e.g. Peano (1890) $\exists$ cts surjection $\underset{\text{interval}}{[0,1]} \to \underset{\text{square}}{[0,1]^2}$
The theorem implies this is not injective.
(cts. bij. compact → Hausdorff $\Rightarrow$ homeo)

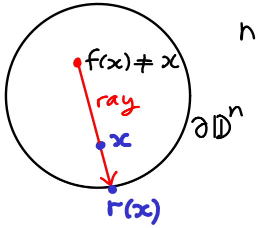
Pf By Lemma, $F_n(S^n \xrightarrow{\cong} S^m)$ is an iso $\underset{\mathbb{Z}}{F_n(S^n)} \xrightarrow{\cong} F_n(S^m)$ of gps.
$\nwarrow$ 0 if $n \neq m$ □

We will build such F_*:
reduced homology $\tilde{H}_*$ s.t. $\begin{cases} \tilde{H}_* = H_* & \text{for } * \neq 0 \\ \tilde{H}_0(X) \cong \mathbb{Z}^{\#(\text{path-conn. components}) - 1} \end{cases}$

Theorem **Brower fixed point thm**

$f : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ continuous $\Rightarrow f$ has a fixed point $(f(p) = p$ some $p)$

Proof Suppose not. Let $r(x) = (\text{ray from } f(x) \text{ to } x) \cap \partial\mathbb{D}^n$



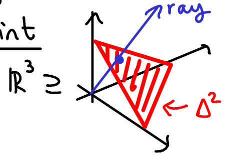
notice: • $r : \mathbb{D}^n \longrightarrow \partial\mathbb{D}^n = S^{n-1}$ continuous
• $r|_{\partial\mathbb{D}^n} = id_{S^{n-1}}$

$$S^{n-1} = \partial\mathbb{D}^n \xrightarrow{\text{inclusion } i} \mathbb{D}^n \xrightarrow{r} S^{n-1} \qquad r \circ i = id$$

$\underset{\text{apply } F_{n-1}}{\Longrightarrow} F_{n-1}(r) \circ F_{n-1}(i) = id \Longrightarrow F_{n-1}(i)$ injective $\underset{\cong\mathbb{Z}}{F_{n-1}(S^{n-1})} \to \underset{=0}{F_{n-1}(\mathbb{D}^n)}$ ↯ □

Example (Brower) $A = n \times n$ matrix, $A_{ij} > 0$ real $\Longrightarrow \exists$ evalue $\lambda > 0$ with real evector $(v_1, \ldots, v_n)$ with $v_i \geq 0$

Hint



$X = \{\text{rays in "positive octant"}\}$ $x \in \mathbb{R}^n : x_i > 0 \ \forall i$
notice $AX \subseteq X$
notice $X \cong \Delta^n = \{x \in \text{octant} : \Sigma x_i = 1\} \cong \mathbb{D}^n$
ray $\mapsto$ ray $\cap\, \Delta^n$

1. ALGEBRAIC PRELIMINARIES : CHAIN COMPLEXES

Graded abelian groups

Def A **$\mathbb{Z}$-graded abelian group** C is an abelian group together with a direct sum decomposition

$$\boxed{C = \bigoplus_{n \in \mathbb{Z}} C_n} \leftarrow \text{abelian group}$$

Convention: always grade by $\mathbb{Z}$ unless say otherwise.

A graded ab. gp. A is a **graded subgp** of C if • subgp • $A_n \subseteq C_n$.

A **homomorphism** $h : C \to D$ of gr. ab. gps is hom of gps s.t. $h(C_n) \subseteq D_n$

A **hom of degree k** is hom with $h(C_n) \subseteq D_{n+k}$

Shift by k: $\mathbb{Z}$-gr. ab. gp. $C[k]$ with

$$\boxed{C[k]_n = C_{k+n}}$$

Notice: $C[k]_0 = C_k$ is now in degree zero, so shifted down by k

$\Rightarrow$ Can view gr. hom of deg k as a gr. hom $h : C \longrightarrow D[k]$

FACT Finitely generated abelian groups are classified:

$$G \cong \mathbb{Z}^r \oplus \underbrace{\mathbb{Z}/_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}/_{p_k^{n_k}}}$$

free part — called **rank G** — **torsion part**

$n_i \in \mathbb{Z}$
p_i primes (possibly not distinct)

compare finite dimensional vectorspaces/field $\mathbb{F}$: $V \cong \mathbb{F}^r$ $\leftarrow r = \dim V$

Chain complexes

Def A **chain complex** (C_*, ∂_*) is a gr. ab. gp. C together with a hom ∂ of degree -1 such that $\partial \circ \partial = 0$.

∂_*: **differential** or **boundary homomorph**

Thus:

$$\boxed{\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[rr, bend right, "\partial_n \circ \partial_{n+1} = 0"'] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r] & \cdots \end{tikzcd}}$$

hence $\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n$

$\operatorname{Im} \partial_{n+1} \overset{..}{=} B_n$ **n-boundaries**, $\operatorname{Ker} \partial_n \overset{..}{=} Z_n$ **n-cycles**

n-chains = elements of C_n

Now consider "cycles modulo boundaries":

Def The **homology** of (C_*, ∂_*) is the gr. ab. gp.

$$\boxed{H_n(C_*, \partial_*) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}}$$

often abbreviate by $H_*(C)$

Two cycles are called **homologous** if they differ by a boundary

Def A **chain map** $h : (C_*, \partial_*) \longrightarrow (\widetilde{C}_*, \widetilde{\partial}_*)$ is a graded hom such that $\boxed{h \circ \partial_* = \widetilde{\partial}_* \circ h}$

Example A chain **subcomplex** $C_* \subseteq \widetilde{C}_*$ is a graded subgp with $\partial_* =$ restriction of $\widetilde{\partial}_*$ to C_*.

So the inclusion $\operatorname{incl} : (C_*, \partial_*) \longrightarrow (\widetilde{C}_*, \widetilde{\partial}_*)$ is a chain map.

Also get **quotient complex** $\widetilde{C}_*/C_*$

with $\widetilde{\partial}_* [\widetilde{c}] = [\widetilde{\partial}_* \widetilde{c}]$ (well-defined: $\widetilde{\partial}_* C_* = \partial_* C_* \subseteq C_*$)

Lemma A chain map induces a hom on homology

$$H_*(h) = h_* : H_*(C_*, \partial_*) \longrightarrow H_*(\widetilde{C}_*, \widetilde{\partial}_*)$$

$$[x] \longmapsto [h(x)]$$

Proof $\operatorname{Ker} \partial_n \xrightarrow{\text{hom}} \operatorname{Ker} \widetilde{\partial}_n$

$x \longmapsto h(x)$ since $\widetilde{\partial}(h(x)) = h(\underbrace{\partial x}_{=0}) = 0$

Need $\operatorname{Im} \partial_n \longrightarrow \operatorname{Im} \widetilde{\partial}_n$ to get well-defined hom

$$H_n(C) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_n \longrightarrow \operatorname{Ker} \widetilde{\partial}_n / \operatorname{Im} \widetilde{\partial}_n = H_n(\widetilde{C})$$

$\Longrightarrow$ Need $h(b) = \widetilde{\partial}(\text{something})$.

$\uparrow$ boundary $b = \partial c$

Proof by "diagram chasing":

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d] & C_n \arrow[r, "\partial_n"] \arrow[d, "h_n"] & C_{n-1} \arrow[r] \arrow[d, "h_{n-1}"] & \cdots \\ \cdots \arrow[r] & \widetilde{C}_{n+1} \arrow[r, "\widetilde{\partial}_{n+1}"'] & \widetilde{C}_n \arrow[r, "\widetilde{\partial}_n"'] & \widetilde{C}_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} c \arrow[r, "\partial"] \arrow[d, "h"'] & \partial c = b \arrow[d, "h"] \\ hc \arrow[r, "\widetilde{\partial}"] & \widetilde{\partial}(hc) = h\partial c = h(b) \end{tikzcd}$$

$\square$

Def (C_*, ∂_*) is **exact** (or **acyclic**) if $H_*(C) = 0$

so $\operatorname{Im} \partial_{n+1} = \operatorname{Ker} \partial_n$

In general **exact sequence** means $\boxed{\text{"Im(previous map) = Ker (next map)"}}$

A **short exact sequence (SES)** is an exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

Easy exercise

$$\left(0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0\right) \Leftrightarrow \begin{cases} i & \text{injective} \\ \pi & \text{surjective} \\ B/_{i(A)} \cong C \text{ via } [b] \longmapsto \pi(b) \end{cases}$$

exact

Examples

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r, "\text{mod } 2"] & \mathbb{Z}/_2 \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\text{inclusion}"] & \mathbb{Z} \oplus \mathbb{Z}/_2 \arrow[r, "\text{project}"] & \mathbb{Z}/_2 \arrow[r] & 0 \end{tikzcd}$$

Note A, C do not determine B.

Snake Lemma A SES of chain complexes and chain maps yields a **long exact sequence (LES) on homology**:

$$\boxed{\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{\pi_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*[-1]} \cdots}$$

degree -1 map $H_*(C) \to H_*(A)[-1]$ called **connecting map**

(So **exact triangle**:
$$\begin{tikzcd} H_*(A) \arrow[rr] & & H_*(B) \arrow[dl] \\ & H_*(C) \arrow[ul, "{[-1]}"] & \end{tikzcd}$$
)

Pf simplify notation by identifying A with $i(A) \subseteq B$: $\overset{\in A}{a} \equiv \overset{\in B}{i(a)}$, $A \overset{i}{\hookrightarrow} B$, $\partial a \equiv i \partial a = \partial i a$

$\Longrightarrow$ now $A_* \subseteq B_*$ inclusion of subcomplex:

$$0 \longrightarrow (A_*, \partial_*) \xrightarrow{\text{incl}} (B_*, \partial_*) \xrightarrow{\pi} (C_*, \tilde{\partial}_*) \longrightarrow 0$$

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0 \\ & & \exists b \arrow[r, "\text{surj.}"] \arrow[d] & \underset{c}{\text{cycle}} = \pi(b) \arrow[d] & \\ & \partial b \arrow[r] & \partial b \arrow[r] & \tilde{\partial} c = 0 & \end{tikzcd}$$

$\nwarrow$ lifts to A by exactness

Define $\delta : H_*(C) \longrightarrow H_*(A)[-1]$
$c \longmapsto \partial b$ $\nwarrow$ where $b \in \pi^{-1}(c)$

(typically b is not in A, so $\underline{\partial b \text{ need not be a bdry in } A}$)

Well-defined?
- $\pi^{-1}(c) = \{b + a : a \in A\}$ and $\partial(b+a) = \partial b + \underbrace{\partial a}_{\substack{\text{boundary} \\ \text{in } A}}$ ✓
- cycle $\to$ cycle : $\partial(\partial b) = 0$ ✓
- boundary $\to$ boundary :

$$\begin{tikzcd} \exists \beta \arrow[r, "\text{surj.}"] \arrow[d] & x \in C_{n+1} \arrow[d] \\ \partial \beta \arrow[r] & \text{boundary } c = \tilde{\partial} x \arrow[d] \\ & 0 \end{tikzcd}$$

$\Rightarrow$ can pick $b = \partial \beta$

$\Rightarrow \partial b = \partial \partial \beta = 0$ ✓

Exactness at $H_n(C)$ (exercise : check exactness at H_*A, H_*B):

Need $\operatorname{Im} \pi_* = \operatorname{Ker} \delta$:

$\subseteq$: $\delta(\pi_* \underset{\text{cycle}}{b}) = \partial b = 0$ ✓

$\supseteq$:

$$\begin{tikzcd} \exists a \arrow[dd] & & \overset{\text{not necessarily cycle!}}{b} \arrow[r] \arrow[dd] & \overset{\in \operatorname{Ker} \delta}{c} = \pi_* b \arrow[dd] \\ & & & \\ \partial a \overset{\text{assumption } \delta c = 0 \in H_* A}{=} \delta c = \partial b \arrow[rr] & & \partial b \arrow[r] & 0 \end{tikzcd}$$

$\pi_*(b-a) \overset{\pi_* A = 0}{=} c$

$\underbrace{\partial(b-a)}_{\text{thus cycle!}} = \partial b - \partial a = 0$

$\Rightarrow c = \pi_*(b-a) \in \operatorname{Im} \pi_*$ ☐

Rmk $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ SES $\Rightarrow$ the **connecting map of LES** is

$$\boxed{\begin{aligned} \delta : H_*(C) &\longrightarrow H_*(A)[1] \\ c &\longmapsto i^{-1}(\partial b) \end{aligned}}$$

$\forall b \in B$ with $\pi(b) = c$.

Lemma The construction of δ is natural (i.e. functorial)

Pf

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "\pi"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & \tilde{A} \arrow[r, "\tilde{i}"] & \tilde{B} \arrow[r, "\tilde{\pi}"] & \tilde{C} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} \overset{=\delta c}{a} \arrow[r] \arrow[d, "f"] & \partial b \arrow[d, "g"] \\ fa \arrow[r] & g\partial b \end{tikzcd} \qquad \begin{tikzcd} b \arrow[r] \arrow[d, "g"] & c \arrow[d, "h"] \\ gb \arrow[r] & hc \end{tikzcd}$$

$$\Rightarrow \delta h c = \tilde{i}^{-1} \tilde{\partial} g b = \tilde{i}^{-1} g \partial b = f a = f \delta c$$ ☐

Exercise Deduce the LES is natural, so

$$\begin{tikzcd} \cdots \arrow[r] & H_* A \arrow[r, "i_*"] \arrow[d, "f_*"] & H_* B \arrow[r, "\pi_*"] \arrow[d, "g_*"] & H_* C \arrow[r, "\delta"] \arrow[d, "h_*"] & H_{*-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\ \cdots \arrow[r] & H_* \tilde{A} \arrow[r] & H_* \tilde{B} \arrow[r, "\tilde{\pi}_*"] & H_* \tilde{C} \arrow[r, "\delta"] & H_{*-1}(\tilde{C}) \arrow[r] & \cdots \end{tikzcd}$$

5-Lemma

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha", "\cong"'] & B \arrow[r] \arrow[d, "\beta", "\cong"'] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta", "\cong"'] & E \arrow[d, "\varepsilon", "\cong"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

exact rows $\Longrightarrow$ γ also iso.

Pf exercise (diagram chase) $\square$

Splitting Lemma

Cor $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ SES of abelian gps

If $B \underset{\exists\gamma}{\overset{\beta}{\rightleftarrows}} C$ s.t. $\beta \circ \gamma = id_C$ then the SES **splits**: $B \cong A \oplus C$

(converse is obvious)

Pf

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A\oplus C \arrow[r] \arrow[d, "\alpha+\gamma"] & C \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

$\square$

Exercise If $A \underset{\exists\mu}{\overset{\alpha}{\rightleftarrows}} B$ s.t. $\mu \circ \alpha = id_A$ then it splits: $B \xrightarrow[\mu\oplus\beta]{\cong} A \oplus C$

Exercise If C is a free abelian group $(C \cong \bigoplus_{i\in I} \mathbb{Z})$ then the SES splits.

Rmk A free $\not\Rightarrow$ splits, e.g. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \longrightarrow 0$

Cultural Rmk Splitting Lemma generalises the **rank-nullity theorem** from linear algebra: $V \xrightarrow{\beta} W$ linear map of vector spaces $\Rightarrow \mathrm{Im}\beta \oplus \mathrm{Ker}\beta \cong V$

Pf $0 \to \mathrm{Ker}\,\beta \xrightarrow{incl} V \xrightarrow{\beta} \mathrm{Im}\,\beta \to 0$ is SES, and splits since $\mathrm{Im}\,\beta$ free.

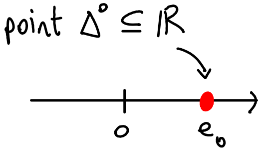
2. Δ-COMPLEXES AND SIMPLICIAL HOMOLOGY

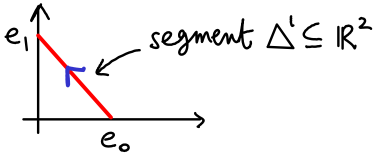
Standard n-simplex $\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \begin{array}{l} t_i \geq 0 \\ \sum t_i = 1 \end{array} \right\}$

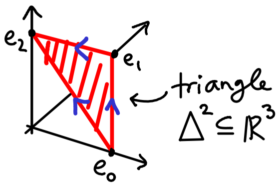
$(t_0, \dots, t_n) = \sum t_i e_i$

$e_0, \dots, e_n$ standard basis of $\mathbb{R}^{n+1}$ $(e_0 = (1,0,\dots,0), \dots$

Examples







Def For $\{v_0, \dots, v_n\} \subseteq \mathbb{R}^{n+k}$ s.t. ← any $k \geq 0$

$v_1 - v_0, \dots, v_n - v_0$ $\mathbb{R}$-linearly independent

$[v_0, \dots, v_n]$ = **n-simplex** spanned by $v_0, \dots, v_n$

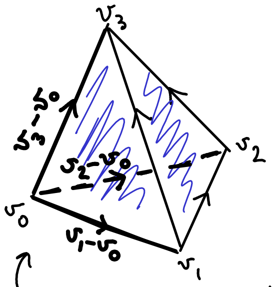
= convex hull of $v_0, \dots, v_n$

$= \{ \sum t_i v_i : t_i \geq 0$ and $\sum t_i = 1\}$

= Image of linear homeo $\sigma : \Delta^n \longrightarrow \mathbb{R}^{n+k}$

$\sigma(e_i) = v_i$

canonical homeomorphism



(Solid prism: includes inside)

Will often blur the distinction between map σ and its image,

$\sigma \equiv [\sigma e_0, \dots, \sigma e_n]$

but the ordering of the v_i will be important (so the map σ is more precise)

We encode this extra data by orienting the edges $v_i \bullet\!\!\longrightarrow\!\!\bullet v_j$ if $i<j$

Def **d-dimensional faces** $[v_{i_0}, \dots, v_{i_d}]$ for $i_0 < \dots < i_d$

Example 0-dim faces are the vertices $v_0, \dots, v_n$

facets = (n−1)-dimensional faces

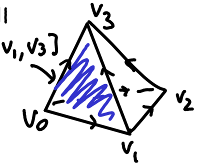
$= [v_0, \dots, \hat{v}_k, \dots, v_n]$ where we omit v_k

$= \{ \sum t_i v_i \in [v_0, \dots, v_n] : t_k = 0\}$

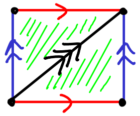
$= \text{Image } \sigma|_{\Delta_k^{n-1}} : \Delta_k^{n-1} \longrightarrow \mathbb{R}^{n+k}$

$\Delta_k^{n-1} = \{t \in \Delta^n : t_k = 0\}$

$[v_0, v_1, \hat{v}_2, v_3] = [v_0, v_1, v_3]$



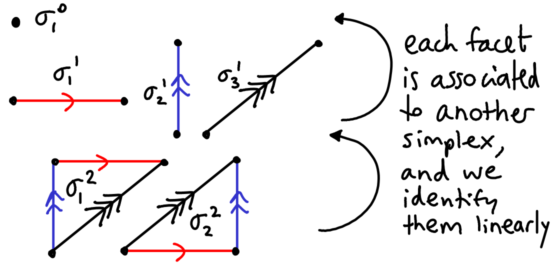
Example Can build a torus out of simplices:

$T^2 =$ 

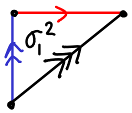
1 0-simplex $\bullet\ \sigma_1^0$

3 1-simplices σ_1^1, σ_2^1, σ_3^1

2 2-simplices σ_1^2, σ_2^2



$T^2 =$ quotient space of $\bigsqcup \sigma_i^n$ / canonical homeos associated to the facets

for example identify facet  of  with σ_2^1  via linear homeo (orientation-preserving)

Def Δ-**complex** is determined by data

- indexing set I_n, for each $n \in \mathbb{N}$
- choice of n-simplex σ_α^n (not necessarily standard) for each $\alpha \in I_n$
- gluing data: for each $\alpha \in I_n$, $0 \le i \le n$ associate some $\beta(\alpha, i) \in I_{n-1}$

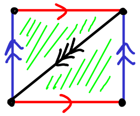
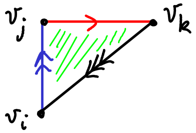
The Δ-complex is the quotient space:

$$\boxed{\bigsqcup_{\substack{\alpha\in I_n\\ n\in\mathbb{N}}} \sigma_\alpha^n \Big/ \begin{array}{l}i\text{-th facet of } \sigma_\alpha^n \text{ is identified with } \sigma^{n-1}_{\beta(\alpha,i)}\\ \text{via the order-preserving canonical linear homeo}\end{array}}$$

A **Δ-complex structure** on a top. space X is a homeo from a Δ-complex to X.

Explicit description of the facet identification

$$\begin{tikzcd}
\{\sum s_i w_i\} = [w_0,\dots,w_{n-1}] \arrow[r] & {[v_0,\dots,v_n] = \{\sum t_i v_i\}} \arrow[dr, phantom, "\cup"] & \\
 & & \{s_0v_0+\dots+s_{i-1}v_{i-1}+s_iv_{i+1}+\dots+s_{n-1}v_n\} = [v_0,\dots,\hat{v}_i,\dots,v_n] \\
\Delta^{n-1} \arrow[uu, "\sigma^{n-1}_{\beta(\alpha,i)}"] \arrow[r] & \Delta^{n-1}_i \subseteq \Delta^n \arrow[uu, "\sigma^n_\alpha"'] \arrow[ur, "\sigma^n_\alpha|_{\Delta^{n-1}_i}"'] & \\
(s_0,\dots,s_{n-1}) \arrow[r, mapsto] & (s_0,\dots,s_{i-1},0,s_i,\dots,s_{n-1}) &
\end{tikzcd}$$

Non-example  

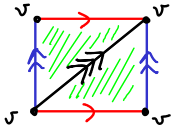
vertices are not totally ordered: $i<j<k<i$ ↯

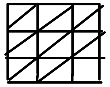
Compare Part A Topology course:

Remark A **simplicial complex** (more precisely, the "topological realisation" of a simpl. complex) is a Δ-complex in which each d-dim face is uniquely determined by d distinct vertices.

A homeo from such a complex to X is a **triangulation** of X.

Non-example

$T^2 =$  both 2-simplices have vertices v, v, v

whereas $T^2 =$  is a triangulation.

Simplicial chain complex

Def For a Δ-complex X,

$C_n^\Delta(X) =$ free abelian group generated by the set X_n of n-simplices of X

$$= \left\{ \sum_{\alpha\in I_n} c_\alpha \cdot \sigma_\alpha^n \ : \ \begin{array}{l} c_\alpha \in \mathbb{Z} \\ \text{only finitely many } c_\alpha \neq 0\end{array} \right\}$$

$$\boxed{\partial_n \sigma_\alpha = \sum (-1)^i \cdot \sigma_{\beta(\alpha,i)}}$$

so: $$\boxed{\partial_n [v_0,\dots,v_n] = \sum (-1)^i \cdot [v_0,\dots,\hat{v}_i,\dots,v_n]}$$

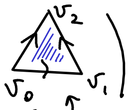
and extend linearly

We will show $\partial \circ \partial = 0$, so get **simplicial homology**:

$$\boxed{H_*^\Delta(X) = H_*(C_*^\Delta, \partial_*)}$$

Examples

$\partial_1\left(\overrightarrow{v_0 \quad v_1}\right) = \overset{\bullet}{-v_0} \quad \overset{\bullet}{+v_1}$

∂_2 () $= + \overrightarrow{v_1 v_2} - \overrightarrow{v_0 v_2} + \overrightarrow{v_0 v_1}$

$\partial_2 \circ \partial_1$ (this) $= +(v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0$

$\partial \circ \partial = 0$ fails for  (not Δ-complex), try!

Later:
The $(-1)^i$ signs keep track of whether the orientation agrees/disagrees with geometric boundary orientation, so  versus ∂  = 

Lemma $\partial \circ \partial = 0$

Pf $\partial_{n-1}(\partial_n [v_0,\dots,v_n]) = \sum (-1)^i \partial_{n-1} [v_0,\dots,\hat{v}_i,\dots,v_n]$

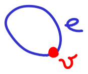
$$= \sum_{j<i} (-1)^i (-1)^j [v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]$$

antisymmetric if swap i,j

$$+ \sum_{j>i} (-1)^i \underline{\underline{(-1)^{j-1}}} [v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]$$

$$= 0$$

□

Example $S^1 =$  Δ-complex:

X_0: 1 0-simplex • $e_1^0 = e_{\beta(1,0)} = e_{\beta(1,1)}$

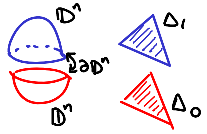
X_1: 1 1-simplex → e_1^1

$$0 \to \underset{\mathbb{Z}e}{\overset{\shortparallel}{C_1^\Delta}} \to \underset{\mathbb{Z}v}{\overset{\shortparallel}{C_0^\Delta}} \to 0$$

$e \longmapsto v - v = 0$

$$H_*^\Delta(S^1) = \begin{cases} \mathbb{Z} & * = 0,1 \\ 0 & \text{else} \end{cases}$$

Example Δ-cx structure on S^n:



$S^n = \Delta^n \sqcup \Delta^n / \text{glue along } \partial\Delta^n$

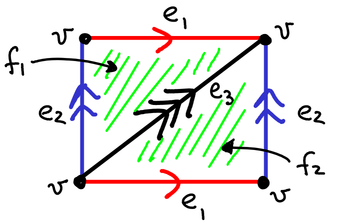
call this Δ_1 — this Δ_0

One can deduce: but messy!

pick any vertex

$$H_* S^n = \begin{cases} \mathbb{Z}\langle \text{point}\rangle \cong \mathbb{Z} & * = 0 \\ \mathbb{Z}\langle \Delta_1 - \Delta_0 \rangle \cong \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$$

Example

$X = T^2 =$ 

$$0 \to \underset{\mathbb{Z}f_1 + \mathbb{Z}f_2}{\overset{\shortparallel}{C_2^\Delta}} \longrightarrow \underset{\mathbb{Z}e_1+\mathbb{Z}e_2+\mathbb{Z}e_3}{\overset{\shortparallel}{C_1^\Delta}} \longrightarrow \underset{\mathbb{Z}v}{\overset{\shortparallel}{C_0^\Delta}} \longrightarrow 0$$

$f_1 \longmapsto e_1 - e_3 + e_2$

$f_2 \longmapsto e_2 - e_3 + e_1$

$e_1, e_2, e_3 \longmapsto v - v = 0$

$$H_*^\Delta(T^2) = \begin{cases} \mathbb{Z}v & * = 0 \\ (\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3)/\mathbb{Z}(e_1 - e_3 + e_2) & * = 1 \\ \mathbb{Z}\cdot(f_1 - f_2) & * = 2 \\ 0 & \text{else} \end{cases}$$

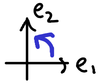
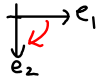
← freely generated by e_1, e_2

$$\cong \begin{cases} \mathbb{Z} & * = 0,2 \\ \mathbb{Z}^2 & * = 1 \\ 0 & \text{else} \end{cases}$$

(Smith normal form of ∂_2: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} \xrightarrow{\text{row op.}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{col. op.}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ so after $\mathbb{Z}$-isos of C_2, C_1 we get $\mathbb{Z}^2 \to \mathbb{Z}^3$, $(a,b) \to (a,0,0)$)

Remark about orientations (see also my B3.2 Geometry of Surfaces notes)

For vector space an **orientation** is a choice of basis modulo linear endomorphisms of $\det > 0$

Example $\mathbb{R}^2$  right-hand orientation (positive) $\xrightarrow[\det < 0]{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$  left-hand orientation (negative)

Fact $GL(n,\mathbb{R})$ has 2 path-components $\begin{array}{l} A : \det A > 0 \\ A : \det A < 0 \end{array}$ so can always continuously deform a basis to another <u>within same orientation</u>

Canonical orientation on $\mathbb{R}^n$: $e_1, \dots, e_n$ standard basis ← **"positive orientation"**

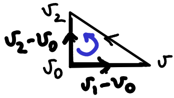
Example $[v_0,\dots,v_n]$ simplex $\Rightarrow$ $v_1 - v_0, \dots, v_n - v_0$ is a basis of vector subspace

$$V = \{\textstyle\sum a_i v_i : \sum a_i = 0\} \subseteq \mathbb{R}^{n+k}$$

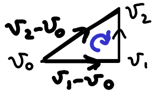
hence a choice of orientation of V, and each transposition of vertices $v_0,\dots,v_n$ switches the orientation class.

If $v_0, \dots, v_n \in \underline{\underline{\mathbb{R}^n}}$ then $V = \mathbb{R}^n$ so simplex's orientation can be compared with $\mathbb{R}^n$-orient^n.

Example In $\mathbb{R}^2$:



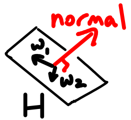
Positively oriented



negatively oriented

- No canonical choice of orientation for abstract vector space. Need choose basis $v_1, \dots, v_n$ then declare another basis positively oriented if the change of basis matrix has det > 0.

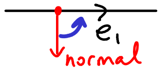
- For hyperplane $H \subseteq \mathbb{R}^n$ with choice of normal can declare orientation



of basis $w_1, \dots, w_{n-1}$ of H positive if normal, $w_1, \dots, w_{n-1}$ is positive $\mathbb{R}^n$-basis

convention "outward normal first"

Example



$H \subseteq \mathbb{R}^2$ $\Rightarrow$ e_1 positive basis for H

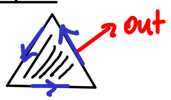
$(\text{normal}, e_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\det = +1 > 0$

Example $\Delta^n \subseteq \mathbb{R}^{n+1}$ with normal $(1, 1, \dots, 1)$ is positively oriented.

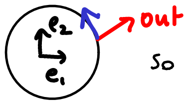
UPSHOT For an n-simplex $[v_0, \dots, v_n]$ in $\underline{\underline{\mathbb{R}^n}}$, each facet lies in a hyperplane and have canonical choice of normal: outward normal. Hence facets are canonically oriented.

Example

$\mathbb{R}^2 \supseteq$



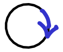
in smooth world : $\mathbb{D}^2$



so $\partial \mathbb{D}^2 = S^1$

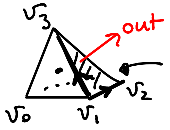
anticlockwise

Any reflection of $\mathbb{R}^n$ will swap orientation : after $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ get



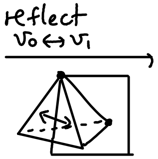
clockwise

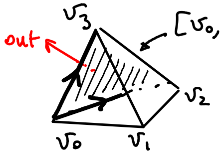
Example



$[\hat{v}_0, v_1, v_2, v_3]$

out, $v_2 - v_1$, $v_3 - v_1$ positive $\mathbb{R}^3$-basis





$[v_0, \hat{v}_1, v_2, v_3]$

out, $v_2 - v_0$, $v_3 - v_0$ negative $\mathbb{R}^3$-basis

UPSHOT $(-1)^i$ in $(-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k]$ in definition of simplicial ∂ is there to ensure that orientations are consistent (crucial for $\partial\partial = 0$)

Lemma $H_*^\Delta(X) \cong \bigoplus H_*^\Delta(X_i)$ where X_i are the path-components of X.

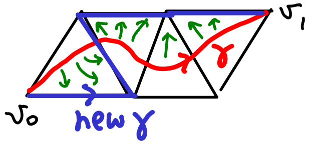
Pf $\bigoplus C_*^\Delta(X_i) \longrightarrow C_*^\Delta(X)$, $\oplus c_i \longmapsto \sum c_i$ is chain isomorphism since any simplex $\sigma : \Delta^k \to X$ has path-connected image, so $\subseteq X_i$ some i. ☐

since Δ^k path-conn.

Theorem X has Δ-cx structure $\Rightarrow$ $H_0^\Delta(X) \cong \bigoplus_{\substack{\text{path-conn.} \\ \text{components}}} \mathbb{Z}$

Pf By Lemma, WLOG X path-connected

- vertex v $\Rightarrow$ $v \in C_0(X) \xrightarrow{\partial} C_{-1}(X) \equiv 0$ $\Rightarrow$ $[v] \in H_0(X)$
- vertices $v_0, v_1 \in X$ $\Rightarrow$ $\exists$ path γ from v_0 to v_1



$\Rightarrow$ can homotope path so that go along edges (continuously deform)

$\Rightarrow$ γ is sum of 1-chains s.t. $\partial\gamma = v_1 - v_0$

$\Rightarrow$ $[v] \in H_0(X)$ independent of choice of v

$\Rightarrow$ $H_0(X) = \langle [v] \rangle$

- $H_0(X) = \langle [v] \rangle \leftarrow \mathbb{Z}$ is injective?

$n v \leftarrow\!\shortmid n$ Suppose $n v = \partial c$ some $c \in C_1(X)$

consider the augmentation hom

$$\begin{tikzcd} C_1(X) \arrow[r, "\partial"] & C_0(X) \arrow[r, "\varepsilon"] & \mathbb{Z} \\ & \sum n_i \sigma_i \arrow[r, mapsto] & \sum n_i \end{tikzcd}$$

0-simplices

notice composite is 0 since $\partial\left(\overset{\text{1-simplex}}{\underset{\sigma_0 \quad \sigma_1}{\bullet\!\longrightarrow\!\bullet}}\right) = \sigma_1 - \sigma_0 \xrightarrow{\varepsilon} 1 - 1 = 0$

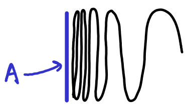
$\Rightarrow$ $n = \varepsilon(n v) = \varepsilon \partial c = 0$. ☐

Rmk X top. space $\Rightarrow$ path conn. component $\subseteq$ connected component since path-conn. $\Rightarrow$ connected. For Δ-cx, these are same (since connected + locally path-conn. $\Rightarrow$ path-connected).

However for top. space they need not be the same, e.g. Topologist's sine curve

$\{(x, \sin\frac{1}{x}) : x \in (0,1]\} \cup \underbrace{0 \times [0,1]}_{A} \subseteq \mathbb{R}^2$

2 path-conn. components



- connected
- not path-connected
- not locally path-connected

3. SINGULAR HOMOLOGY

Motivation Not obvious that H^Δ_* is functorial: $\Delta^k \xrightarrow{\sigma} X \xrightarrow{f} Y$ ← continuous map

then $f\circ\sigma$ typically not a simplex:

Solution 1: only allow **simplicial maps** $f: X\to Y$ (so $f\circ\sigma$ simplex $\forall\sigma$)

Solution 2: show that any cts map $f: X\to Y$ can be approximated arbitrarily well by a simplicial map after having performed barycentric subdivision on X, Y enough times. Also any two such approximations induce the same map $H^\Delta_*(X)\to H^\Delta_*(Y)$

Solution 3: develop new homology $H_*(X)$ which allows any cts map $\Delta^n\to X$ and prove $H^\Delta_*(X)\cong H_*(X)$ for Δ-complexes X.

WILL DO THIS.

Def **Singular n-simplex** is any $\boxed{\text{continuous map } \sigma: \Delta^n \longrightarrow X}$ ← X is any top. space

Singular n-chains $C_n(X)$ = free abelian group generated by σ

$$= \left\{ \sum_{\substack{\text{singular}\\ n\text{-simplices }\sigma}} c_\sigma\cdot\sigma \;:\; c_\sigma\in\mathbb{Z} \quad \text{only finitely many } c_\sigma\neq 0 \right\}$$

$$\boxed{\partial_n\sigma = \sum (-1)^i\cdot \sigma|_{\Delta^n_i}} \leftarrow \text{i-th facet} \quad \text{(and extend linearly)}$$

We will show $\partial\circ\partial=0$, so get **singular homology**:

$$\boxed{H_*(X) = H_*(C_*, \partial_*)}$$

For Δ-complex X have inclusion of subcomplex $C^\Delta_*\to C_*$

$\Longrightarrow$ induces $H^\Delta_*(X) \longrightarrow H_*(X)$ Fact: isomorphism (proof later, see cellular $H^{CW}_*\cong H_*$)

Corollary $H^\Delta_*(X)$ is independent of choice of Δ-cx structure on X

Lemma $\partial\circ\partial = 0$

Proof

$$\partial_{n+1}(\partial_n\sigma) = \partial_{n+1}\Big(\sum(-1)^i\sigma|_{\Delta^{n-1}_i}\Big) \leftarrow [e_0,\dots,\hat{e}_i,\dots,e_n]$$

$$= \sum_{j<i}(-1)^i(-1)^j\sigma|_{[e_0,\dots,\hat{e}_j,\dots,\hat{e}_i,\dots,e_n]}$$

$$+\sum_{j>i}(-1)^i\underline{\underline{(-1)^{j-1}}}\sigma|_{[e_0,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_n]}$$

$$=0$$

□

Example $X = \text{point} \Longrightarrow C_n(X) = \mathbb{Z}\cdot(\sigma_n: \Delta^n\to X$ the constant map$)$

$$\partial\sigma_n = \sum(-1)^i\sigma_n|_{\Delta^{n-1}_i} = \underbrace{\sum(-1)^i}_{\left\{\begin{smallmatrix}1 & \text{if } n \text{ even}\\ 0 & \text{" odd}\end{smallmatrix}\right.}\sigma_{n-1} \;\Bigg\}\; \begin{matrix}\Rightarrow \cdots\xrightarrow{1}\mathbb{Z}\xrightarrow{0}\mathbb{Z}\xrightarrow{1}\underset{C_1}{\underset{\wr\wr}{\mathbb{Z}}}\xrightarrow{0}\underset{C_0}{\underset{\wr\wr}{\mathbb{Z}}}\to 0\\ \Longrightarrow H_*(pt)\cong\begin{cases}\mathbb{Z} & *=0\\ 0 & \text{else}\end{cases}\end{matrix}$$

Lemma $\boxed{H_*(X)\cong\bigoplus H_*(X_i)}$ where X_i are path-components of X

Pf Image of cts map $\Delta^n\longrightarrow X$ is path conn. so lies in some X_i. □

Cor $\boxed{H_0(X) = \bigoplus_{X_i}\mathbb{Z}}$

Pf By Lemma, WLOG X path-connected. $\Delta^0 = pt\longrightarrow X$ ← generators of $C_0(X)$ is cycle since $C_{-1}(X)=\phi$

Given 2 points $x,y\in X$, a path $\Delta^1=[0,1]\xrightarrow[\gamma]{} X$, $\gamma(0)=x, \gamma(1)=y$ is also a 1-chain!

So $x-y=\partial\gamma$, so x,y are homologous. Finally if $n\cdot[x]=0\in H_0(X)$ then $nx=\partial c$ some $c\in C_1(X)$ = generated by paths. Now run the augmentation hom. trick like we did for H^Δ_0: $n=\varepsilon(nx)=\varepsilon\partial c = 0$ as $\varepsilon\circ\partial=0$. □

Naturality (i.e. functoriality)

Lemma $f: X\longrightarrow Y$ continuous

$\Longrightarrow H_*(f) = f_*: H_*(X)\longrightarrow H_*(Y)$ induced by chain map

induced map

$f_*: C_*(X)\longrightarrow C_*(Y)$

$\boxed{f_*(\sigma) = f\circ\sigma}$ and extend linearly

$$\begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[dr, "f_*\sigma"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Pf $\partial_n(f_*\sigma) = \sum(-1)^n f\circ\sigma|_{\Delta^{n-1}_i} = f_*\big(\sum(-1)^n\sigma|_{\Delta^{n-1}_i}\big) = f_*(\partial_n\sigma)$ □

Properties
1) $X\xrightarrow{f}Y\xrightarrow{g}Z \Longrightarrow (g\circ f)_* = g_*\circ f_*$
2) $id_* = id$

Pf 1) $(g\circ f)_*\sigma = g\circ f\circ\sigma = g_*(f\circ\sigma) = g_*(f_*\sigma)$ ✓
2) $id_*\sigma = id\circ\sigma = \sigma$ ✓ □

Cor $H_*: \left\{\begin{matrix}\text{topological spaces}\\ \&\\ \text{cts maps}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{graded abelian groups}\\ \&\\ \text{graded homs}\end{matrix}\right\}$ is a functor

Cor $X\cong Y \Rightarrow H_*(X)\cong H_*(Y)$

4. CHAIN HOMOTOPIES AND HOMOTOPY INVARIANCE

Algebra : chain homotopies

$f_*, g_* : (C_*, \partial_*) \longrightarrow (\tilde{C}_*, \tilde{\partial}_*)$ chain maps

Def f_*, g_* are **chain homotopic** if $\exists$ (degree $+1$) hom $h : C_* \longrightarrow \tilde{C}_*[1]$ s.t.

$$\boxed{\tilde{\partial} \circ h + h \circ \partial = f - g}$$

h is called a **chain homotopy**

Consequence $f_* = g_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$ on homology

Pf

$$\begin{tikzcd} C_{n+1} \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[dl, "h_{n-1}"'] \\ \tilde{C}_{n+1} \arrow[r, "\tilde{\partial}_{n+1}"'] & \tilde{C}_n \arrow[r] & \tilde{C}_{n-1} \end{tikzcd}$$

c cycle $\in C_n$

$$\Rightarrow f_n(c) - g_n(c) = \underbrace{\tilde{\partial}_{n+1} \circ h_n(c)}_{\text{boundary}} + h_{n-1} \circ \underbrace{\partial_n(c)}_{=0}$$

$$\Rightarrow f_n(c) = g_n(c) \text{ in } H_*(\tilde{C})$$ □

Theorem $i_0 : X \longrightarrow X \times I,\ i_0(x) = (x, 0)$ where $I = [0,1]$

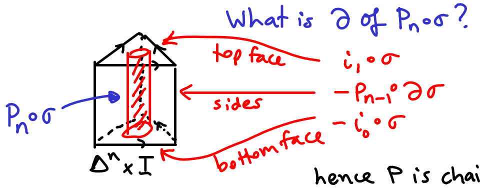
$i_1 : X \longrightarrow X \times I,\ i_1(x) = (0, x)$

$\Longrightarrow i_{0*}, i_{1*} : C_*(X) \to C_*(X \times I)$ are chain hpic.

Key idea Need "prism operator" which cuts $\Delta^n \times I$ into a sum Γ_n of $(n+1)$-simplices in $\Delta^n \times I$:

$$\begin{tikzcd} (\sigma : \Delta^n \to X) \arrow[r, mapsto] \arrow[dr, bend right, "\text{prism operator } P_n"'] & \sigma \times id : \Delta^n \times I \to X \times I \\ & \Delta^{n+1} \arrow[u, "\Gamma_n"'] \\ & (\sigma \times id) \circ \Gamma_n : \Delta^{n+1} \to X \times I \end{tikzcd}$$

What is ∂ of $P_n \circ \sigma$?



$i_1 \circ \sigma$

$-P_{n-1} \circ \partial \sigma$

$-i_0 \circ \sigma$

hence P is chain hpy

Pf ← Non-examinable

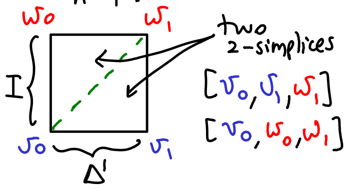
bottom facet $\Delta^n \times 0 = [v_0, \ldots, v_n]$ ← $v_i = e_i \times 0$

top facet $\Delta^n \times 1 = [w_0, \ldots, w_n]$ ← $w_i = e_i \times 1$

$\} \subseteq \Delta^n \times [0,1] \subseteq \mathbb{R}^{n+1}$

Examples

$n = 1$:

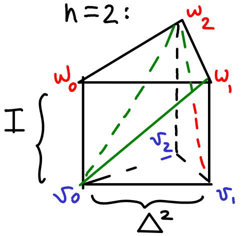


two 2-simplices

$[v_0, v_1, w_1]$

$[v_0, w_0, w_1]$

$n = 2$:



three 3-simplices:

$[v_0, v_1, v_2, w_2]$

$[v_0, v_1, w_1, w_2]$

$[v_0, w_0, w_1, w_2]$

Let $s_i = [v_0, \ldots, v_i, w_i, \ldots, w_n]$

Claim The s_i cover $\Delta \times [0,1]$ and give Δ-cx structure on $\Delta^n \times I$

Pf $\sum_{k \le i} t_k v_k + \sum_{k \ge i} s_k w_k = (t_0, \ldots, t_{i-1}, \underline{t_i + s_i}, s_{i+1}, \ldots, s_n, s_i + \cdots + s_n)$

So given $(x_0, \ldots, x_n, a) \in \Delta^n \times I$, equate and solve:

$t_0 = x_0, \ldots, t_{i-1} = x_{i-1},\ s_{i+1} = x_{i+1}, \ldots, s_n = x_n$, and $\begin{cases} s_i = a - x_{i+1} - \cdots - x_n \\ t_i = x_i - s_i \end{cases}$

Note $x_k \geq 0, \Sigma x_k = 1, a \in [0,1]$ hence $\Sigma t_k + \Sigma s_k = 1$ ✓ $\begin{cases} t_k \geq 0 \text{ for } k < i \\ s_k \geq 0 \text{ for } k > i \end{cases}$ ✓

but $\left.\begin{matrix} s_i \geq 0 \\ t_i \geq 0 \end{matrix}\right\} \Leftrightarrow \begin{cases} a \geq x_{i+1} + \cdots + x_n \\ x_i + x_{i+1} + \cdots + x_n \geq a \end{cases}$. Thus a solution exists if we pick $i = \min\{k : a \geq x_{i+1} + \cdots + x_n\}$

There are multiple solutions if $x_{i+1} = x_{i+2} = \cdots = x_j = 0$, but that is as expected: those points of $\Delta^n \times I$ belong to the faces of $s_i, s_{i+1}, \ldots, s_j$. □

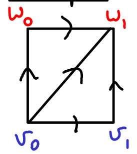
Def

$\Gamma_n = \sum (-1)^i s_i \in C_{n+1}(\Delta^n \times [0,1])$ ← geometrically this "represents" $\Delta^n \times I$ as a simplicial chain

$$\Rightarrow \partial \Gamma_n = \sum_i \sum_{j \le i} (-1)^i (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_i, w_i, \ldots, w_n] + \sum_i \sum_{j \ge i} (-1)^i (-1)^{j+1} [v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j, \ldots, w_n]$$

} geometrically this "represents" $\partial(\Delta^n \times I) = (\partial\Delta^n \times I) \sqcup (\Delta^n \times \partial I)$

Example



$\Gamma_1 = [v_0, w_0, w_1] - [v_0, v_1, w_1]$ "is the square"

$\partial \Gamma_1 = [w_0, w_1] - \cancel{[v_0, w_1]} + [v_0, w_0] - [v_1, w_1] + \cancel{[v_0, w_1]} - [v_0, v_1]$

"is ∂ of square"

"inside facets" cancel

Prism operator $P : C_n(X) \longrightarrow C_n(X\times[0,1])$

$$\boxed{P(\sigma) = (\sigma\times id)_*(\Gamma_n)}$$

$\sigma : \Delta^n \to X$

$\sigma\times id : \Delta^n\times[0,1] \to X\times[0,1]$

$(\sigma\times id)(x,t) = (\sigma(x),t)$

$$\partial P(\sigma) = \partial(\sigma\times id)_*(\Gamma_n) = (\sigma\times id)_*(\partial\Gamma_n)$$

$$= \sum_i \sum_{j\leq i} (-1)^i(-1)^j[i_0\sigma e_0, \dots, \widehat{i_0\sigma e_j}, \dots, i_0\sigma e_i, i_1\sigma e_i, \dots, i_1\sigma e_n]$$

$$+ \sum_i \sum_{j\geq i} (-1)^i(-1)^{j+1}[i_0\sigma e_0, \dots, i_0\sigma e_i, i_1\sigma e_i, \dots, \widehat{i_1\sigma e_j}, \dots, i_1\sigma e_n]$$

$$= i_{1*}\sigma - i_{0*}\sigma - \underbrace{P\partial\sigma}$$

$i=j=0$ 1st sum

$i=j=n$ 2nd sum

$((\partial\sigma)\times id)_*\Gamma_{n-1}$

$\sum_i (-1)^i[v_0, \dots, v_i, w_i, \dots, w_{n-1}]$

⊛

$(\sigma\times id)(v_i)$
$= (\sigma\times id)(e_i,0)$
$= (\sigma(e_i),0)$
$= i_0(\sigma)(e_i)$
$(\sigma\times id)(w_i)$
$= (\sigma(e_i),1)$
$= i_1(\sigma)(e_i)$

$\sigma = [\sigma e_0, \dots, \sigma e_n]$

now use ⊛ and
$\partial\sigma = \sum(-1)^k[\sigma e_0, \dots, \widehat{\sigma e_k}, \dots, \sigma e_n]$. □

Homotopy invariance

$f_0, f_1 : X \longrightarrow Y$

Def $f_0 \simeq f_1$ **homotopic** if $\exists$ continuous map

$$F : X\times[0,1] \longrightarrow Y$$

s.t. $f_0 = F\circ i_0$
$f_1 = F\circ i_1$.

← called **homotopy**

Idea Think of this as a continuous family of maps $f_t = F(\cdot,t) : X\to Y$ from f_0 to f_1.

Exercise $\simeq$ is an equivalence relation.

Homotopic **relative to $A\subseteq X$** if $F(a,t) = f_0(a) = f_1(a)$ all $a\in A$, all t.
write "$f\simeq g$ **rel A**"

Def $X\simeq Y$ **homotopy equivalent** spaces if $\exists$ maps

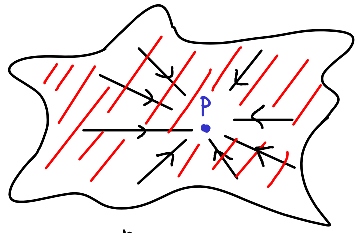
$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$ with $g\circ f \simeq id$, $f\circ g \simeq id$

Rmk homeo $\Rightarrow$ hpy equivalent

Def X **contractible** if $X \simeq pt$

equivalently $(X \xrightarrow{id} X) \simeq (X \xrightarrow{const} \text{point} \in X)$

Example
- $\mathbb{R}^n \simeq pt$
  $F(x,t) = tx$ then $f_0 \equiv 0$, $f_1 = id$.
- (star-shaped subsets of $\mathbb{R}^n$) $\simeq pt$



← contains line segments to a specific point p

WLOG $p=0$ & use same F
↑
translate

e.g. disc D^n

Theorem $f_0 \simeq f_1 \Rightarrow f_{0*} = f_{1*} : H_*X \longrightarrow H_*Y$

Pf $f_{1*} - f_{0*} = F_* i_{1*} - F_* i_{0*}$ (where F = homotopy, i_0, i_1 as in previous Thm)

$= F_*(i_{1*} - i_{0*})$

previous Thm $\nearrow$ $= F_*(\partial P + P\partial)$

$= \partial\circ(F_*\circ P) + (F_*\circ P)\circ\partial$

F_* chain map $\nearrow$ $\Rightarrow F_*\circ P$ is chain hpy from f_{0*} to f_{1*} □

Cor $X\simeq Y \Longrightarrow H_*X \cong H_*Y$

Pf $f_* g_* = id_*$, $g_* f_* = id_*$ □

Example X contractible $\Rightarrow H_*X \cong H_*(pt) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Relative homology

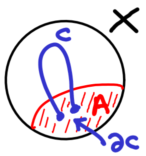
Def (X,A) **pair of spaces** if $A \subseteq X$ topological subspace

$\Rightarrow$ $i = \text{incl}: A \hookrightarrow X$ induces a subcx $i_*: C_*A \to C_*X$

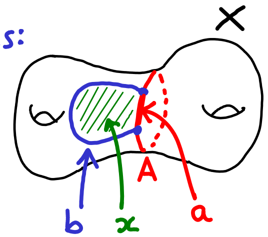
$\Rightarrow$ C_*X/C_*A quotient chain cx (recall $\partial[x] = [\partial x]$)

$$\boxed{H_*(X,A) = H_*(C_*X/C_*A)}$$

Idea: **relative cycles**: $c \in C_*X$ s.t. $\partial c \in C_*A$



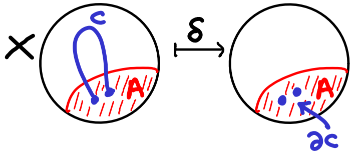
relative boundaries: $b \in C_*X$ s.t. $\exists\, x \in C_{*+1}X$ $\partial x = b + a_{\in C_*A}$



$\Rightarrow$ $0 \to C_*A \to C_*X \to C_*X/C_*A \to 0$ SES

Cor $\boxed{\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} \cdots}$

LES for the pair



$\delta c = \underset{\underset{C_*A}{\cap}}{\partial c}$

Need not be ∂a some $a \in C_*A$

Reduced homology

$\boxed{\tilde{H}_*X = \ker H_*X \longrightarrow H_*(pt)}$ induced by map $X \to pt$

equivalently H_* of **augmented chain complex**

$$\cdots \to C_n(X) \to \cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

augmentation $\varepsilon(\Sigma n_i \cdot p_i) = \Sigma n_i$ ($n_i \in \mathbb{Z}$, points $p_i \in X$)

can view $C_{-1}(X) = \mathbb{Z} \cdot (\text{map } \emptyset \to X)$ where allow the empty simplex $\emptyset$

Example $\tilde{H}_*(\mathbb{R}^n) \cong \tilde{H}_*(\mathbb{D}^n) \cong \tilde{H}_*(pt) = 0$

Check
- $H_*\tilde{X} = H_*X \quad * \neq 0$, and $H_0\tilde{X} \cong H_0X \oplus \mathbb{Z}$
- $f: X \to Y \Rightarrow f_*: \tilde{H}_*X \longrightarrow \tilde{H}_*Y$

Lemma (X,A) pair $\Rightarrow \exists$ LES

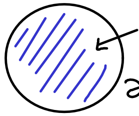
$$\boxed{\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \to H_n(X,A) \to \tilde{H}_{n-1}(A) \xrightarrow{i_*} \cdots \to H_0(X,A) \to 0}$$

Pf use augmented ch. cx. and $\mathbb{Z}/\mathbb{Z} = 0$ □

Cor $H_*(X, pt) \cong \tilde{H}_*(X)$

Pf $\tilde{H}_*(pt) = 0$. □

Example LES: $\tilde{H}_*(S^{n-1}) \longrightarrow \tilde{H}_*(\mathbb{D}^n) = 0$, $[-1]$, $H_*(\mathbb{D}^n, S^{n-1})$



$\mathbb{D}^n \subseteq \mathbb{R}^n$

$\partial\mathbb{D}^n = S^{n-1}$

$\Rightarrow \boxed{H_*(\mathbb{D}^n, S^{n-1}) \cong \tilde{H}_{*-1}(S^{n-1})}$

Naturality of the LES for pairs

Def A **map of pairs of spaces** $(X,A) \xrightarrow{f} (Y,B)$ means $f: X \to Y$ and $f(A) \subseteq B$.

Lemma

$$\begin{tikzcd}
\cdots \arrow[r] & H_*A \arrow[r] \arrow[d, "f_*"'] & H_*X \arrow[r] \arrow[d, "f_*"'] & H_*(X,A) \arrow[r] \arrow[d] & H_{*-1}A \arrow[r] \arrow[d, "f_*"'] & \cdots \\
\cdots \arrow[r] & H_*B \arrow[r] & H_*Y \arrow[r] & H_*(Y,B) \arrow[r] & H_{*-1}B \arrow[r] & \cdots
\end{tikzcd}$$

Pf

$$\begin{tikzcd}
0 \arrow[r] & C_*A \arrow[r] \arrow[d, "f_*"'] & C_*X \arrow[r] \arrow[d, "f_*"'] & C_*X/C_*A \arrow[r] \arrow[d, "f_*"'] & 0 \\
0 \arrow[r] & C_*B \arrow[r] & C_*Y \arrow[r] & C_*Y/C_*B \arrow[r] & 0
\end{tikzcd}$$

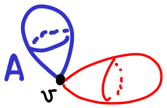
$\Rightarrow$ claim follows by naturality of LES induced by SESs of chain cxes. □

5. EXCISION THEOREM AND QUOTIENTS

(X,A) pair

Def $r: X \to X$ **retraction** onto A if $\begin{cases} r(X) = A \\ r|_A = id_A \end{cases}$

Example



$X = \underbrace{S^2}_{A} \vee S^2$ = two spheres glued at one point v (**wedge sum**)

$r: X \to A$ map second sphere to v

Example In Pf of Brower fixed pt thm we built a retraction r by contradiction

Cor r retraction $\Rightarrow$ $r_*: H_*X \longrightarrow H_*A$ surjective

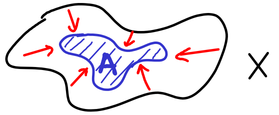
$\text{incl}_*: H_*A \longrightarrow H_*X$ injective

Pf

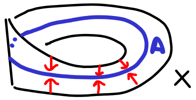
$$\begin{tikzcd}
A \arrow[r, "\text{incl}"] \arrow[rr, bend right, "id_A"'] & X \arrow[r, "r"] & A
\end{tikzcd}$$

now use H_* functorial □

Def $r : X \to X$ **deformation retraction** onto A if $\begin{cases} r \text{ retraction} \\ r \simeq \mathrm{id}_X \ \text{ rel } A \end{cases}$



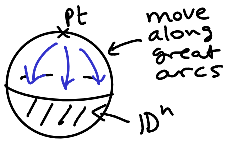
Example $X =$ Möbius strip
$A =$ equator



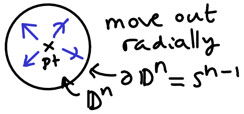
Lemma r def. retr. $\Rightarrow$ · $A \xrightarrow[\simeq]{\text{incl}} X$ is a homotopy equivalence.
· incl_* and r_* are isos on H_*, so $H_*A \cong H_*X$

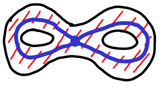
Pf $A \overset{\text{incl}}{\underset{r}{\rightleftarrows}} X$ $\quad \text{incl} \circ r = r \simeq \mathrm{id}_X$, $\quad r \circ \text{incl} = r|_A = \mathrm{id}_A$ □

Example $S^n \setminus pt$ def. retracts to $\mathbb{D}^n \cong$ lower hemisphere



$\Rightarrow S^n \setminus pt \simeq \mathbb{D}^n$
$\Rightarrow S^n \setminus [2 \text{ points}] \simeq \mathbb{D}^n \setminus pt \simeq S^{n-1}$



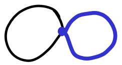
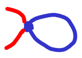
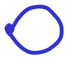
$\Rightarrow S^n \setminus \{3 \text{ points}\} \underset{\text{def.ret.}}{\simeq}$  $\underset{\text{ret.}}{\simeq}$  $S^{n-1} \vee S^{n-1}$

Excision theorem

$E \subseteq A \subseteq X$ subspaces with $\overline{E} \subseteq A^\circ$ $\Longrightarrow$ $(X \setminus E, A \setminus E) \longrightarrow (X, A)$ induces iso

$$\boxed{H_*(X \setminus E, A \setminus E) \xrightarrow{\cong} H_*(X, A)}$$

Proof Later.

Example $X = S^1 \vee S^1 =$  $\supseteq A =$  $\supseteq E =$  $\cong S^1$

$\Rightarrow H_*(X, A) \underset{\text{exc. thm.}}{\cong} H_*($  $) \underset{\text{hpy invce}}{\cong} H_*(\mathbb{D}^1, \partial \mathbb{D}^1) \cong \widetilde{H}_0(S^0) \cong \mathbb{Z}$ (2 points)

Rephrasing of Excision Thm

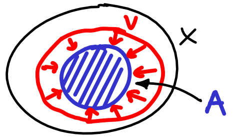
$X = A^\circ \cup B^\circ$ $\Longrightarrow$ $\boxed{H_*(X, A) \cong H_*(B, A \cap B)}$ induced by inclusion $(X, A) \leftarrow (B, A \cap B)$
$(A, B \subseteq X$ subspaces$)$

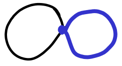
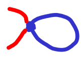
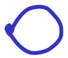
Pf Take $B = X \setminus E$ so $A \cap B = A \setminus E$. □

Idea why excision holds: $C_*(A) + C_*(B) \longrightarrow C_*(X)$ is a homotopy equivalence and $C_*(A) \cap C_*(B) = C_*(A \cap B)$. Idea: can subdivide chains in X many times, and small enough chains lie either in A or in B (or in both).

Good pairs and quotients

(X, A) pair
- **Quotient** $X/A = X/\sim$ ← equiv. relation $x \sim y \Leftrightarrow x = y$ or $x, y \in A$
- (X, A) **good pair** if $\begin{cases} A \neq \emptyset \\ A \text{ closed} \\ A \text{ deformation retract of nbhd } V \text{ of } A \end{cases}$

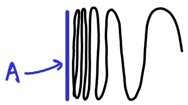


Example $X = S^1 \vee S^1 =$  $\supseteq V =$  $\supseteq A =$  $\cong S^1$

$X/A \cong$  ← all points of A are identified with the node

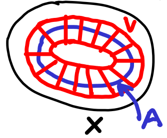
Non-example **Topologist's sine curve**

$\{(x, \sin\frac{1}{x}) : x \in (0,1]\} \cup \underbrace{0 \times [0,1]}_{A} \subseteq \mathbb{R}^2$



connected
not path-connected
not locally connected
not locally path-connected

Cultural Rmk
Smooth submanifold $\subseteq$ smooth manifold is a good pair (tubular neighbourhood theorem)



Cor (X, A) good $\Rightarrow (X, A) \longrightarrow (X/A, pt)$ induces iso

$$\boxed{H_*(X, A) \longrightarrow H_*(X/A, pt) = \widetilde{H}_*(X/A)}$$

Pf good $\Rightarrow \exists$ nbhd V of A, and $A \xrightarrow[\text{incl}]{\simeq} V$.

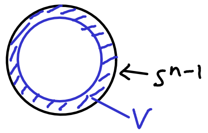
LES for pairs & 5-Lemma since $A \simeq V$, $A/A \simeq V/A$

excision

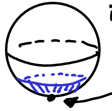
$$\begin{tikzcd}
H_n(X, A) \arrow[r, "\cong"] \arrow[d, "\text{quot.}"'] & H_n(X, V) \arrow[d, "\text{quot.}"'] & H_n(X \setminus A, V \setminus A) \arrow[l, "\cong"'] \arrow[d, "="', "\mathrm{id}_* = \text{identity}"] \\
H_n(X/A, \underbrace{A/A}_{\text{call this point } p}) \arrow[r, "\cong"] & H_n(X/A, V/A) & H_n(X/A \setminus p, V/A \setminus p) \arrow[l, "\cong"']
\end{tikzcd}$$

Hence all arrows are isos. □

Example $\mathbb{D}^n \supseteq S^{n-1}$ good:



$\xrightarrow{\text{quotient}}$



$\Longrightarrow H_*(\mathbb{D}^n, S^{n-1}) \underset{\text{Cor}}{\cong} \widetilde{H}_*(\mathbb{D}^n / S^{n-1}) \cong \widetilde{H}_*(S^n)$ ← $\mathbb{D}^n / S^{n-1} \cong S^n$

Recall we proved $\widetilde{H}_*(\mathbb{D}^n, S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})$ (from LES & $\widetilde{H}_*(\mathbb{D}^n)=0$)

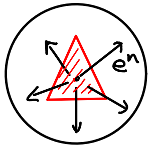
$\Rightarrow$ inductively, using Example $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1}) \cong \dots \cong \widetilde{H}_{k-n}(\underset{\text{2 points}}{S^0}) = \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{else} \end{cases}$

$H_0(2\text{pts}) = \mathbb{Z}\oplus\mathbb{Z}$

Generator of $H_n(S^n) \cong \widetilde{H}_n(\mathbb{D}^n/S^{n-1}) \cong H_n(\mathbb{D}^n, S^{n-1}) \cong \mathbb{Z}$

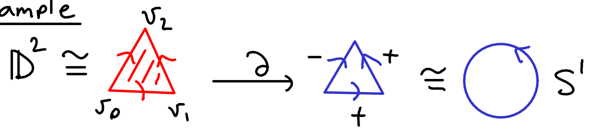
Observe $\exists$ homeo $e^n : \Delta^n \cong \mathbb{D}^n$ (homework) inducing Δ-cx structure on S^{n-1}:

$$\boxed{\partial\Delta^n \cong \partial\mathbb{D}^n = S^{n-1}}$$



stretch ctsly outwards from barycentre(Δ^n)

Example



Upshot ($n \geq 2$)

$$\boxed{\begin{aligned} H_n(\mathbb{D}^n, S^{n-1}) &= \mathbb{Z}\cdot e^n \\ H_{n-1}(S^{n-1}) &= \mathbb{Z}\cdot\partial e^n \\ H_n(\mathbb{D}^n/S^{n-1}) &= \mathbb{Z}\cdot[e^n] \end{aligned}}$$

LES for $n-1 \geq 1$, so $n \geq 2$

by Cor

Exercise Recall another Δ-cx structure on S^n:

$$\boxed{S^n = \Delta^n \sqcup \Delta^n / \text{glue along } \partial\Delta^n}$$

call this Δ_1 this Δ_0

then $\boxed{H_n S^n = \mathbb{Z}\cdot(\Delta_1 - \Delta_0)}$ and $H_n(S^n, \Delta_0) \overset{\text{exc.}}{\cong} H_n(\Delta_1, \partial\Delta_1) \cong H_n(\mathbb{D}^n, \partial\mathbb{D}^n)$

$\Delta_1 - \Delta_0 \to \Delta_1$

Another remark about orientations

Fact $\{\text{homeos } \Delta^n \to \mathbb{D}^n\}$ has 2 path-components

Above we chose a path-component by constructing e^n.

If r is any reflection in $\mathbb{R}^{n+1}$ then $e^n \circ r$ is in the other path-component

e.g. swap 2 coordinates in Δ^n

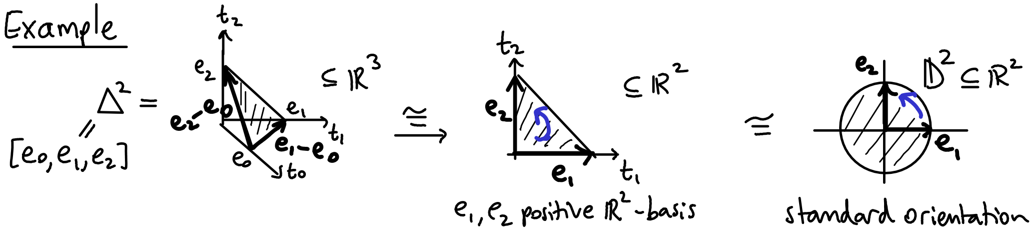
$$H_n(S^n) \cong H_n(\mathbb{D}^n, S^{n-1}) \xrightarrow{\cong} \mathbb{Z}$$
$$e^n \longmapsto +1$$
$$e^n \circ r \longmapsto -1$$

We will see later in the course that this corresponds to a choice of orientation of $\mathbb{D}^n$ and S^n.

Our choice is consistent with the inclusion $\mathbb{D}^n \subseteq \mathbb{R}^n$ (with the positive (canonical) orientation of $\mathbb{R}^n$) and the inclusion

$$(\Delta^n \subseteq \underline{\underline{\mathbb{R}^{n+1}}}) \xrightarrow{\cong} \{(t_1, \dots, t_n) \in \underline{\underline{\mathbb{R}^n}} : t_i \geq 0, \sum_{i=1}^n t_i \leq 1\}$$
$$(\underline{\underline{t_0}}, \dots, t_n) \longmapsto (\underline{\underline{t_1}}, \dots, t_n)$$
$$t_i \geq 0, \sum t_i = 1$$

Example



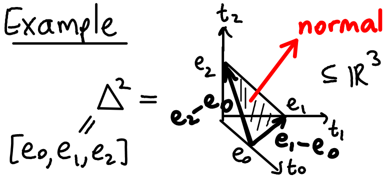
e_1, e_2 positive $\mathbb{R}^2$-basis

standard orientation

Our choice is also consistent with the "normal first" convention for orienting hyperplanes with a given choice of normal:

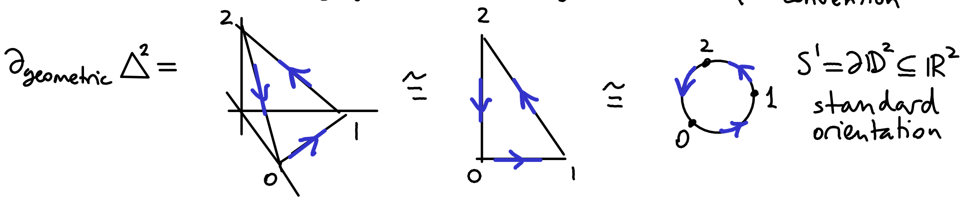
$\Delta^n \subseteq \text{hyperplane } \{(t_0, \dots, t_n) : \sum t_j = 1\} \subseteq \underline{\underline{\mathbb{R}^{n+1}}}$ normal $(1, 1, \dots, 1)$ (so pointing to ∞ in positive quadrant)

Example



normal, $e_1 - e_0$, $e_2 - e_0$ positive $\mathbb{R}^3$-basis

Consistent also with the geometric boundary orientation (outward normal first convention)



Compare $\partial\Delta = +[\hat{e}_0, e_1, e_2] - [e_0, \hat{e}_1, e_2] + [e_0, e_1, \hat{e}_2]$

This $-[e_0, e_2]$ is not equal to singular chain $[e_2, e_0]$ since they are different maps and we take free abelian group generated by maps.

But $[e_0, e_2] + [e_2, e_0]$ is homologous to 0 (Homework).

Locality (or "small simplices theorem")

$\mathcal{U} = \{\text{subspaces } U_i \subseteq X\}$ whose interiors cover X:

$$X = \bigcup U_i^\circ$$

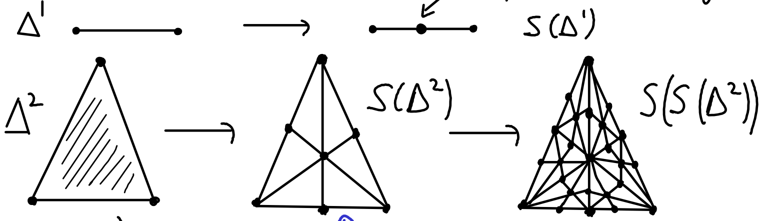
Def $C_*^{\mathcal{U}}(X) \subseteq C_*(X)$ subcx generated by n-simplices σ with $\sigma(\Delta^n) \subseteq U_i$ some i

Theorem $$\boxed{H_*(C_*^{\mathcal{U}}(X)) \cong H_*(C_*(X)) = H_* X}$$

Sketch Pf ① **Barycentric subdivision**

↑ Non-examinable

barycentre of $[v_0, .., v_n]$ is $\frac{1}{n+1}(v_0 + ... + v_n)$



$\Rightarrow$ chain map S: $C_*(X) \longrightarrow C_*(X)$

$\sigma \longmapsto \sigma \circ S$

and $S(C_*^{\mathcal{U}}) \subseteq C_*^{\mathcal{U}}$

subdivide the boundary (inductively by dimension) then draw the new faces obtained by convex combinations involving the new vertices and the barycentre

② S chain hpic to id:

$T: C_n(X) \longrightarrow C_{n+1}(X)$

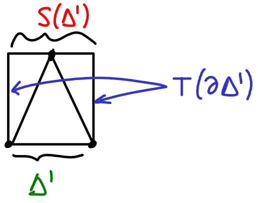
$T(\sigma): \Delta^n \times I \xrightarrow{\text{project}} \Delta^n \xrightarrow{\sigma} X$

exercise: $\partial T + T\partial = S - id$

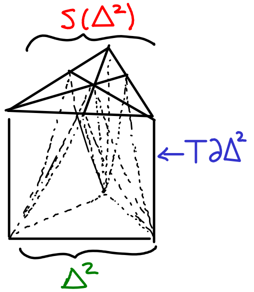
$$\Rightarrow \boxed{S_*: H_*(X) \xrightarrow{id} H_*(X)}$$

↑ Idea:

$\Delta^1 \times I =$



$\Delta^2 \times I =$



③ $\forall$ n-simplex $\sigma: \Delta^n \longrightarrow X$, apply $S(\cdot)$ enough times so that σ(each n-simplex of subdivision) $\subseteq U_i$ some i

③ $\Rightarrow$ $\forall$ cycle c, $\exists\, n$ s.t. $S^n(c) \in C_*^{\mathcal{U}}(X)$ cycle

$\Rightarrow H_*^{\mathcal{U}}(C) \longrightarrow H_*(X)$ surjective

$[S^n(c)] \longmapsto S_*^n[c] = [c]$ by ②

③ $\Rightarrow$ $\forall$ bdry $c = \partial b$, $\exists\, n$ s.t. $S^n(b) \in C_*^{\mathcal{U}}(X)$

claim: $H_*^{\mathcal{U}}(C) \longrightarrow H_*(X)$ injective

suppose $[c] \longmapsto 0$

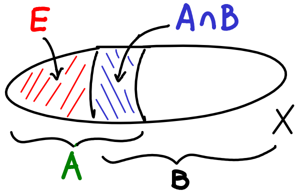
then $c = \partial b$ for $b \in C_*(X)$

now $S^n c, S^n b \in C_*^{\mathcal{U}}(X)$ for large n

$\Rightarrow \partial S^n b = S^n \partial b = S^n c$ in $C_*^{\mathcal{U}}(X)$

$\Rightarrow [c] \underset{②}{=} S_*^n[c] = [S^n c] = [\partial S^n b] = 0$ in $H_*^{\mathcal{U}}(X)$ ✓ □

Proof of excision theorem



Let $B = X \setminus E$

use $\mathcal{U} = \{A, B\}$

so $C_*^{\mathcal{U}}(X) = C_*(A) + C_*(B) \subseteq C_*(X)$

$$\Rightarrow C_*(X\setminus E)\big/C_*(A \setminus E) = C_*(B)\big/C_*(A \cap B) \cong C_*^{\mathcal{U}}(X)\big/C_*(A)$$

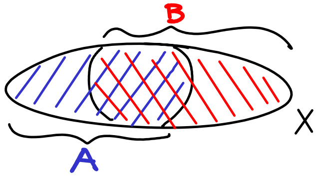
$\Rightarrow$ Compare LES's:

$$\begin{tikzcd}
 & & H_*(X \setminus E, A \setminus E) \arrow[d, phantom, "\cong" rotate=90] & & \\
H_*(A) \arrow[r] \arrow[d, equal] & H_*(C_*^{\mathcal{U}}X) \arrow[r] \arrow[d, "\text{locality}"', "\cong"] & H_*(C_*^{\mathcal{U}}X/C_*A) \arrow[r] \arrow[d, "\text{iso by 5-lemma}"] & H_{*-1}(A) \arrow[r] \arrow[d, equal] & H_{*-1}(C^{\mathcal{U}}X) \arrow[d, "\text{locality}"', "\cong"] \\
H_*(A) \arrow[r] & H_*(X) \arrow[r] & H_*(C_*X/C_*A) \arrow[r] \arrow[d, equal] & H_{*-1}(A) \arrow[r] & H_{*-1}(X) \\
 & & H_*(X, A) & &
\end{tikzcd}$$

□

6. MAYER-VIETORIS SEQUENCE ← Key computational tool

$X = \underbrace{A \cup B}_{\text{any subspaces}}$ s.t. $X = A^\circ \cup B^\circ$



MV Theorem ∃ LES :

$$\boxed{\cdots \to H_*(A\cap B) \xrightarrow{i_*} H_*(A)\oplus H_*(B) \to H_*(X) \to H_{*-1}(A\cap B) \xrightarrow{i_*[-1]} \cdots}$$

& same holds for $\tilde{H}_*$.

Pf SES $0 \to C_*(A\cap B) \to C_*(A)\oplus C_*(B) \to C^{\mathcal{U}}_*(X) \to 0$ $\mathcal{U}=\{A,B\}$

$\sigma \longmapsto (\sigma, -\sigma)$

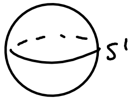
$(\alpha, \beta) \longmapsto \alpha+\beta$

⟹ induces the LES (using locality $H^{\mathcal{U}}_* X \cong H_* X$). □

Exercise connecting map is $\delta: H_*(X) \to H_{*-1}(A\cap B)$

$[\alpha+\beta] \longmapsto [\partial\alpha] = -[\partial\beta]$

Example S^2



$A \simeq pt$

$B \simeq pt$

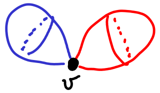
$A\cap B \simeq S^1$

$$\cdots \to \underset{0}{\underset{\shortparallel}{H_2(pt)\oplus H_2(pt)}} \to \underset{\text{hence } \mathbb{Z}}{\underset{\uparrow}{H_2 S^2}} \to \underset{\mathbb{Z}}{\underset{\cong}{H_1(S^1)}} \to \underset{0}{\underset{\shortparallel}{H_1(pt)\oplus H_1(pt)}} \to \cdots$$

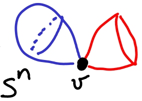
Exercise Compute $H_* S^n = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$ using MV

Example **wedge sum** of X, Y with basepoints $x \in X$, $y \in Y$ $X \vee Y = X \times Y / x \sim y$

$X = S^n \vee S^n =$

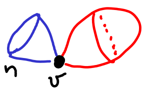


$A =$



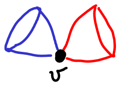
$\simeq S^n$

$B =$



$\simeq S^n$

$A \cap B =$



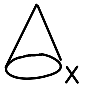
$\simeq pt$

$$0 \to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\cong} H_n(X) \to 0 \to \cdots \to 0 \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to \underset{\mathbb{Z}}{\underset{\cong}{H_0(X)}} \to 0$$

$$1 \longmapsto (1,-1)$$

Similarly $\boxed{H_*(X \vee Y) \cong H_*(X) \oplus H_*(Y) \text{ for } * \neq 0}$ if ∃ contractible nbhds of $x \in X$, of $y \in Y$.

Cones and suspensions



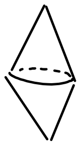
$\underset{\simeq pt}{CX} = (X \times [0,1]) \,/\, (x,s) \sim (y,t)$ iff equal or $s=t=1$

$\Sigma X = (X \times [0,1]) \,/\, (x,s) \sim (y,t)$ iff equal or $s=t=0$ or $s=t=1$

Example $CS^n \cong D^{n+1}$, $\Sigma S^n \cong S^{n+1}$.

Lemma $\boxed{H_*(\Sigma X) \cong \tilde{H}_{*-1}(X)}$

Pf

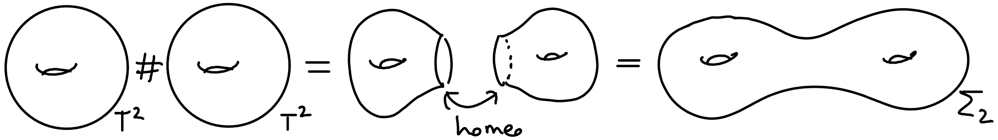


$A \simeq pt$, $B \simeq pt$

$A\cap B \simeq X$ now apply MV. □

Connected sum

M, N connected n-manifolds ⟹ $M\#N = \underbrace{(M \setminus \overset{\text{open}}{n\text{-ball}}) \cup (N \setminus \overset{\text{open}}{n\text{-ball}})}_{\text{identify } \partial\text{balls via a homeo}}$



Fact compact connected orientable surfaces are homeo to $\underset{\substack{\text{genus}\\ g=0}}{\underset{\uparrow}{S^2}}$ or $T^2\#\cdots\#T^2$ ← $g = \#$ copies, called Σ_g

and " " non-orientable ones : $\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2$.

Exercise (Homework) For M, N compact connected

By MV, $\boxed{H_*(M\#N) \cong H_*(M)\oplus H_*(N) \text{ for } 1 \leq * \leq n-2}$

If M or N orientable: $*=n-1$ also works

If both non-orientable: $*=n-1$ one of $\mathbb{Z}/2$ summands becomes $\mathbb{Z}$

$H_0(M\#N) \cong \mathbb{Z}$ since connected

fact: $H_n(M\#N)$ is $\mathbb{Z}$ or 0 ← else

↑ if M,N both orientable (see later in course)

Cor 1) $\chi(M\#N) = \chi(M) + \chi(N) - \begin{cases} 2 & \text{for } n \text{ even} \\ 0 & \text{" " odd} \end{cases}$

2) $H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}^{2g} & *=1 \\ \mathbb{Z} & *=2 \end{cases}$ ← genus g

7. DEGREE OF MAPS OF SPHERES

$$f : S^n \longrightarrow S^n \quad \Rightarrow \quad f_n : H_nS^n \longrightarrow H_nS^n$$

$$\begin{tikzcd} H_nS^n \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_nS^n \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \\ 1 \arrow[r, mapsto] & \mathbf{deg(f)} \in \mathbb{Z} \end{tikzcd}$$

$\Rightarrow f_* : \widetilde{H}_*S^n \to \widetilde{H}_*S^n$ is $\deg(f)\cdot \mathrm{id}$

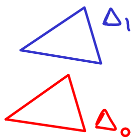
Properties
1) $\deg(\mathrm{id}) = 1$
2) $\deg(f\circ g) = \deg f \cdot \deg g$
3) $f \simeq g \Rightarrow \deg f = \deg g$
4) $f \simeq \text{const} \Rightarrow \deg f = 0$
5) f homeomorphism $\Rightarrow \deg f = \pm 1$ ← (sign depends on whether f is orientation-preserving or reversing)

Pf
$\mathrm{id}_* = \mathrm{id}$, $(f\circ g)_* = f_* \circ g_*$, $f \simeq g \Rightarrow f_* = g_*$, $\text{const}_* = 0$, f homeo $\Rightarrow f_n$ iso. □

Examples

1) $S^n = \Delta^n \times 1 \cup \Delta^n \times 0 \leftarrow$ call this Δ_0 / $(b,1) \sim (b,0)$ if $b \in \partial\Delta$

call this Δ_1



recall $H_nS^n = \mathbb{Z}\cdot(\Delta_1 - \Delta_0)$

reflection: $r : S^n \longrightarrow S^n$, $r(x,t) = (x, 1-t)$

so $\Delta_0 \leftrightarrow \Delta_1$ swapped by r, so $r_*(\Delta_1 - \Delta_0) = -(\Delta_1 - \Delta_0)$

$\Rightarrow \deg(r) = -1$

2) antipodal map $-\mathrm{id} : S^n \to S^n$ viewing $S^n \subseteq \mathbb{R}^{n+1}$

$\Rightarrow \boxed{\deg(-\mathrm{id}) = (-1)^{n+1}}$

Pf $-\mathrm{id} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} \circ \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \circ \cdots \circ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ composition of $n+1$ reflections each homotopic to r. □

3) $A \in O(n) \Rightarrow A : S^n \to S^n \Rightarrow \deg A = \det A \in \{\pm 1\}$

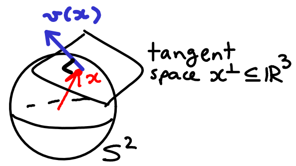
Pf fact $SO(n)$ is path-connected so $A \in SO(n)$ is $\simeq \mathrm{id}$ so $\deg A = \det A = +1$
The other path-component of $O(n)$ is $r \circ O(n)$ where r is any reflection. □

4) $\boxed{f \text{ not surjective} \Rightarrow \deg f = 0}$

Pf If $y \notin \mathrm{Im} f \Rightarrow$
$$\begin{tikzcd} H_n(S^n) \arrow[rr, "f_*"] \arrow[dr, "f_*"'] & & H_n(S^n) \\ & H_n(S^n \setminus y) \cong H_n(\mathbb{R}^n) = 0 \arrow[ur] & \end{tikzcd}$$
□

Application to vector fields on S^n

$v : S^n \longrightarrow \mathbb{R}^{n+1}$ tangent vector field on S^n
so $v(x) \perp x$



Cor **Hairy ball theorem** $\boxed{\exists \text{ nowhere zero v.f. on } S^n \iff n \text{ odd}}$

(case $n=2$: "you cannot comb a ball of hair without creating a tuft")

Pf Suppose $v(x) \neq 0 \quad \forall x$

$\Rightarrow$ hpy $F : S^n \times [0,1] \longrightarrow S^n$

$$F(x,t) = \cos(\pi t)\, x + \sin(\pi t)\, \frac{v(x)}{\|v(x)\|}$$

$\Rightarrow F_0 = \mathrm{id}$, $F_1 = -\mathrm{id}$

$\Rightarrow 1 = \deg F_0 = \deg F_1 = (-1)^{n+1}$

$\Rightarrow n$ odd

For n even: $v(x) = (-x_2, x_1, \ldots, -x_{2k}, -x_{2k-1})$ □

Cultural Remark Adams in 1962 proved using alg. topology:
$(\max \#\text{pointwise linearly independent vector fields on } S^n) = 2^b + 8a - 1$
where $n+1 = 2^{4a+b}\cdot(\text{odd number})$, $0 \leq b \leq 3$, $a, b \in \mathbb{N}$, $n \geq 1$. ↖ get 0 if n even $\Rightarrow$ Cor ✓

Local degree

$$\begin{tikzcd} f : S^n \arrow[r] & S^n \\ x \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & y = f(x) \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

(★) Suppose points $\neq x$ near x do not map to y:
$\exists$ nbhds $x \in U \cong D^n$, $y \in V \cong D^n$ s.t. $\boxed{(U, U\setminus x) \xrightarrow{f} (V, V\setminus y)}$
↖ call this $f|_x$ **local map at x**

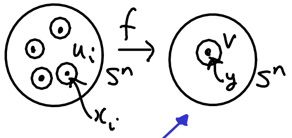
$$\begin{tikzcd} \Rightarrow (f|_x)_* : H_n(U, U\setminus x) \arrow[r, "f_*"] \arrow[d, "\cong"', "\text{excise } S^n\setminus U"] & H_n(V, V\setminus y) \arrow[dd, equal] \\ H_n(S^n, S^n\setminus x) \arrow[d, phantom, "\cong" rotate=90] \arrow[d, phantom, "\underbrace{}_{\cong\, pt}"'] & \\ \widetilde{H}_nS^n \cong \mathbb{Z} \arrow[r] & \mathbb{Z} \\ 1 \arrow[r, mapsto] & \mathbf{\deg_x f} \end{tikzcd}$$

Lemma $f: S^n \to S^n$, $f^{-1}(y) = \{x_1, \dots, x_k\}$

$$\Rightarrow \boxed{\deg f = \sum \deg_{x_i} f}$$

Pf

$$\begin{tikzcd}
\tilde{H}_n S^n \arrow[r, "f_*"] \arrow[d, "\text{quotient}"'] \arrow[dr, phantom, "\circlearrowright"] & \tilde{H}_n(S^n) \arrow[d, phantom, "\cong \text{ exc.}"] \\
H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) \arrow[r, "f_*"'] \arrow[d, "\text{exc. } S^n \setminus \cup U_i"', "\cong"] & H_n(V, V \setminus y) \\
\bigoplus H_n(U_i, U_i \setminus x_i) \arrow[ur, "\Sigma (f|_{x_i})_*"'] &
\end{tikzcd}$$



Rmk can use same V for all i by taking $\tilde{V} = \cap V_i$, $\tilde{u}_i = f^{-1}(V) \cap U_i$

map to each summand is exc. of $S^n \setminus U_i$ so iso.

is:

$$\begin{tikzcd}
1 \in \mathbb{Z} \arrow[r, "\deg f"] \arrow[d] & \mathbb{Z} \arrow[d, phantom, "\cong"] \\
(1, \dots, 1) \in \bigoplus_i \mathbb{Z} \arrow[r, "\Sigma \deg_{x_i} f"'] & \mathbb{Z}
\end{tikzcd}$$

. $\square$

Example $p: \mathbb{C} \to \mathbb{C}$ polynomial $a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, $a_i \in \mathbb{C}$ ($a_n \neq 0$)

$\Rightarrow f: S^2 = \mathbb{CP}^1 \to \mathbb{CP}^1 = S^2$ (where view $\mathbb{CP}^1 = \mathbb{C} \cup \infty \cong S^2$ stereographic projection)

$z \mapsto p(z)$

$\infty \mapsto \infty$

$\Rightarrow$ hpy $F(z,t) = a_n z^n + t(a_{n-1} z^{n-1} + \dots + a_0)$

$F_0 = a_n z^n$ and $F_1 = f$

$$\Rightarrow \deg f = \deg(a_n z^n) \overset{\text{Lemma}}{=\!=} \sum_{k=0}^{n-1} \underbrace{\deg_{w_k} a_n z^n}_{=1} \leftarrow w_k = e^{\frac{2\pi i k}{n}}$$

$= n$

$=$ degree of the poly p.

orient^n preserving homeo near w_k (holomorphic maps are always orientation preserving)

Cor (Fundamental Thm of Algebra) $n \geq 1 \Rightarrow p$ has a root

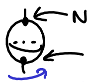
Pf $p^{-1}(0) = \emptyset \Rightarrow f^{-1}(0) = \emptyset \Rightarrow \deg f = 0$ ↯ $\square$

Cultural Rmk For smooth $f: S^n \to S^n$

$\deg f =$ (the number of preimages of a generic point.)

(i.e. almost any point works)

Example $S^2 \to S^2$



rotate by $\frac{2\pi}{d}$ about vertical axis

$\Rightarrow \deg = d = \#$ preimages of a point

except if pick North/South pole

8. CELLULAR HOMOLOGY

Def **CW complex** X is sequence $\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq X^2 \subseteq \dots$

s.t.

- X^0 is any set
- $\boxed{X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I_n} \mathbb{D}^n_\alpha \Big/ x \sim \varphi_\alpha(x)}$

n-discs labelled by some index set I_n

$\varphi_\alpha : \partial \mathbb{D}^n \to X^{n-1}$ **attaching map** (any continuous map, often not injective)

n-skeleton

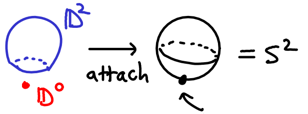
$\Longrightarrow X = \bigcup_{n \geq 0} X^n$ top. space with **weak topology**:

$U \subseteq X$ open $\Longleftrightarrow$ $U \cap X^n \subseteq X^n$ open $\forall n$.

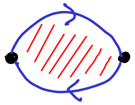
($\Longleftrightarrow$ $\varphi_\alpha^{-1}(U) \subseteq \mathbb{D}^n$ open $\forall \varphi_\alpha$)

Call X **n-dimensional** if $X = X^n$ and this is the least such n.

Example $S^n = (\mathbb{D}^0 \sqcup \mathbb{D}^n) / (\mathbb{D}^0 \sim \partial \mathbb{D}^n)$

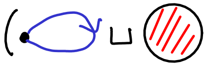
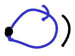


boundary $S^1 = \partial \mathbb{D}^2$ identified with •

Example $X = \mathbb{R}P^2 =$ 

$X^0 = \bullet = \mathbb{D}^0$

$X^1 =$  $= S^1 = (\mathbb{D}^0 \sqcup \mathbb{D}^1)/(\bullet \sim \varphi_1(x))$, $\partial \mathbb{D}^1 = S^0 = \{0,1\} \xrightarrow{\varphi_1} X^0 = \bullet$

$X^2 =$ () / (wrap ∂ of  twice around )

$= (X^1 \sqcup \mathbb{D}^2) \Big/ \left(\overset{\partial \mathbb{D}^2 = S^1}{\overset{\Downarrow}{z}} \sim z^2 \in X^1 = S^1 \right)$ $\quad \partial \mathbb{D}^2 = S^1 \xrightarrow{\varphi_2} S^1$, $\varphi_2(z) = z^2$

Fact If we homotope φ_α, we get a homotopy equivalent space

Example If use another degree 2 map φ_2 above, get $X \simeq \mathbb{R}P^2$.

X is partitioned as a set by interiors of **n-cells** $\boxed{e^n_\alpha = \text{Image}(\mathbb{D}^n_\alpha \to X)}$

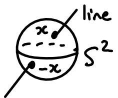
$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I_\alpha} \overset{\circ}{e}{}^n_\alpha$$

$$= \Big(\bigsqcup_{\alpha \in I_0} e^0_\alpha \Big) \sqcup \Big(\bigsqcup_{\alpha \in I_1} \overset{\circ}{e}{}^1_\alpha \Big) \sqcup \Big(\bigsqcup_{\alpha \in I_2} \overset{\circ}{e}{}^2_\alpha \Big) \sqcup \dots$$

$\leftarrow$ **Rmk** interior $\mathbb{D}^0 = \mathbb{D}^0$ so $\overset{\circ}{e}{}^0_\alpha = e^0_\alpha$

Examples

real projective space $\mathbb{RP}^n = S^n/(\mathbb{Z}/_2\text{-action by } \pm id)$ ← $x \sim -x$



$X^k = \mathbb{RP}^k$ inductively

$X^n = X^{n-1} \cup e^n$ with $\varphi : S^{n-1} \to X^{n-1} = \mathbb{RP}^{n-1} = S^{n-1}/\pm id$

$x \longmapsto [x] = [-x]$

complex projective space

$\mathbb{CP}^n = (S^{2n+1} \subseteq \mathbb{C}^n)/(S^1\text{-action by } \lambda \cdot Id)$ ← $x \sim \lambda x$ for $\lambda \in S^1 \subseteq \mathbb{C}^*$

$X^0 = X^1 = pt = \mathbb{CP}^0$

$X^2 = X^3 = X^0 \cup e^2 = \mathbb{CP}^1$, $\varphi : S^1 \to pt$ ← $\mathbb{CP}^1 \cong$  $\cong S^2$

$X^4 = X^5 = X^2 \cup e^4 = \mathbb{CP}^2$, $\varphi : S^3 \to \mathbb{CP}^1 = S^3/(S^1\text{-action})$

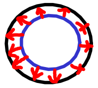
$x \longmapsto [x] = [\lambda x], \forall \lambda \in S^1$

$X^{2n} = X^{2n-2} \cup e^{2n} = \mathbb{CP}^n$, $\varphi : S^{2n-1} \to X^{2n-2} = \mathbb{CP}^{n-1}$

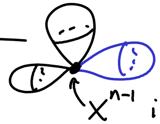
$x \longmapsto [x]$

Observe: For X CW complex, for $n \geqslant 1$: ← (For $n = 1$ $(X^0, X^{-1}) = (X^0, \emptyset)$, $X^0/X^{-1} = X^0$)

- (X^n, X^{n-1}) is a good pair ← (since $\exists$ nbhd of $\partial\mathbb{D}^n$ that deformation retracts to $\partial\mathbb{D}^n$)



- $\boxed{X^n/X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n}$ ←



$S^n_\alpha = \mathbb{D}^n_\alpha/\partial\mathbb{D}^n_\alpha$

X^{n-1} identified to a point

Def **Cellular complex** for X a CW cx,

$$\boxed{\begin{gathered} C^{CW}_n(X) = H_n(X^n, X^{n-1}) \cong H_n(X^n/X^{n-1}) \cong H_n(\bigvee_{\alpha \in I_n} S^n) \\ = \text{free abelian gp gen. by the } n\text{-cells } e^n_\alpha \end{gathered}}$$

since $\Delta^n \cong \mathbb{D}^n \longrightarrow (e^n \subseteq X^n) \longrightarrow \mathbb{D}^n_\alpha/\partial\mathbb{D}^n_\alpha = S^n_\alpha$ generate

← as usual we use the standard orientations of $\Delta^n, \mathbb{D}^n, S^n$.

Will build **cellular differential** d, prove $d \circ d = 0$,

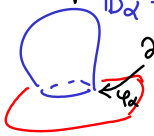
$\Rightarrow$ get $\boxed{H^{CW}_*(X) = H_*(C^{CW}_*(X), d)}$

$\boxed{d\, e^n_\alpha = \sum_{\beta \in I_{n-1}} d^n_{\alpha\beta} \cdot e^{n-1}_\beta}$ now describe the coefficients $d^n_{\alpha\beta} \in \mathbb{Z}$ and why that is a finite sum.

$$\begin{tikzcd}
d : C^{CW}_n \arrow[rr] \arrow[d, equal] & & C^{CW}_{n-1} \arrow[d, equal] \\
H_n(X^n, X^{n-1}) \arrow[r, "\delta^n_*"', "\text{connecting map of LES}"'] & H_{n-1}(X^{n-1}) \arrow[r, "q^{n-1}_*", "\text{quotient map}"'] & H_{n-1}(X^{n-1}, X^{n-2}) \\
e^n_\alpha \arrow[r, mapsto] & (\partial\mathbb{D}^m_\alpha \to \varphi_\alpha(S^{m-1})) \arrow[r, mapsto] & \sum d_{\alpha\beta}\, e^{n-1}_\beta
\end{tikzcd}$$

Recall LES

$$\begin{tikzcd}
H_*(X^{n-1}) \arrow[rr] & & H_*(X^n) \arrow[dl, "q^n_*"] \\
& H_*(X^n, X^{n-1}) \arrow[ul, "\delta^n_*", "{[-1]}"'] &
\end{tikzcd}$$



here it is important that we chose identifications $\Delta^n \cong \mathbb{D}^n$, $S^n \cong \mathbb{D}^n/\partial\mathbb{D}^n$ compatibly with orientations.

Therefore:

$$\boxed{d^n_{\alpha\beta} = \deg(\underset{\partial\mathbb{D}^n_\alpha}{\overset{\|}{S^{n-1}}} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \cong \bigvee_{I_{n-1}} S^{n-1} \xrightarrow{\text{Quotient by } \bigvee_{I_{n-1}\setminus\beta} S^{n-1}} \underset{\mathbb{D}^{n-1}_\beta/\partial\mathbb{D}^{n-1}_\beta}{\overset{\|}{S^{n-1}}})}$$

Rmk Only finitely many $d^n_{\alpha\beta} \neq 0$ (for fixed α) because φ_α, q are continuous and S^{n-1} compact, so get a compact image in $\bigvee_\beta S^{n-1}$, therefore cannot surject onto ∞ many S^{n-1}_β.

Lemma $d \circ d = 0$

Pf $d_n = q^{n-1}_{n-1} \circ \delta^n_n$

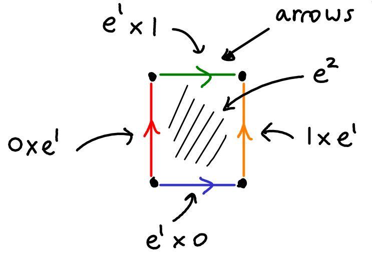
$d_{n-1} \circ d_n = q^{n-2}_{n-2} \circ \overbrace{\delta^{n-1}_{n-1} \circ q^{n-1}_{n-1}}^{=0 \text{ by LES}} \circ \delta^n_n$ $\square$

Cor $\boxed{\operatorname{rank} H^{CW}_n(X) \leqslant \#n\text{-cells}}$

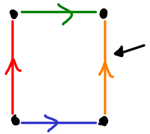
Pf $\#n\text{-cells} = \operatorname{rank} C^{CW}_n(X) \geqslant \operatorname{rank} \operatorname{Ker} d^{CW}_n \geqslant \operatorname{rank} H^{CW}_n(X)$ $\square$

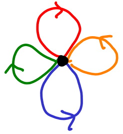
Example $I \times I$ $\quad I = [0,1]$ $\quad \mathbb{D}^1 = [-1,1]$

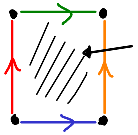
arrows here tell us how we map $[-1,1] \to$ edge (so orientation)



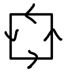
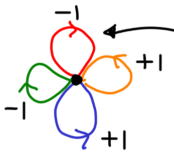
$X^0 =$  $= 4$ 0-cells

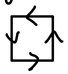
$X^1 =$  ← 4 1-cells $\quad e^1\times 0$, $1 \times e^1$, $e^1 \times 1$, $0 \times e^1$

$X^1/X^0 =$  orientations of cells tell us how to orient the circles

$X^2 =$ 

$e^2 : \mathbb{D}^2 \cong$  $\longrightarrow X^1$

$\partial e^2 : S^1 \cong$  $\longrightarrow X^1/X^0 =$ 

degree -1 because top edge of  maps to  by an orientation-reversing homeomorphism.

$$\Rightarrow \partial e^2 = +e^1 \times 0 + 1 \times e^1 - e^1 \times 1 - 0 \times e^1$$

$\big(= (\partial e^1) \times e^1 - e^1 \times (\partial e^1)$ ← we come back to this later$\big)$

Example $\mathbb{RP}^n$ $\quad$ recall: 1 cell in each dim, $\varphi : S^k \to X^k = \mathbb{RP}^k$, $x \longmapsto [\pm x]$

$S^{k-1} =$ 

$$\Delta_1 \xrightarrow{\varphi} X^{k-1}/X^{k-2} = \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \overset{\Delta_1/\partial\Delta}{\cong} S^{k-1} \qquad \deg = +1$$

$$\Delta_2 \longrightarrow -\mathrm{id}(\Delta_1) \qquad \deg = (-1)^k$$

$$\Rightarrow d^k_{\alpha\beta} = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

$$C^{CW}_*(\mathbb{RP}^n) \quad 0 \to \underset{k=n}{\mathbb{Z}} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{2} \underset{k=1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \to \underset{-1}{0}$$

2 if n even, 0 " odd $\qquad$ 0 2 resp.

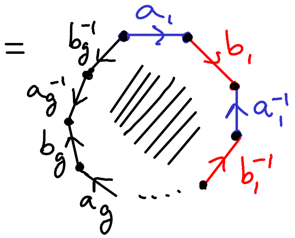
$$H^{cw}_*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & * = 0, \text{ also } * = n \text{ if odd} \\ \mathbb{Z}/2 & * = 1,3,5,\dots < n \\ 0 & \text{else} \end{cases}$$

Example S^n: $\quad C^{CW}_*(S^n)$: $n \geq 2$: $0 \to \mathbb{Z}\cdot\mathbb{D}^n \xrightarrow{0} 0 \to \cdots \to 0 \to \mathbb{Z}\cdot\mathbb{D}^0 \to 0$

$n = 1$: $0 \to \mathbb{Z}\mathbb{D}^1 \xrightarrow{0} \mathbb{Z}\mathbb{D}^0 \to 0$

$\deg\varphi = 0$

$$\Rightarrow H^{cw}_*(S^n) \cong \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{else} \end{cases}$$

Example $\Sigma_g =$  ← genus g surface

boundary identifications $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$

← Notice all vertices are identified, call vertex v

$\partial a_i = v - v = 0$

$\partial b_i = v - v = 0$

$$0 \longrightarrow \underset{\mathbb{Z}\cdot\mathbb{D}}{\overset{\mathbb{Z}}{\cong}} \xrightarrow{\partial} \underset{\mathbb{Z}\langle a_1, b_1, \dots, a_g, b_g\rangle}{\overset{\mathbb{Z}^{2g}}{\cong}} \xrightarrow{\partial = 0} \underset{\mathbb{Z}\cdot v}{\overset{\mathbb{Z}}{\cong}} \longrightarrow 0$$

$$\mathbb{D} \longmapsto -a_1 - b_1 + a_1 + b_1 + \cdots - a_g - b_g + a_g + b_g = 0$$

$$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0,2 \\ \mathbb{Z}^{2g} & * = 1 \\ 0 & \text{else} \end{cases}$$

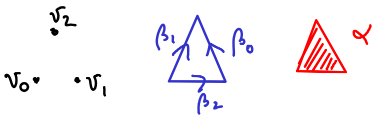
↖ signs: compare edge orientation with anticlockwise orientation of $\partial\mathbb{D}$

Lemma X Δ-cx structure $\Longrightarrow$ induces CW-cx structure on X and $(C^{CW}_*(X), d^{cw}) \cong (C^{\Delta}_*(X), d^{\Delta})$

$$\Longrightarrow \boxed{H^{CW}_*(X) \cong H^{\Delta}_*(X)}$$

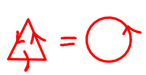
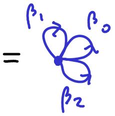
Pf $X^n = \underset{\mathbb{D}^n}{\overset{\cong}{\bigcup}} n\text{-simplices of } X$ and degrees are ± 1 depending on orient^n so can identify d^{cw} and d^{Δ}. ☐

Example $X = \text{triangle} = \Delta^2$



$X^0 \qquad X^1 \qquad X^2$

$$\Rightarrow d^{\Delta}\alpha = \beta_0 - \beta_1 + \beta_2$$

 $\xrightarrow{\varphi} X^1/X^0 =$ 

$d_{\alpha\beta_2} = d_{\alpha\beta_0} = +1$, $\quad d_{\alpha\beta_1} = -1$

$$\Rightarrow d^{cw}\alpha = d^{\Delta}\alpha \quad ✓ \quad ☐$$

Theorem X CW cx (or Δ-cx) $\Longrightarrow$ $\boxed{H_*^\Delta(X) \cong H_*(X)}$

$\Rightarrow$ H_*^Δ, H_*^{CW} independent of choice of CW-cx/Δ-cx structure.

Pf ① $H_*(X^n, X^{n-1}) \cong \tilde{H}_*(X^n/X^{n-1}) \cong \tilde{H}_*(\bigvee_\alpha S^n) \cong \bigoplus_\alpha \tilde{H}_* S^n$ lives in degree n

$= 0 \iff * = n$

LES for (X^n, X^{n-1}) $\Rightarrow$ $H_*(X^{n-1}) \to H_*(X^n)$ iso for $* < n-1$, $* > n$

② for $*<n$: $H_*(X^n) \cong H_*(X^{n+1}) \cong H_*(X^{n+2}) \cong \dots \cong H_*(X)$

by ①

by compactness each sing. chain lands in X^N, some N

③ for $* > n$: $H_*(X^n) \cong H_*(X^{n-1}) \cong H_*(X^{n-2}) \cong \dots \cong H_*(X^{-1}) = 0$

④ LES: $\dots \to \underset{\overset{\|}{0}}{H_n(X^{n-1})} \to H_n(X^n) \xrightarrow[q_n^n]{} H_n(X^n, X^{n-1}) \to \dots$ by ②

$\Rightarrow$ q_n^n injective $\forall n$

⑤ LES: $\dots \xrightarrow{q^{n+1}_{n+1}} H_{n+1}(X^{n+1}, X^n) \xrightarrow{\delta^{n+1}_{n+1}} H_n(X^n) \to H_n(X^{n+1}) \to \overset{①}{0}$

UPSHOT

$$H_n(X) \overset{②}{\cong} H_n(X^{n+1}) \overset{⑤}{\cong} H_n(X^n) / \operatorname{im} \delta^{n+1}_{n+1} \overset{④}{\cong} \underbrace{(q_n^n H_n(X^n))}_{\substack{\operatorname{im} q_n^n \\ \| \\ \operatorname{Ker} \delta_n^n \overset{④}{=} \operatorname{Ker} \underbrace{q^{n-1}_{n-1} \circ \delta^n_n}_{d_n^{cw}}}} / \operatorname{im} \underbrace{q_n^n \circ \delta^{n+1}_{n+1}}_{d^{cw}_{n+1}} \cong H_n^{CW}(X)$$

1st iso thm

exactness LES

□

Rmk by ① $\boxed{H_k \text{ not affected if attach } (k+2)\text{-cells or higher}}$

by ② $\boxed{\text{Inclusion } X^n \to X \text{ induces iso } H_*(X^n) \to H_*(X) \text{ for } * < n}$

Cor X n-dimensional cell cx $\Longrightarrow$ $H_*(X) = 0$ for $* > n$

Axioms for homology: Eilenberg-Steenrod axioms

It was no accident that H_*^Δ, H_*^{CW}, H_* all agreed.

Def A **generalised homology theory (GHT)** is a **functor** F: TopPairs = (Category of pairs of spaces and maps of pairs) $\longrightarrow$ Graded Abelian Gps

with a **natural transformation** $\delta : F_*(X,A) \longrightarrow F_{*-1}(X, \emptyset)$ satisfying: ← abbreviated: $F_{*-1}(X)$

1) **homotopy invariance**: $f \simeq g \Rightarrow F(f) = F(g)$

2) **exactness**: $\exists$ LES $\boxed{\dots \to F_*(A) \xrightarrow[F(\text{incl}: A \to X)]{} F_*(X) \xrightarrow[F(\text{incl}: (X,\emptyset) \to (X,A))]{} F_*(X,A) \xrightarrow{\delta} F_{*-1}(A) \to \dots}$

3) **additivity**: $(X,A) = \bigsqcup (X_i, A_i)$, $\text{incl}_i : (X_i, A_i) \longrightarrow (X,A)$

then $\boxed{\Sigma F(\text{incl}) : \bigoplus F(X_i, A_i) \xrightarrow{\cong} F(X,A)}$

4) **excision**: $\boxed{\bar{E} \subseteq A^\circ \subseteq X \Rightarrow F(X \setminus E, A \setminus E) \xrightarrow[F(\text{incl})]{\cong} F(X,A)}$

Remark (4) $\iff$ $X = A^\circ \cup B^\circ$, incl: $(A, A \cap B) \longrightarrow (X,B)$

then $\boxed{F(\text{incl}) : F(A, A\cap B) \xrightarrow{\cong} F(X,B)}$

Pf $A = X \setminus E$, $B = A$ noticing that $(X \setminus E)^\circ \cup A^\circ = X$

$E = B \setminus A$ noticing that $\bar{E} \subseteq \bar{B} \setminus A^\circ \subseteq B^\circ \setminus A^\circ \subseteq B^\circ = A^\circ$ — $X = A^\circ \cup B^\circ$ so $\partial B \subseteq A^\circ$

Rmk In (3), the topology on the disjoint union $\bigsqcup (X_i, A_i)$ is defined by: $U \subseteq \bigsqcup (X_i, A_i)$ open $\iff$ $U \cap X_i \subseteq X_i$ open $\forall i$

FACT **Theorem**

a) $(F, \delta_F), (G, \delta_G)$ GHTs, $\alpha : F \to G$ a natural transformation commuting with δ_F, δ_G such that $\alpha_{\text{point}} : F(\text{point}) \to G(\text{point})$ is an iso, then α is an iso.

b) If (F, δ_F) GHT satisfies (5) **dimension**: $\boxed{F_*(\text{point}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}}$

Then $\exists$ natural iso $F \cong H_*$ (such an F is called a **homology theory**)

Rmk In (b) if require $F_0(\text{point}) = G$ an abelian group (instead of $\mathbb{Z}$)

$\Longrightarrow$ $F(X,A) \cong H_*(X,A;G)$ = (homology with coefficients in G) ← later in course

9. COHOMOLOGY

(C_*, ∂_*) chain cx s.t. C_* free $\mathbb{Z}$-module ← $C_* \cong \bigoplus_\alpha \mathbb{Z}$

Def n-cochains $\boxed{C^n = \mathrm{Hom}(C_n, \mathbb{Z})}$

cobondary map $\partial^n : C^n \longrightarrow C^{n+1}$

$\partial^n(\phi) = \phi \circ \partial_{n+1}$

$$\begin{tikzcd} C^n \arrow[d, "\phi"'] & C^{n+1} \arrow[l, "\partial_{n+1}"'] \arrow[dl, "\partial\phi"] \\ \mathbb{Z} & \end{tikzcd}$$

Notice ∂^* is degree +1 map (not −1)

$$\boxed{H^m(C_*, \partial_*) = \frac{\mathrm{Ker}\, \partial^m \leftarrow \text{cocycles}}{\mathrm{Im}\, \partial^{m-1} \leftarrow \text{coboundaries}}}$$

Rmk If use negative grading, (C^{-*}, ∂^{-*}) is a chain complex with homology so many results from H_* carry over to H^*. So abusively write

$$\boxed{H^*(C_*, \partial_*) = H_*(\mathrm{Hom}(C_*, \mathbb{Z}), \partial^*)}$$

Remarks about dualisation

$\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$ generated by projection maps $\pi_i(x_1, \dots, x_n) = x_i$

$\alpha : \mathbb{Z}^n \to \mathbb{Z}^m$, $x \mapsto A \cdot x$ ← m×n matrix

$$\begin{tikzcd} \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z}) \arrow[l, "\text{dual}"', "\alpha^*"] \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z}^n & \mathbb{Z}^m \arrow[l, "\text{transpose }(A)"] \end{tikzcd}$$

$\Rightarrow$ as above; $\alpha^*\phi = \phi \circ \alpha$

Def X space $\Rightarrow$ singular cohomology $\boxed{H^*(X) = H^*(C^*(X), \partial^*)}$

dualise $C_* = C_*(X)$

similarly define H^*_Δ, H^*_{CW}

Example $\mathbb{RP}^3$: $C^{CW}_*(\mathbb{RP}^3)$: $0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

dualise: $C^*_{CW}(\mathbb{RP}^3)$: $0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longrightarrow 0$

$$H^*(\mathbb{RP}^3) \cong H^*_{CW}(\mathbb{RP}^3) = \begin{cases} \mathbb{Z} & * = 0, 3 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

← notice $H_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$ has moved to grading 2.

Functoriality

$f : X \to Y \Rightarrow f_* : C_*X \to C_*Y$

$\Rightarrow C^*X \xleftarrow{f^*} C^*Y$ is dual: $\boxed{f^*\phi = \phi \circ f_*}$ ← called pull-back

Lemma f^* is a cochain map

$\Rightarrow \boxed{f^* : H^*Y \to H^*X}$

Pf
$$\begin{aligned} (\partial^* \circ f^*)(\phi) &= \partial^*(\phi \circ f_*) \\ &= (\phi \circ f_*) \circ \partial \\ &= (\phi \circ \partial) \circ f_* \quad \text{as } f_* \text{ chain map} \\ &= f^* \circ (\phi \circ \partial) \\ &= f^* \circ (\partial^* \phi) \\ &= (f^* \circ \partial^*)(\phi) \end{aligned}$$

Properties
- $\mathrm{id}^* = \mathrm{id}$
- $(f \circ g)^* = g^* \circ f^*$ notice order!

$\Rightarrow \boxed{H^* : \mathrm{Top} \longrightarrow \mathrm{GradedAbGps}}$ contravariant functor

Exercise $H^0(X) = \prod_{\pi_0 X} \mathbb{Z}$ where $\pi_0 X = \{\text{path-components of } X\}$

Homotopy invariance

Lemma $f_*, g_* : C_* \longrightarrow \tilde{C}_*$ chain hpic $\Rightarrow f^* = g^* : H^*\tilde{C} \to H^*C$

free

Pf $f_* - g_* = \tilde{\partial} \circ h + h \circ \partial$ some $h : C_* \longrightarrow \tilde{C}_*[1]$

$f^* - g^* = h^* \circ \tilde{\partial}^* + \partial^* \circ h^*$ for dual $h^* : \tilde{C}^* \longrightarrow C^*[-1]$.

(notice degree −1, not +1) □

Def h^* called cochain homotopy

Cor $f \simeq g : X \to Y \Rightarrow f^* = g^* : H^*Y \longrightarrow H^*X$ □

Algebra : dual of SES

Lemma $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ exact, A,B,C free

$\Longrightarrow$ $0 \leftarrow A^* \xleftarrow{i^*} B^* \xleftarrow{j^*} C^* \leftarrow 0$ " , A^*,B^*,C^* free

Pf C free $\Rightarrow$ $\exists$ splitting $B \underset{s}{\overset{j}{\rightleftarrows}} C$ $j\circ s = id$

↑ Pick preimages b_i for basis e_i of C, then $s(e_i) = b_i$

$\Rightarrow$ $A \oplus C \xrightarrow[i\oplus s]{\cong} B$

Rmk inverse is $B \cong A \oplus C$
$b \longmapsto i^{-1}(b - s(b)) \oplus j(b)$

dual $\Rightarrow$ $A^* \oplus C^* \xleftarrow[i^*\oplus s^*]{\cong} B^*$ and $s^* \circ j^* = id$

→ so i^* surj → so j^* inj

$\Rightarrow$ $0 \leftarrow A^* \xleftarrow{i^*} B^* \underset{s^*}{\overset{j^*}{\leftrightarrows}} C^* \leftarrow 0$

where $0 = (j\circ i)^* = i^* \circ j^*$ so $\operatorname{Im} j^* \subseteq \operatorname{Ker} i^*$

Prove $\supseteq$: $i^* b = 0 \Rightarrow b - j^* s^* b \in \operatorname{Ker} i^* \cap \operatorname{Ker} s^* = \{0\}$

since $s^* j^* = id$

$\Rightarrow b = j^* s^* b \in \operatorname{Im} j^*$

$\Rightarrow \operatorname{Ker} i^* = \operatorname{Im} j^*$ □

Excision, LES, Mayer–Vietoris

By previous Lemma get dual results :

Excision $\bar{A} \subseteq V^\circ \subseteq X \Rightarrow H^*(X\setminus A, V\setminus A) \overset{i^*}{\xleftarrow{\cong}} H^*(X,V)$

LES for pair (X,A) $\dots \xleftarrow{q^*[+1]} H^{*+1}(X,A) \xleftarrow{\delta} H^*(A) \xleftarrow{i^*} H^*(X) \xleftarrow{q^*} H^*(X,A) \leftarrow \dots$

M.V. $X = A^\circ \cup B^\circ \Rightarrow \dots \leftarrow H^{*+1}(X) \leftarrow H^*(A\cap B) \xleftarrow{i_A^* \oplus -i_B^*} H^*(A) \oplus H^*(B) \xleftarrow{j_A^* \oplus j_B^*} H^*(X) \leftarrow \dots$

where
$$\begin{tikzcd} & A \arrow[dr, "j_A"] & \\ A\cap B \arrow[ur, "i_A"] \arrow[dr, "i_B"'] & & X \\ & B \arrow[ur, "j_B"'] & \end{tikzcd}$$
are the obvious maps

Axioms for cohomology These are analogous to the axioms for homology except we reverse all arrows, and we change axiom (3): $\prod$ instead of $\oplus$

additivity : $(X,A) = \bigsqcup (X_i, A_i)$, $incl_i : (X_i, A_i) \longrightarrow (X,A)$

then $\boxed{\prod F(incl) : \prod F(X_i,A_i) \xleftarrow{\cong} F(X,A)}$

10. CUP PRODUCT

Theorem $H^*(X)$ ↖space is ① **unital** ② **graded-commutative** ring via

$\smile : H^k(X) \times H^\ell(X) \longrightarrow H^{k+\ell}(X)$ determined by

$$\boxed{\begin{aligned} \smile : C^k(X) \times C^\ell(X) &\longrightarrow C^{k+\ell}(X) \\ (\phi \smile \psi)(\sigma) &= \phi(\sigma|_{[e_0,\dots,\underline{\underline{e_k}}]}) \cdot \psi(\sigma|_{[\underline{\underline{e_k}},\dots,e_{k+\ell}]}) \end{aligned}}$$

① $1 \in C^0(X)$ constant function $\Rightarrow$ $1 \smile \phi = \phi \smile 1$

② $\phi \smile \psi = (-1)^{\deg\phi \cdot \deg\psi} \psi \smile \phi$

Useful trick

If X is CW-cx, then $C_*^{CW}(X) \xrightarrow[\simeq]{\text{inclusion}} C_*(X)$, so $C^*_{CW}(X) \xleftarrow[\simeq]{\text{restriction}} C^*(X)$. So to define/determine a class in $H^*(X)$ it is enough to define its values on CW chains (provided it is a CW-cycle). So doing: $H^k_{CW} \times H^\ell_{CW} \xrightarrow{\cong} H^k \times H^\ell \xrightarrow{\smile} H^{k+\ell}$.

Proof of Theorem

$$\begin{aligned} \partial^*(\phi\smile\psi)(\sigma) &= (\phi\smile\psi)(\partial\sigma) \\ &= (\phi\smile\psi)\ \textstyle\sum (-1)^i\ \sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_n]} \quad n = k+\ell \\ &= \sum_{i\leq k} (-1)^i \phi(\sigma|_{[e_0,\dots,\hat{e}_i,\dots,\underline{\underline{e_{k+1}}}]}) \cdot \psi(\sigma|_{[\underline{\underline{e_{k+1}}},\dots,e_n]}) \\ &\quad + \sum_{i>k} (-1)^i \phi(\sigma|_{[e_0,\dots,e_k]}) \cdot \psi(\sigma|_{[e_k,\dots,\hat{e}_i,\dots,e_n]}) \cdot \underbrace{(-1)^{i-k}(-1)^{k-i}}_{1} \\ &= ((\partial^*\phi)\smile\psi(\sigma)) + (-1)^k\ \phi\smile\partial^*\psi \end{aligned}$$

induces $[\phi]\smile[\psi] = [\phi\smile\psi]$:

well-defined : • cycles→cycle : $\partial(\phi\smile\psi) = \overset{=0}{\widetilde{(\partial\phi)}}\smile\psi \pm \phi\smile\overset{=0}{\widetilde{(\partial\psi)}} = 0$ ✓

• $[\phi] = [\phi + \partial\alpha]$ so need $[(\partial\alpha)\smile\psi] = 0$

$(\partial\alpha)\smile\psi \underset{\partial\psi=0}{\overset{\uparrow}{=}} \partial(\alpha\smile\psi)$ ✓

• Similarly $[\phi]\smile[\partial\beta] = 0$

bilinear, associative, distributive: true at chain level unital: $(\partial 1)(\sigma) = 1(\sigma|_{[e_0]}) - 1(\sigma|_{[e_1]}) = 1-1=0$

$(1\cup\psi)(\sigma) = 1(\sigma|_{[e_0]})\cdot\psi(\sigma|_{[e_0,\ldots,e_n]}) = \psi(\sigma)$ ($\phi\cup 1 = \phi$ similar)

graded-comm. sketch proof:

Let $r : C_n(X)\longrightarrow C_n(X)$, $r(\sigma) = \varepsilon_n\,\overline{\sigma}$ where: $\varepsilon_n = (-1)^{\frac{n(n+1)}{2}}$

and $\overline{\sigma}|_{[v_0,\ldots,v_n]} = \sigma|_{[v_n,\ldots,v_0]}$ ← reverse order of vertices: is product of $\underbrace{n+(n-1)+\cdots+1}_{n(n+1)/2}$ transpositions

(idea: each transposition is a reflection in hyperplane, so reverses orientation of simplex, so insert ε_n to compensate)

one checks:
- r chain map
- $\underbrace{r^*\varphi\cup r^*\psi}_{\varepsilon_k\varepsilon_\ell} = \underbrace{r^*(\psi\cup\varphi)}_{\varepsilon_{k+\ell}}$ differ by $(-1)^{k\ell}$
- $r\simeq id$ so can drop $r^* = id$ on cohomology

($r - id = P\partial + \partial P$ with

$P\sigma = \sum(-1)^i\,\varepsilon_{n-i}\,(\sigma\circ\pi)|_{[v_0,\ldots,v_i,\underline{\underline{w_n}},\ldots,w_i]}$)

v_i, w_i like for prism operator

projection $\Delta^n\times I\xrightarrow{\pi}\Delta^n$

□

Naturality of cup product

Lemma $f: X\longrightarrow Y \implies f^* : H^*Y\longrightarrow H^*X$ hom of unital rings

Pf
$$\begin{aligned} f^*(\varphi\cup\psi)(\sigma) &= (\varphi\cup\psi)(f_*\sigma)\\ &= \varphi(f_*\sigma|_{[e_0,\ldots,e_k]})\cdot\psi(f_*\sigma|_{[e_k,\ldots,e_n]})\\ &= ((\varphi\circ f_*)\cup(\psi\circ f_*))(\sigma)\\ &= (f^*\varphi\cup f^*\psi)(\sigma)\end{aligned}$$

unital: $f^*(1) = 1\circ f_* = 1$ □

UPSHOT $\boxed{H^* : \text{Top}\longrightarrow\left\{\begin{array}{l}\text{Graded-commutative unital rings}\\ \text{with graded unital ring homs}\end{array}\right\}\ \text{contravariant functor.}}$

Warning An (iso)morphism $H^*(Y)\longrightarrow H^*(X)$ of groups will also preserve the ring structure if f^* is induced by a map of spaces $X\to Y$ (by above Lemma).

$\implies$ Cor The excision theorem iso on cohomology is an iso of rings.

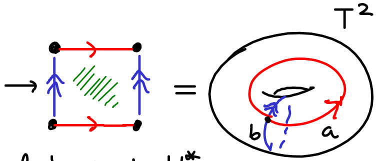
However the connecting hom in M.V. or LES cannot possibly be a ring hom since it drops gradings by 1 ($\Rightarrow\delta(a\cup b)$ and $\delta(a)\cup\delta(b)$ have different grading!)

Example $H^1(T^2)\times H^1(T^2)\xrightarrow{\cup}H^2(T^2)$ bilinear form $\mathbb{Z}^2\times\mathbb{Z}^2\to\mathbb{Z}$ with matrix $\begin{pmatrix}0&+1\\-1&0\end{pmatrix}$

Pf recall:

*		$H_*(T^2)$	$H^*(T^2)$
0	$\mathbb{Z}$	$\mathbb{Z}\cdot pt$	$\mathbb{Z}\cdot 1$
1	$\mathbb{Z}^2$	$\mathbb{Z}a\oplus\mathbb{Z}b$	$\mathbb{Z}a^*+\mathbb{Z}b^*$
2	$\mathbb{Z}$	$\mathbb{Z}\cdot D$	$\mathbb{Z}\cdot D^*$

where $D:\Delta^2\cong$ square →



$1, a^*, b^*, D^*$ are dual basis in H^*

Identify $H^*(T^2)\cong H^*_\Delta(T^2)$ so at chain level:

$a^*: C_1^{cw}(X)\longrightarrow\mathbb{Z}$, $a\longmapsto 1$, $b\longmapsto 0$

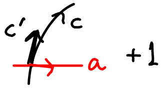
$b^*: C_1^{cw}(X)\longrightarrow\mathbb{Z}$, $a\longmapsto 0$, $b\longmapsto 1$

$D^*: C_2^{cw}(X)\longrightarrow\mathbb{Z}$, $D\longmapsto 1$

$\Rightarrow b^*(\underset{C_1^{cw}}{\underset{\cap}{c}}) =$ # a intersects c counted with orientation signs

$a^*(\underset{C_1^{cw}}{\underset{\cap}{c}}) = -$ # b intersects c counted with signs.

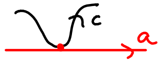
Fact Same holds for smooth singular 1-chains $c:\Delta^1\cong I\longrightarrow T^2$ which intersect a transversely: velocity vectors a', c' span $\mathbb{R}^2$



Otherwise ill-defined:

c not smooth

and

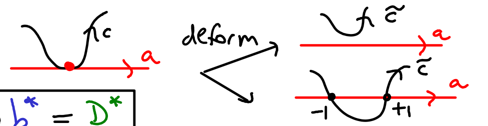


a,c not transverse (tangency)

are bad.

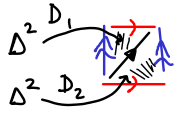
trick need first pick homologous representative which is smooth & transverse, by continuously deforming the chain (continuous map $\simeq id$ so id on H_*)

Example



both cases: $a^*(\tilde{c}) = 0$

claim $\boxed{a^*\cup b^* = D^*}$



$(a^*\cup b^*)(\underbrace{D_1+D_2}_{\text{homologous to T}}) = a^*(D_1|_{[e_0,e_1]})\cdot b^*(D_1|_{[e_1,e_2]}) +$ same for D_2

$= a^*(a)\,b^*(b) + a^*(b)\,b^*(a)$

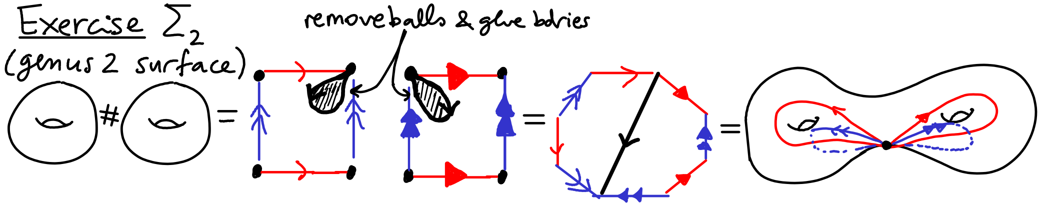
$= 1$

Graded-comm. $\Rightarrow b^*\cup a^* = -D^*$, $a^*\cup a^* = (-1)^{1\cdot1}a^*\cup a^*$ so $=0$, sim'ly $b^*\cup b^* = 0$. □

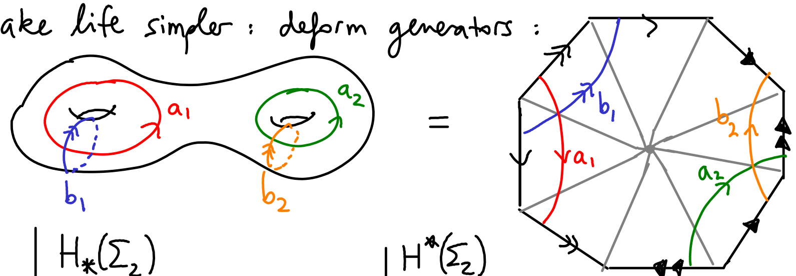
Idea $\cup$ just counts (signed) geometric intersection # of corresponding curves.

Why "$a\cap a = 0$"? Can deform a to make it disjoint from a:

Exercise Σ_2 (genus 2 surface)



Make life simpler : deform generators :



		$H_*(\Sigma_2)$	$H^*(\Sigma_2)$
0	$\mathbb{Z}$	$\mathbb{Z}\cdot pt$	$\mathbb{Z}\cdot 1$
1	$\mathbb{Z}^4$	$\mathbb{Z}a_1 + \mathbb{Z}b_1 + \mathbb{Z}a_2 + \mathbb{Z}b_2$	$\mathbb{Z}\langle a_1^*, b_1^*, a_2^*, b_2^* \rangle$ ← dual basis
2	$\mathbb{Z}$	$\mathbb{Z}\cdot D$	$\mathbb{Z}\cdot D^*$

Notice on $C_1^{CW}(\Sigma_2)$: $a_i^*(c) = -\underset{\text{signed count}}{\#}(b_i \text{ intersects } c)$

$b_i^*(c) = \#(a_i$ " " $)$

Exercise $a_i^* \cup b_j^* = \underset{\left\{\begin{smallmatrix}1 & i=j\\ 0 & i\neq j\end{smallmatrix}\right.}{\delta_{ij}} \cdot D^* = - b_j^* \cup a_i^*$

$a_i^* \cup a_i^* = b_i^* \cup b_i^* = 0$

} so same as geometric intersection numbers of corresponding curves.

Cultural Rmk on general theory (Intersection Theory/Differential Topology)

M^m oriented m-mfd
$N^n \subseteq M^m$ oriented compact n-dim submfd $\Rightarrow$ $H_n(N) \xrightarrow{incl_*} H_n(M)$ ← see later in course
$\exists\, [N] \longmapsto [N]$

N, M also smooth (see Differential Geometry course) $\Rightarrow$ $\omega_N \in H^{m-n}(M)$ counts # intersecns with N (with signs)

$N_1^{n_1}, N_2^{n_2} \subseteq M$ compact oriented smooth submfds
$n_1 + n_2 = m$
(so complementary dimensions) $\Rightarrow$ $\omega_{N_1} \cup \omega_{N_2} = \underbrace{\#(N_1 \cap N_2)}_{\text{geometric intersection }\#} \cdot [M]^*$

may require ← deforming unless N_1, N_2 are already transverse

Fact (Thom 1954)
Not all $a \in H^j(M)$ arise as ω_N for connected compact oriented codim=j smooth submfd N
But $\exists\, N \in \mathbb{N}$ s.t. $N\cdot a$ does arise. They do arise for $H^*(M;\mathbb{Q})$, $H^*(M,\mathbb{R})$, $H^*(M;\mathbb{Z}/2)$

11. KÜNNETH FORMULA AND PRODUCT SPACES

Algebra : tensor products

R ring (comm. with 1) ← e.g. abelian groups = $\mathbb{Z}$-mods, vector spaces/$\mathbb{F}$ = $\mathbb{F}$-mods

Def A, B R-modules $\Rightarrow$ **Tensor product** is R-module

$$\boxed{A \underset{R}{\otimes} B = \langle (a,b) : a \in A, b \in B \rangle / \text{relations of bilinearity \& rescaling}}$$

(or $A \otimes B$) R-mod generated; write $a \otimes b$ for its class

bilinearity : $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$
$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$

rescaling : $r \cdot (a \otimes b) = (ra) \otimes b = a \otimes (rb)$ $\quad \forall r \in R$

- So general element looks like $\sum a_k \otimes b_k$ (finite sum) ← NOT UNIQUELY!
- Don't confuse with $A \times B$: e.g. $0 \otimes b = 0$ $\forall b$

Rmk Can define $A \underset{R}{\otimes} B$ also by a universal property : for all R-mods C,

$$\boxed{Hom_R(A \underset{R}{\otimes} B; C) \xrightarrow[\text{natural}]{\cong} \{R\text{-bilinear maps } A \times B \to C\}}$$

Using above description of $A \otimes B$: $\varphi \longmapsto (A \times B \to C, (a,b) \mapsto \varphi(a \otimes b))$

Example ($R = \mathbb{F}$) $\underset{\text{basis } v_i}{V}, \underset{\text{basis } w_j}{W}$ v.s./$\mathbb{F}$ $\Rightarrow$ $V \otimes W$ v.s./$\mathbb{F}$ basis $v_i \otimes w_j$
$\dim_{\mathbb{F}} V \otimes W = \dim V \cdot \dim W$

Exercise V, W finite dim/$\mathbb{F}$ $\Rightarrow$ $V^* \otimes W \cong Hom_{\mathbb{F}}(V, W)$
Hint $f : V \to \mathbb{F}$, $w \in W$, $f \otimes w \mapsto (V \to W, v \mapsto f(v)\cdot w)$

Examples ($R = \mathbb{Z}$)
- $\mathbb{Z}^n \otimes \mathbb{Z}^m \cong \mathbb{Z}^{n\cdot m}$
- $\mathbb{Z}/n \otimes \mathbb{Z} \cong \mathbb{Z}/n$ ← $1 \otimes x = x \otimes 1$
- $\mathbb{Z}/2 \otimes \mathbb{Z}/3 = 0$ ← $1 \otimes x = 3 \otimes x = 1 \otimes 3x = 0$
- $\mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong \mathbb{Z}/2$ ← $1 \otimes 1 = 3 \otimes 1 = 1 \otimes 3$, $1 \otimes 2 = 2 \otimes 1 = 0$

Examples
- $A \otimes B \cong B \otimes A$
- $(\underset{i}{\oplus} A_i) \otimes (\underset{j}{\oplus} B_j) \cong \underset{i}{\oplus}\underset{j}{\oplus} (A_i \otimes B_j)$
- $A \otimes R \cong A$ (so "$\cdot \underset{R}{\otimes} R$ does nothing")
- $A \otimes R/d \cong A/d\cdot A$

} hence now know $A \otimes B$ for any f.g. R-mods A, B.

for example $\mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong (\mathbb{Z}/4)/2\cdot\mathbb{Z}/4 \cong \mathbb{Z}/2$ ← (Rmk $(\mathbb{Z}/n)/m\cdot\mathbb{Z}/n \cong \mathbb{Z}/gcd(m,n)$)

More generally : $\begin{cases} R/_I \underset{R}{\otimes} R/_J \cong R/_{(I+J)} & \text{for ideals } I, J \subseteq R. \\ A \otimes_R R/_J \cong A/_{J\cdot A} \end{cases}$

Warning $\cdot\otimes A$ often not an exact functor, i.e. does not preserve exact sequences
indeed it can ruin injectivity: $0\to\mathbb{Z}\xrightarrow{\cdot 2}\mathbb{Z}$ now take $\cdot\otimes\mathbb{Z}/2$ get $0\to\mathbb{Z}/2\xrightarrow{0}\mathbb{Z}/2$.

Fact $\boxed{\cdot\otimes_{\mathbb{Z}}\mathbb{Q} \text{ and } \cdot\otimes_{\mathbb{Z}}\mathbb{R} \text{ are exact functors on } \mathbb{Z}\text{-mods}}$ ← More generally $\cdot\otimes_R \mathrm{Frac}(R)$ is exact on R-mods where Frac R is fraction field, and R is an integral domain "localisation is an exact functor"

example A f.g. $\mathbb{Z}$-mod $\Rightarrow A\cong\mathbb{Z}^r\oplus\mathbb{Z}/_{d_1}\oplus\cdots\oplus\mathbb{Z}/_{d_k}$ some $d_i\neq 0$
$\Rightarrow A\otimes\mathbb{Q}\cong\mathbb{Q}^r \Rightarrow \boxed{r=\operatorname{rank}A=\dim_{\mathbb{Q}}(A\otimes\mathbb{Q})}$

Tensor product of chain cxes

$C_*, \tilde{C}_*$ chain cxes of R-mods

$$(C_*\otimes\tilde{C}_*)_n = \bigoplus_{i+j=n} C_i\otimes\tilde{C}_j$$

$$\boxed{\partial(x\otimes y) = (\partial x)\otimes y + (-1)^{\deg x}\, x\otimes\tilde{\partial}y}$$ "Leibniz rule"

Think of ∂ as an operator of deg $=-1$ acting from left

since ∂ "jumps over x" get $(-1)^{\deg\partial\cdot\deg x}$

Exercise $\partial\circ\partial=0$ ← would fail without sign

$$Z_i\otimes\tilde{Z}_j\subseteq Z_{i+j}(C_*\otimes\tilde{C}_*) \text{ and } \left.\begin{array}{l} Z_i\otimes\tilde{B}_j \\ B_i\otimes\tilde{Z}_j\end{array}\right\}\subseteq B_{i+j}(C_*\otimes\tilde{C}_*)$$

Cor $\exists$ natural maps $\boxed{\begin{array}{l} H_i(C_*)\otimes H_j(\tilde{C}_*)\longrightarrow H_{i+j}(C_*\otimes\tilde{C}_*) \\ \sum[c_k]\otimes[\tilde{c}_k]\longmapsto\sum[c_k\otimes\tilde{c}_k]\end{array}}$

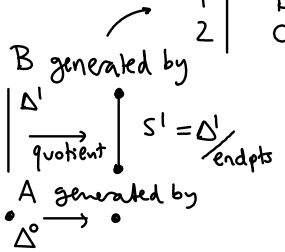
FACT:
Algebraic Künneth Thm
$C_*, H_*(C_*)$ f.g. free $\overset{\text{PID}}{R}$-mods (no assumption on $\tilde{C}_*$)

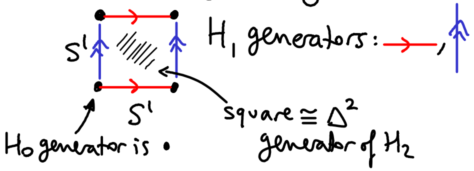
$$\Rightarrow \boxed{\bigoplus_{i+j=n} H_i(C_*)\otimes H_j(\tilde{C}_*) \cong H_n(C_*\otimes\tilde{C}_*)}$$ via

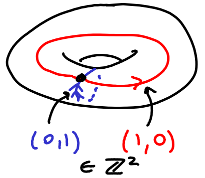
Example

$*$	$H_*(S^1)$
0	$A\cong\mathbb{Z}$
1	$B\cong\mathbb{Z}$
2	0

$*$	$H_*(S^1\times S^1)$ ← torus	
0	$A\otimes A$	$\cong\mathbb{Z}$
1	$(A\otimes B)\oplus(B\otimes A)$	$\cong\mathbb{Z}^2$
2	$B\otimes B$	$\cong\mathbb{Z}$
3	0	







Product spaces

$\underset{e_\alpha}{X}, \underset{e_\beta}{Y}$ CW-cxes $\Rightarrow X\times Y$ CW-cx with cells $e_\alpha\times e_\beta$ attaching maps

$$\begin{tikzcd}[column sep=tiny] \partial(D^i_\alpha\times D^j_\beta)=(\partial D^i_\alpha)\times D^j_\beta \arrow[d, "\varphi_\alpha\times id"'] & \cup & D^i_\alpha\times(\partial D^j_\beta) \arrow[d, "id\times\varphi_\beta"] \\ X^{i-1}\times Y^j \arrow[dr] & & X^i\times Y^{j-1} \arrow[dl] \\ & (X\times Y)^{i+j-1} & \end{tikzcd}$$

Algebra: Euler characteristic

C finitely generated graded abelian gp (so $\mathbb{Z}$-mod)
(more generally: R-mod for PID R)

Def Euler characteristic $\chi(C)=\sum(-1)^i\operatorname{rank}C_i$

Example/Motivation X finite CW-cx then take $C=C_*^{CW}(X)$ to get
$$\boxed{\chi(X)=\#(0\text{-cells})-\#(1\text{-cells})+\#(2\text{-cells})-\cdots}$$

Lemma If C_* f.g. chain cx $\Rightarrow \boxed{\chi(C_*)=\chi(H_*(C_*))}$ $(=\sum(-1)^i\operatorname{rank}H_i(C))$

Pf Observation: $\operatorname{rank}C_i=\dim_{\mathbb{Q}}(C_i\otimes_{\mathbb{Z}}\mathbb{Q})$ ← for R-mods, do $\dim_{\mathbb{F}}(C_i\otimes_R\mathbb{F})$ with $\mathbb{F}=\mathrm{Frac}(R)$
$\Rightarrow$ WLOG assume C_i are vector spaces/field $\mathbb{F}$.
Abbreviate $|C_i|=\dim_{\mathbb{F}}C_i$.

$0\to Z_i\to C_i\xrightarrow{\partial}B_{i+1}\to 0 \overset{\text{Rank-nullity thm}}{\Rightarrow} |C_i|=|Z_i|+|B_{i+1}|$
$0\to B_i\to Z_i\to H_i\to 0 \Rightarrow |H_i|=|Z_i|-|B_i|$
$\Rightarrow |C_i|-|H_i|=|B_{i+1}|-|B_i|$

$\Rightarrow\chi(C_*)-\chi(H_*)=\sum(-1)^i|B_{i+1}|-\sum(-1)^i|B_i|=\sum(-1)^i(-|B_i|+|B_i|)=0.$ □

Cor X space $\Rightarrow \boxed{\begin{array}{l}\chi(X)=\sum(-1)^i\operatorname{rank}H_i(X) \\ \quad\;=\sum(-1)^i\operatorname{rank}C_i(X)\end{array}}$ ← if finite rank $H_*(X)$ ← if finite rank $C_*(X)$

So $\chi(X)$ is invariant up to hpy equivalence! Example $\chi\left(\begin{smallmatrix}\text{platonic}\\ \text{solid}\end{smallmatrix}\right)=\chi(S^2)=2$

Cor $\chi(X\times Y)=\chi(X)\cdot\chi(Y)$ $\forall$ finite CW-cxes X, Y

Pf $\sum(-1)^k\operatorname{rank}H_k^{CW}(X\times Y)=\sum(-1)^k\operatorname{rank}C_k^{CW}(X\times Y)$
$=\sum_k\sum_{i+j=k}(-1)^{i+j}\operatorname{rank}C_i^{CW}(X)\cdot\operatorname{rank}C_j^{CW}(Y)$. □

Lemma $d(e^i_\alpha\times e^j_\beta)=(de^i_\alpha)\times e^j_\beta+(-1)^i e^i_\alpha\times(de^j_\beta)$

hence $\boxed{C_*^{CW}(X\times Y)\cong C_*^{CW}(X)\otimes C_*^{CW}(Y)}$

(hence if $H_*(Y)$ free then by Künneth $H_*(X\times Y)\cong H_*(X)\otimes H_*(Y)$)

Pf $(\partial \mathbb{D}^i_\alpha)\times\mathbb{D}^j_\gamma \xrightarrow[\varphi_\alpha\times id]{} X^{i-1}\times Y^j \longrightarrow X^{i-1}\times Y^j \big/ \underbrace{(X\times Y)^{i+j-2}\cap(X^{i-1}\times Y^j)}_{\overset{\| \leftarrow \text{easy check}}{X^{i-2}\times Y^j \,\cup\, X^{i-1}\times Y^{j-1}}}$

$\star :=$

This proof is Non-examinable

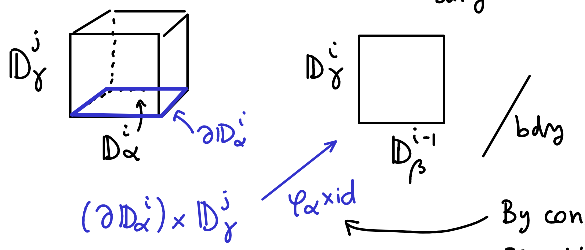
$$\left.\begin{aligned} X^{i-1} &= X^{i-2}\cup(\mathbb{D}^{i-1}_\beta \cup \dots)/\sim \\ Y^j &= Y^{j-1}\cup(\mathbb{D}^j_\gamma \cup \dots)/\sim \end{aligned}\right\} \text{get } \sim \text{ from attaching maps}$$

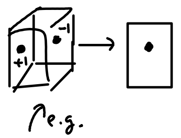
$$X^{i-1}\times Y^j = X^{i-2}\times Y^j \cup X^{i-1}\times Y^{j-1} \cup (\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma \cup \dots)\big/\sim$$

$$\Rightarrow \star = (\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma\cup\dots)\,/\,\text{boundaries}$$

$$= \frac{\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma}{\partial(\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma)} \vee \dots$$

$$(\partial\mathbb{D}^i_\alpha)\times\mathbb{D}^j_\gamma \xrightarrow[\varphi_\alpha\times id]{} \mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma\big/_{\text{bdry}} \vee \dots$$





By considering local degrees now we see we get degree $= d_{\alpha\beta}$ for this.

$\Rightarrow$ get contribution $(de^i_\alpha)\times e^j_\beta$ ✓

similarly

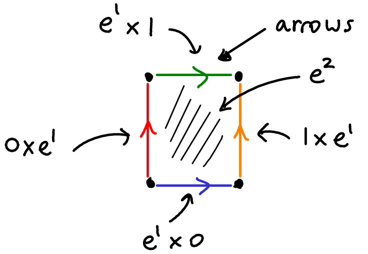
$\mathbb{D}^i_\alpha\times\partial\mathbb{D}^j_\gamma \xrightarrow[id\times\varphi_\gamma]{} \mathbb{D}^i_\alpha\times\mathbb{D}^{j-1}_\delta\big/_{\text{bdry}}$ $\Rightarrow$ degree $(-1)^i d_{\gamma\delta}$ so get $(-1)^i e^i_\alpha\times de^j_\beta$

$(-1)^i$ caused by orientations:

Could reorder factors: $\mathbb{D}^i_\alpha\times\mathbb{D}^j_\gamma \cong \mathbb{D}^j_\gamma\times\mathbb{D}^i_\alpha$ by $\begin{pmatrix}0 & Id_j\\ Id_i & 0\end{pmatrix}$ whose det $= (-1)^{ij}$. Then $\partial\mathbb{D}^j_\gamma\times\mathbb{D}^i_\alpha \to \mathbb{D}^{j-1}_\delta\times\mathbb{D}^i_\alpha/_{\text{bdry}}$ gives degree $d_{\gamma\delta}$.

Swap factors $\mathbb{D}^{j-1}_\delta\times\mathbb{D}^i_\alpha/_{\text{bdry}}$ by $\begin{pmatrix}0 & Id_i\\ Id_{j-1} & 0\end{pmatrix}$, det $= (-1)^{i(j-1)}$. Total sign $= (-1)^i$.

Example Recall after definition of H^{CW}_* we had example $I\times I$:



arrows here tell us how we map $[-1,1]\to$ edge (so orientation)

$$\partial e^2 = +e^1\times 0 + 1\times e^1 - e^1\times 1 - 0\times e^1$$
$$= (\partial e^1)\times e^1 \underset{\uparrow\, (-1)^{\dim e^1}}{-} e^1\times(\partial e^1) \quad ✓$$

A further comment on orientation sign $(-1)^i$

$\mathbb{D}^i\times\mathbb{D}^j \cong \underset{[v_0,\dots,v_i]}{\overset{\|}{\Delta^i}}\times\underset{[w_0,\dots,w_j]}{\overset{\|}{\Delta^j}}$ $\leftarrow$ viewed in $\mathbb{R}^i, \mathbb{R}^j$ project $\mathbb{R}^{i+1}\to\mathbb{R}^i$ $(t_0,\dots,t_i)\mapsto(t_1,\dots,t_i)$

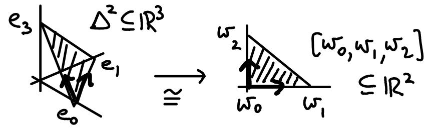
$$\partial(\mathbb{D}^i\times\mathbb{D}^j) \cong \underbrace{\partial\Delta^i}\times\Delta^j \cup \Delta^i\times\underbrace{\partial\Delta^j}$$

$$\partial\Delta^i = \sum_k(-1)^k[v_0,\dots,\hat{v}_k,\dots,v_i] \qquad \partial\Delta^j = \underline{\underline{\sum(-1)^k}}[w_0,\dots,\hat{w}_k,\dots,w_j]$$

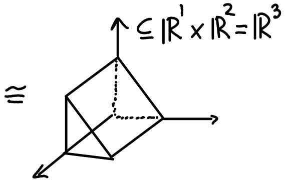
would be correct orientation sign for basis $w_1-w_0,\dots,\widehat{w_k-w_0},\dots,w_j-w_0$ but actually we have $[v_0,\dots,v_i]\times[w_0,\dots,\hat{w}_k,\dots,w_j]\subseteq\mathbb{R}^i\times\mathbb{R}^j$ and $(-1)^{i+k}$ is the orientation sign for the basis $v_1-v_0,\dots,v_i-v_0,w_1-w_0,\dots,\widehat{w_k-w_0},\dots,w_j-w_0$ for the hyperplane in $\mathbb{R}^{i+j+1}$ containing

$\Rightarrow$ need $(-1)^i$ to fix orientation sign.

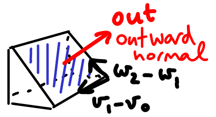
Example $\Delta^1\times\Delta^2$



$\Delta^1\times\Delta^2 \cong$



$[v_0,v_1]\times[\hat{w}_0,w_1,w_2]$



out, v_1-v_0, w_2-w_1 is negative $\mathbb{R}^3$-basis

out, w_2-w_1 is positive $\mathbb{R}^2$-basis

differ due to $(-1)^i$, $i=1$.

Projections $X \times Y \xrightarrow{p_X} X$, $X \times Y \xrightarrow{p_Y} Y$

FACT:

Künneth Theorem If $H_n(Y)$ finitely generated, free $\forall n$

no conditions on X

e.g. $Y \simeq$ finite CW cx

automatic if use field coefficients

$H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y)$	$H^n(X \times Y) \cong \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y)$ $p_X^* a \cup p_Y^* b \longleftarrow a \otimes b$ ✪

Recall for cellular homology this on generators is: $e^i_\alpha \times e^j_\beta \longmapsfrom e^i_\alpha \otimes e^j_\beta$

This is hom of rings if use following product

$$(a \otimes b) \cdot (\tilde{a} \otimes \tilde{b}) = (-1)^{|b| \cdot |\tilde{a}|} (a \cup a) \otimes (b \cup \tilde{b})$$

gradings

think of it as "exchanging order of $b, \tilde{a}$"

Rmk

An indirect proof the Thm is to write down two generalised cohomology theories $F(X,A) = H^*(X,A) \otimes H^*(Y)$ and $G(X,A) = H^*(X \times Y, A \times Y)$, and consider the natural transformation $\alpha : F \to G$ given by ✪, notice for $\underset{A=\emptyset}{X=pt}$ both F, G give $H^*(Y)$.

Example $X = S^n$, $Y = S^m$ $n \neq m$

$$H_*(S^n \times S^m) \cong \begin{cases} \mathbb{Z} & * = 0, n, m, n+m \\ 0 & \text{else} \end{cases} \cong H^*(S^n \times S^m)$$

$a_n \cup a_m = a_{n+m}$ where $a_i = \text{dual}(e^i)$

$$H_*(S^n \times S^n) \cong \begin{cases} \mathbb{Z} & * = 0, 2n \\ \mathbb{Z}^2 & * = n \\ 0 & \text{else} \end{cases} \cong H^*(S^n \times S^n)$$

$a_n \cup a_n = a_{2n}$

Cor $H^k(T^n) \cong \Lambda^k[x_1, \ldots, x_n]$

n-torus $S^1 \times \cdots \times S^1$

exterior algebra = free abelian gp. on gens. $\{x_{i_1} \wedge \cdots \wedge x_{i_k} : i_1 < \cdots < i_k\}$ so rank $= \binom{n}{k}$

where $x_i = p_i^*(\text{gen. of } H^1(S^1))$

$p_i : T^n \to S^1$ projections to factors.

Pf idea Künneth & induction $(T^n = T^{n-1} \times S^1)$ □

FACT cup product equals composition

$$\cup : H^i(X) \otimes H^j(X) \longrightarrow H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$$

$$(\Delta^i \xrightarrow{\sigma_1} X) \otimes (\Delta^j \xrightarrow{\sigma_2} X) \longmapsto (\underset{\overset{\wr\wr}{\Delta^{i+j}}}{\Delta^i \times \Delta^j} \xrightarrow{\sigma_1 \times \sigma_2} X \times X)$$

$\Delta =$ diagonal map $X \longrightarrow X \times X$, $x \longmapsto (x, x)$

12. UNIVERSAL COEFFICIENTS THEOREM

This proof is Non-examinable

(C_*, ∂_*) chain cx

$$\Longrightarrow 0 \longrightarrow \underset{\circlearrowright \partial = 0}{Z_* = \operatorname{Ker} \partial_*} \xrightarrow{\text{incl}} C_* \xrightarrow{\partial_*} \underset{\partial = 0 \circlearrowright}{B_{*-1}} = \operatorname{Im} \partial_{*-1} \longrightarrow 0 \text{ is SES}$$

FACT: Submodules of a free $\mathbb{Z}$-module are free
Rmk The same holds for R-mods if R is PID

($\mathbb{Z}$-module $\equiv$ abelian gp; free means: $\bigoplus_{\text{indexing set}} \mathbb{Z}$)

(PID = principal ideal domain = integral domain R s.t. every ideal = R·a some a)

Assume C_* free $\mathbb{Z}$-mod

FACT $\Rightarrow$ Z_*, B_* free (as $\operatorname{Ker}\partial^*, \operatorname{Im}\partial^*$ are submods of C_*)

$\Rightarrow$ SES splits, choose splitting $C_* \underset{s}{\overset{\partial_*}{\rightleftarrows}} B_{*-1}$ so $\partial_* \circ s = id$

dual SES

$$\begin{tikzcd}
0 & Z^* \arrow[l] & C^* \arrow[l, "\text{incl}^*"'] & B^{*-1} \arrow[l, "\partial^*"'] & 0 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
0 & Z^n \arrow[l] & C^n \arrow[l] & B^{n-1} \arrow[l, "\partial^*"'] & 0 \arrow[l] \\
0 & Z^{n-1} \arrow[l] \arrow[u, "\partial = 0"] & C^{n-1} \arrow[l] \arrow[u, "\partial"] & B^{n-2} \arrow[l, "\partial^*"'] \arrow[u, "\partial = 0"] & 0 \arrow[l]
\end{tikzcd}$$

note: $\text{incl}^* =$ restrict to Z_* since $\text{incl}^* \circ \phi : Z_* \xrightarrow{\text{incl}} B_* \xrightarrow{\phi} \mathbb{Z}$

(Rmk Although $\partial^* = 0 : B^* \to B^{*+1}$ the map $\partial^* : B^{n-1} \to C^n$ need not $= 0$
$\psi : B_{n-1} \to \mathbb{Z}$
$\Rightarrow \partial^* \psi = \psi \circ \partial : C_n \xrightarrow{\partial} B_{n-1} \xrightarrow{\psi} \mathbb{Z}$)

Connecting map $\delta : Z^{n-1} \to B^{n-1}$ of LES:

$$\begin{tikzcd}
0 & \partial^* \varphi & \varphi|_{B_*} = \phi|_{B_*} \arrow[l, mapsto, "\partial^*"'] \\
\varphi|_{Z_*} = \phi \arrow[u, mapsto] & \exists \varphi \arrow[l, mapsto] \arrow[u, mapsto] &
\end{tikzcd}$$

$B_* \subseteq Z_*$

$$\Longrightarrow \boxed{\delta(\phi) = \phi|_{B_*}}$$

LES $\Longrightarrow$ $(H^* B = B^*, H^* C = C^*$ since $\partial^* = 0)$

$$\begin{tikzcd}
\cdots & Z^n \arrow[l] & H^n C \arrow[l] & B^{n-1} \arrow[l, "\partial^*"'] & Z^{n-1} \arrow[l, "\delta^{n-1}"'] \\
& & & \phi|_{B_{n-1}} & \phi \arrow[l, mapsto]
\end{tikzcd}$$

$$\Longrightarrow 0 \longleftarrow \operatorname{Ker} \delta^n \longleftarrow H^n C \longleftarrow B^{n-1} / \operatorname{Im} \delta^{n-1} \longleftarrow 0$$

$$\operatorname{Ker} \delta^n = \{ \phi \in Z^n : \phi(B_n) = 0 \} \Longrightarrow \text{so: } \phi : Z_n \to \mathbb{Z}$$

$$\begin{tikzcd}
Z_n \arrow[rr, "\phi"] \arrow[dr] & & \mathbb{Z} \\
& Z_n / B_n = H_n(C_*) \arrow[ur] &
\end{tikzcd}$$

$$= \operatorname{Hom}(H_n(C_*), \mathbb{Z})$$

✪

Universal Coefficients Thm:
$0 \longrightarrow \underset{\overset{\parallel}{\textbf{Ext}^1(H_{n-1}(C); \mathbb{Z})}}{B^{n-1} / \operatorname{Im} \delta^{n-1}} \longrightarrow H^n(C) \longrightarrow \operatorname{Hom}(H_n(C_*), \mathbb{Z}) \longrightarrow 0$ is SES
$[\varphi] \longmapsto (\varphi : H_n(C_*) \to \mathbb{Z})$

see next Lemma

and SES splits (but not naturally): $B^{n-1} / \operatorname{Im} \delta^{n-1} \underset{s^*}{\overset{\partial^*}{\rightleftarrows}} H^n(C)$ $s^* \circ \partial^* = id$ (since $\partial \circ s = id$ $\Rightarrow id = (\partial \circ s)^* = s^* \circ \partial^*$)

$$\Rightarrow H^n(C) \cong \operatorname{Hom}(H_n(C), \mathbb{Z}) \oplus \operatorname{Ext}^1(H_{n-1}(C); \mathbb{Z})$$

Lemma $\mathrm{Ext}^1(H_{n-1}(C);\mathbb{Z}) \cong B^{n-1}/\mathrm{Im}\,\delta^{n-1}$ canonically

Algebra background on Extension groups $\mathrm{Ext}^i(M;\mathbb{Z})$

general case

M R-module, R ring (comm. with 1)

$\Rightarrow \exists$ **free resolution**:

$$\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \to 0$$ exact, P_i free R-mods

(pick gens x_α for $M \Rightarrow P_0 = \bigoplus_\alpha R \xrightarrow[\varphi_0]{} M$, $e_\alpha \mapsto x_\alpha$

" " y_β for $\mathrm{Ker}\,\varphi_0 \Rightarrow P_1 = \bigoplus_\beta R \xrightarrow[\varphi_1]{} \mathrm{ker}\,\varphi_0$, $e_\beta \mapsto y_\beta$

continue inductively)

our case

$H_{n-1}(C_*)$ $\mathbb{Z}$-mod

$$\begin{tikzcd}0 \arrow[r] & B_{n-1} \arrow[r, hook] \arrow[d, equal] & Z_{n-1} \arrow[r] \arrow[d, equal] & H_{n-1}(C) \arrow[r] \arrow[d, equal] & 0 \\ & P_1 & P_0 & M & \end{tikzcd}$$

Take $\mathrm{Hom}(\cdot\,;\mathbb{Z})$ and drop $\mathrm{Hom}(M;\mathbb{Z})$

$$0 \to \mathrm{Hom}(P_0;\mathbb{Z}) \xrightarrow{\varphi_1^*} \mathrm{Hom}(P_1;\mathbb{Z}) \xrightarrow{\varphi_2^*} \cdots$$

Is cochain complex but not exact

$\Rightarrow$ take cohomology groups:

Def $\mathrm{Ext}^0(M;\mathbb{Z}) = \mathrm{Ker}\,\varphi_1^*$

$\mathrm{Ext}^1(M;\mathbb{Z}) = \mathrm{Ker}\,\varphi_2^* / \mathrm{Im}\,\varphi_2^*$

$\cdots$

Fact independent of choices P_i, φ_i

$$0 \to B^{n-1} \to Z^{n-1} \to 0$$

Example 1 $\mathrm{Ext}^0(M;\mathbb{Z}) \cong \mathrm{Hom}(M,\mathbb{Z})$

$$\begin{tikzcd}P_1 \arrow[r, "\varphi_1"] \arrow[dr, "0"'] & P_0 \arrow[r, two heads, "\varphi_0"] \arrow[d, "\phi"] & M \arrow[dl, dashed] \\ & \mathbb{Z} & \end{tikzcd}$$

descends: $m \longmapsto \phi(\varphi_0^{-1} m)$

well defined since $\phi(\mathrm{Ker}\,\varphi_0) = 0$, $\mathrm{Ker}\,\varphi_0 = \mathrm{Im}\,\varphi_1$

Example 2 $\mathrm{Ext}^1(M;\mathbb{Z}) =$

$$\left\{\phi : \begin{tikzcd}P_2 \arrow[r] \arrow[dr, "0"'] & P_1 \arrow[r] \arrow[d, "\phi"] & P_0 \\ & \mathbb{Z} & \end{tikzcd}\right\} \Big/ \left\{\phi = \varphi\circ\varphi_1 : \begin{tikzcd}P_1 \arrow[r, "\varphi_1"] \arrow[dr, "\phi"'] & P_0 \arrow[d, "\varphi"] \\ & \mathbb{Z}\end{tikzcd}\right\}$$

Proof of Lemma

By **Example 2**,

$\mathrm{Ext}^1(H_{n-1}(C_*);\mathbb{Z}) =$

$$= \left\{\begin{tikzcd}0 \arrow[r] & B_{n-1} \arrow[r] \arrow[d, "\phi"'] & Z_{n-1} \\ & \mathbb{Z} & \end{tikzcd}\right\}$$ modulo

those arising from restriction

$$\left\{\begin{tikzcd}0 \arrow[r] & B_{n-1} \arrow[r] \arrow[dr, "\phi|_{B_{n-1}}"'] & Z_{n-1} \arrow[d, "\phi"] \\ & & \mathbb{Z}\end{tikzcd}\right\}$$

Thus $B^{n-1}/\mathrm{Im}\,\delta^{n-1}$. □

Rmk If R PID, then $\mathrm{Ker}(P_0 \to M)$ is free (since submod of free mod P_0)

$\Rightarrow$ can pick $P_1 = \mathrm{Ker}(P_0 \to M)$, $P_k = 0$ for $k \geq 2 \Rightarrow \mathrm{Ext}^k(M;\mathbb{Z}) = 0$ $k \geq 2$

(Co)homology with coefficients in a ring/field/module

Motivation

So far we had (C_*, ∂_*) chain cx of abelian groups } in graded sense

group hom

$\Rightarrow H_*(C_*) = \mathrm{Ker}\,\partial_* / \mathrm{Im}\,\partial_*$ abelian group (since $\mathrm{Ker}\,\partial$, $\mathrm{Im}\,\partial$ are)

We cannot use a chain cx of (non-abelian) groups, because $\mathrm{Im}\,\partial_*$ need not be a normal subgroup of $\mathrm{Ker}\,\partial_*$.

However, abelian groups can be thought of as $\mathbb{Z}$-modules, then given any **abelian group G**, define **homology with coeffs in G**

$$\boxed{H_*(C_*;G) = H_*(C_* \otimes_{\mathbb{Z}} G)}$$ with differential $\partial_* \otimes \mathrm{id}$

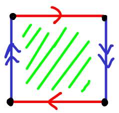
Def X space $\Rightarrow$ $\boxed{H_*(X;G) = H_*(C_*(X) \otimes_{\mathbb{Z}} G)}$

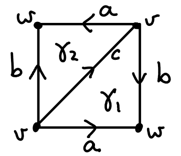
Explanation:

$C_k(X)$ free $\mathbb{Z}$-mod $\cong \bigoplus_{I_k} \mathbb{Z} \Rightarrow C_k(X) \otimes_{\mathbb{Z}} G \cong \bigoplus_{I_k} G$: just replace $\mathbb{Z}$ by G (as $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot \cong \cdot$)

Why care? We hope to get more/new invariants of spaces

Example $X = \mathbb{RP}^2 =$





$*$	$C^\Delta_*(\mathbb{RP}^2;G)$
0	$Gv \oplus Gw$
1	$Ga \oplus Gb \oplus Gc$
2	$G\gamma_1 \oplus G\gamma_2$

for $G = \mathbb{Z}/2$: $$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow[\begin{pmatrix}1&1\\1&1\\1&1\end{pmatrix}]{\partial_2} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow[\begin{pmatrix}1&1&0\\1&1&0\end{pmatrix}]{\partial_1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0$$

$$\Rightarrow H_*(\mathbb{RP}^2;\mathbb{Z}/2) \cong \begin{cases}\mathbb{Z}/2 & *=0\\ \mathbb{Z}/2 & *=1\\ \mathbb{Z}/2 & *=2\\ 0 & \text{else}\end{cases}$$

compare: $H_*(\mathbb{RP}^2) \cong \begin{cases}\mathbb{Z} & *=0\\ \mathbb{Z}/2 & *=1\\ 0 & \text{else}\end{cases}$ ($G = \mathbb{Z}$ case)

Form cochain complex using $\mathrm{Hom}_{\mathbb{Z}}(\cdot, G)$ (= group homs to G) in place of $\mathrm{Hom}(\cdot,\mathbb{Z})$

$$\boxed{\begin{aligned}H^*(C_*;G) &= H_*(\mathrm{Hom}_{\mathbb{Z}}(C_*, G))\\ H^*(X;G) &= H^*(\mathrm{Hom}_{\mathbb{Z}}(C_*(X);G))\end{aligned}}$$

with differential ∂^*: $\partial^*\phi = \phi \circ \partial_*$

X space

so: $H^*(C_*(X);G)$

Universal coefficients thm (Same proof using $\mathrm{Hom}_{\mathbb{Z}}(\cdot,G)$)

$$\boxed{\begin{aligned}0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(C_*);G) \to H^n(C_*;G) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_n(C_*), G) \to 0\\ [\varphi] &\longmapsto (\varphi : H_n(C_*) \to G)\end{aligned}}$$

__Example__ $X=\mathbb{R}P^2$, $G=\mathbb{Z}/2$, apply $\mathrm{Hom}_{\mathbb{Z}}(\cdot,G)$ to the complex in previous example

$$0\leftarrow \mathbb{Z}/2\oplus\mathbb{Z}/2 \xleftarrow[\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}]{\partial^1} \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2 \xleftarrow[\begin{pmatrix}1&1\\1&1\\0&0\end{pmatrix}]{\partial^0} \mathbb{Z}/2\oplus\mathbb{Z}/2 \leftarrow 0$$

$$\Rightarrow H^*(\mathbb{R}P^2;\mathbb{Z}/2)\cong\begin{cases}\mathbb{Z}/2 & *=0\\ \mathbb{Z}/2 & *=1\\ \mathbb{Z}/2 & *=2\\ 0 & \text{else}\end{cases}$$

compare: $H^*(\mathbb{R}P^2)\cong\begin{cases}\mathbb{Z} & *=0\\ 0 & *=1\\ \mathbb{Z}/2 & *=2\\ 0 & \text{else}\end{cases}$ ($G=\mathbb{Z}$ case)

__Can generalise further__:

C_* = chain cx of ...	coefficients in:	
abelian gps ($\mathbb{Z}$-mods)	abelian gp G ($\mathbb{Z}$-mod)	$H_*(C_*;G)=H_*(C_*\otimes_{\mathbb{Z}} G)$
R-modules ↖ ring (comm. with 1)	R-module M	$H_*(C_*;M)=H_*(C_*\otimes_R M)$

__Rmk__ $H_*(C;M)$ will be an R-module since $\ker\partial, \mathrm{Im}\,\partial$ are $\left(\begin{smallmatrix}\partial_* \text{ is } R\text{-linear}\\ \text{hom by assumption}\end{smallmatrix}\right)$

X space $\Rightarrow C_k(X;R)=C_k(X)\otimes_{\mathbb{Z}} R\cong\bigoplus_{I_k} R$: just replace $\mathbb{Z}$ by R (as $\mathbb{Z}\otimes_{\mathbb{Z}}\cdot\cong\cdot$)

$$\Rightarrow \boxed{H_*(X;M)=H_*(C_*(X;R);M)=H_*(C_*(X;R)\otimes_R M)}$$ so just replace each $\mathbb{Z}$ by M in $C_*(X)$

Form __cochain complex__ using $\mathrm{Hom}_R(\cdot,M)$ $\left(=\begin{smallmatrix}R\text{-linear}\\ \text{homs to } M\end{smallmatrix}\right)$ in place of $\mathrm{Hom}(\cdot,\mathbb{Z})$ with differential ∂^*: $\partial^*\phi=\phi\circ\partial_*$

$$\boxed{\begin{aligned}H^*(C_*;M)&=H_*(\mathrm{Hom}_R(C_*,M))\\ H^*(X;M)&=H^*(\mathrm{Hom}_R(C_*(X;R);M))\end{aligned}}$$

X space; so: $H^*(C_*(X;R);M)$

__Rmk__ These are R-mods. If we use $M=R$, then they are also rings via cup product

> __Universal Coefficients Thm__ For R any __PID__, C_* chain cx of R-mods,
>
> $0\to \mathrm{Ext}^1_R(H_{n-1}(C_*);M)\to H^n(C_*;M)\to \mathrm{Hom}_R(H_n(C_*),M)\to 0$ is SES
>
> $\mathrm{Ext}^1_R \cong B^{n-1}/\mathrm{im}\,\delta^{n-1}$ working over R using homs to M; $[\varphi]\longmapsto(\varphi: H_n(C_*)\to M)$
>
> and the SES splits but the splitting is not natural.
>
> Same proof using $\mathrm{Hom}_R(\cdot,M)$

__Example__ $R=\mathbb{F}$ __field__ $\Rightarrow C_*, H_*, H^*$ are vector spaces$/\mathbb{F}$.

__Rmk__ all $\mathbb{F}$-mods (i.e. vector spaces$/\mathbb{F}$) are free $\mathbb{F}$-mods $\cong\bigoplus\mathbb{F}b_i$ (basis b_i)

up to iso they are determined by $\dim_{\mathbb{F}}$ = cardinality of basis.

__Cor__ C_* = chain cx of $\mathbb{F}$-vector spaces $\Rightarrow \boxed{H^n(C_*;\mathbb{F})\cong(H_n(C_*))^*}$ ← dual v.s.: $\mathrm{Hom}_{\mathbb{F}}(H_n(C_*);\mathbb{F})$

__Pf__ Pick any basis v_i for $\mathbb{F}$-v.s. B_{n-1}, extend it to a basis v_i, w_j of Z_{n-1} (also works in ∞ dim case).

$\Rightarrow$ can extend any $\mathbb{F}$-linear map $\psi: B_{n-1}\to\mathbb{F}$ to $\phi: Z_{n-1}\to\mathbb{F}$ just pick any values $\phi(w_j)\in\mathbb{F}$ e.g. $\phi(w_j)=0$.

$\Rightarrow B^{n-1}/\mathrm{im}\,\delta^{n-1}=0$ so $H^n(C_*;\mathbb{F})\longrightarrow\mathrm{Hom}(H_n(C_*);\mathbb{F})$ iso □

__Cor__ $\boxed{H^n(X;\mathbb{F})\cong H_n(X;\mathbb{F})^*}$ ← dual v.s. for any field $\mathbb{F}$.

__Cor__ $\boxed{H^n(X;M)\underset{\substack{\uparrow\\ \text{if } X\simeq\text{CW-cx}}}{\cong} H^n_{CW}(X;M)\underset{\substack{\uparrow\\ \text{if } X\simeq\Delta\text{-cx}}}{\cong} H^n_{\Delta}(X;M)}$

__Pf__ Cor holds for __homology__ and the isos are __natural__ ← i.e. functorial w.r.t. maps.
The universal coeff. thm SES is __natural__. So result holds by 5-Lemma. □

__Algebra: structure thm for f.g. abelian groups and R-mods__

__Fact 1__ A f.g. abelian gp $\Rightarrow A\cong\underbrace{\mathbb{Z}^r}_{\text{free part } F}\oplus\underbrace{\mathbb{Z}/p_1^{n_1}\oplus\cdots\oplus\mathbb{Z}/p_k^{n_k}}_{\text{torsion part } T}$

where $p_i\in\mathbb{Z}$ prime (need not be distinct)
Also r,k,p_i,n_i are unique (up to reordering)

__Example__ $\mathbb{Z}/4=\mathbb{Z}/2^2\not\cong\mathbb{Z}/2\oplus\mathbb{Z}/2$
$\mathbb{Z}/6\cong\mathbb{Z}/2\oplus\mathbb{Z}/3$ (Chinese Remainder Thm)

__Fact 2__ $T\cong\mathbb{Z}/d_1\oplus\cdots\oplus\mathbb{Z}/d_k$ with $d_1|d_2|\ldots|d_k$ ($d_i\in\mathbb{N}$ unique)

__Example__ $\mathbb{Z}/2\oplus\mathbb{Z}/2^2\oplus\mathbb{Z}/3\cong\mathbb{Z}/2\oplus\mathbb{Z}/3\cdot2^2$ $\quad d_1=2$, $d_2=12$

__Fact 3__ M f.g. R-mod, R __PID__, then:

$$\boxed{\begin{aligned}M&\cong F\oplus T\\ F&\cong R^r\\ T&\cong R/d_1\oplus R/d_2\oplus\cdots\oplus R/d_k\end{aligned}}$$

$r\in\mathbb{N}$ unique, called __rank__ of M
$d_1|\cdots|d_k$ non-zero, not invertible
d_i called __invariant factors__
unique up to mult^n by invertible elements ↖ e.g. ±1 if $R=\mathbb{Z}$

__Rmk__ $T=\{m\in M: rm=0 \text{ some } r\neq0\in R\}$ = __torsion elements__
$F\cong M/T$

Torsion shift

Easy Exercise $\boxed{\operatorname{Ext}^*_R(\bigoplus_i M_i ; \prod_j N_j) \cong \prod_i \prod_j \operatorname{Ext}^*_R(M_i ; N_j)}$ ← any R-mods M_i, N_j

↓

Upshot To compute $\operatorname{Ext}^1_R(M,R)$ for $M = R^r \oplus R/d_1 \oplus \cdots$ just need:

$$\boxed{\begin{aligned} \operatorname{Ext}^1_R(R ; R) &= 0 \\ \operatorname{Ext}^1_R(R/d ; R) &\cong R/d \end{aligned}}$$

since $\begin{array}{c} 0 \to R \xrightarrow{id} R \to 0 \\ \;\; \| \quad\;\; \| \\ \;\; P_1 \quad\; P_0 \end{array}$

since $0 \to R \xrightarrow{d} R \xrightarrow{} R/d \to 0$, $\downarrow \phi$ to R ← so choice of $\phi(1) \in R$ modulo ϕ coming from

$$\begin{tikzcd} R \arrow[r, "d"] \arrow[d, "\phi"'] & R \arrow[dl, "\varphi"] \\ R & \end{tikzcd}$$

so $\phi(1) = d\cdot\varphi(1) \in d\cdot R$

$\Rightarrow \boxed{\operatorname{Ext}^1_R(M ; R) \cong \operatorname{Torsion}(M)}$

Exercises

- $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n ; \mathbb{Z}/m) \cong \mathbb{Z}/\gcd(m,n)$
- G abelian gp $\Rightarrow \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}; G) = 0$ and $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/d ; G) \cong G/d\cdot G$ ← $d \neq 0$
- R any ring (comm. with 1) $x \in R$ not zero divisor $\Rightarrow \operatorname{Ext}^*_R(R/(x) ; N) \cong \begin{cases} \{n \in N : x\cdot n = 0\} & * = 0 \\ N/x\cdot N & * = 1 \\ 0 & \text{else} \end{cases}$ (N ← R-mod)

Cor If $H_n(X;R)$ f.g. R-mod $\forall n$, R PID,

$\Rightarrow H_n(X;R) = R^{r_n} \oplus T_n$ (free & torsion parts)

$\Rightarrow \boxed{H^n(X;R) \cong R^{r_n} \oplus T_{n-1}}$ ← torsion moves up!

($\cong$ ↑ not natural)

Pf $0 \to T_{n-1} \to H^n(X;R) \to \operatorname{Hom}(R^{r_n} \oplus T_{n-1} ; R) \to 0$

$\operatorname{Hom}(R^{r_n} \oplus T_{n-1} ; R) \cong \underbrace{(\operatorname{Hom}(R;R))^{r_n}}_{} \oplus \underbrace{\operatorname{Hom}(T_{n-1};R)}_{}$

$R \to R$, $1 \mapsto x$ $\longrightarrow$ $\overset{\cong}{R^{r_n}}$; x determines the hom

0 since $T_{n-1} \to R$, $1 \mapsto 0$ (R is integral domain, so no torsion elts $\neq 0$)

$\Rightarrow 0 \to T_{n-1} \to H^n(X;R) \to \underbrace{R^{r_n}}_{} \to 0$ free, so can split the SES (pick lifts of basis). ← so not canonical □

Example

$*$	$H_*(\mathbb{RP}^3)$	$H^*(\mathbb{RP}^3)$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/2$	0
2	0	$\mathbb{Z}/2$
3	$\mathbb{Z}$	$\mathbb{Z}$

torsion moves up

Universal coefficients Theorem in homology

FACT Theorem C_* chain cx of free R-mods (← PID), M R-module

$$\Rightarrow \boxed{\begin{array}{l} \text{SES } 0 \to H_*(C_*) \otimes_R M \to H_*(C_* \otimes_R M) \to \operatorname{Tor}_1^R(C_{*-1} ; M) \to 0 \\ \qquad\qquad [c] \otimes m \mapsto [c \otimes m] \qquad\qquad \nwarrow \text{defined below.} \\ \text{The SES splits, but the splitting is not natural.} \end{array}}$$

Torsion groups: A, B R-mods (R comm. ring with 1)

pick $\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} A \to 0$ free resolution ← exact sequence, P_i free R-mods

take $\cdot \otimes B$, omit $A \otimes B$ $\Rightarrow \cdots \to P_2 \otimes B \xrightarrow[\varphi_2 \otimes id]{} P_1 \otimes B \xrightarrow[\varphi_1 \otimes id]{} P_0 \otimes B \to 0$ not exact but is chain cx

$\operatorname{Tor}^R_k(A,B) = H_k$ (this complex) ← fact independent of choices of P_i, φ_i

Rmk R PID $\Rightarrow \ker \varphi_0$ free $\Rightarrow$ can pick $P_1 = \ker\varphi_0$, $P_k = 0$ for $k \geq 2$

$\Rightarrow$ only $\operatorname{Tor}^R_0, \operatorname{Tor}^R_1$ can be non-zero

Example $\operatorname{Tor}^{\mathbb{Z}}_*(\mathbb{Z}/a, \mathbb{Z}/b) = ?$

$0 \to \mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z} \xrightarrow[\text{quotient}]{\varphi_0} \mathbb{Z}/a \to 0$ free resolution

take $\cdot \otimes \mathbb{Z}/b$, drop $\mathbb{Z}/a \otimes \mathbb{Z}/b$ $\Rightarrow 0 \to \mathbb{Z}/b \xrightarrow{\cdot a} \mathbb{Z}/b \to 0$ (since $\mathbb{Z} \otimes_{\mathbb{Z}} G \cong G$ any G)

$\operatorname{Tor}^{\mathbb{Z}}_0(\mathbb{Z}/a, \mathbb{Z}/b) = (\mathbb{Z}/b)/a\cdot\mathbb{Z}/b \cong \mathbb{Z}/\gcd(a,b) \cong \mathbb{Z}/a \otimes \mathbb{Z}/b$

$\operatorname{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/a ; \mathbb{Z}/b) = \{x \in \mathbb{Z}/b : a\cdot x = 0\} \cong \mathbb{Z}/\gcd(a,b)$

Facts $\operatorname{Tor}^R_0(A,B) = P_0 \otimes B / \operatorname{Im}(\varphi_1 \otimes id) \cong A \otimes B$

$\operatorname{Tor}^R_*(A,B) \cong \operatorname{Tor}^R_*(B,A)$

Exercise $\operatorname{Tor}^R_*(\bigoplus A_i, \bigoplus B_j) \cong \bigoplus_i \bigoplus_j \operatorname{Tor}^R_*(A_i, B_j)$

$\operatorname{Tor}^R_*(A,B) = 0$ for $* \geq 1$ if A or B is free (use $M \otimes_R R \cong M$)

$\Downarrow$

$\operatorname{Tor}^R_*(R/u ; M) \cong \begin{cases} M/u\cdot M & * = 0 \\ \underline{u\text{-torsion}}(M) = \{x \in M : u\cdot x = 0\} & * = 1 \\ 0 & \text{else} \end{cases}$ ($\cong$ ↑ R PID)

deduce $\operatorname{Tor}^R_1(A,M)$ for f.g. R-mods A (← PID)

Example $H_*(\mathbb{RP}^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \end{cases}$ $\quad H_*(\mathbb{RP}^2) \otimes \mathbb{Z}/2 \cong \begin{cases} \mathbb{Z} \otimes \mathbb{Z}/2 \\ \mathbb{Z}/2 \otimes \mathbb{Z}/2 \\ 0 \end{cases} \cong \begin{cases} \mathbb{Z}/2 \\ \mathbb{Z}/2 \\ 0 \end{cases}$

↖ caused by $\operatorname{Tor}^{\mathbb{Z}}_1(H_1(\mathbb{RP}^2); \mathbb{Z}/2) = \operatorname{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$

Künneth Thm

R PID $\Longrightarrow$ natural SES: (C_* free ch. cx. R-mods, D_* any ch. cx. R-mods) →

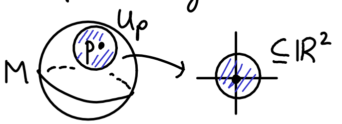
$$\boxed{0 \to \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \to H_n(C_* \otimes D_*) \to \bigoplus_{i+j=n-1} \operatorname{Tor}^R_1(H_i(C), H_j(D)) \to 0}$$

and the SES splits but the splitting is not natural. Example $R =$ field, then this $= 0$.

13. MANIFOLDS : POINCARÉ-LEFSCHETZ DUALITY

- M **n-mfd** is Hausdorff topological space s.t. $\forall p \in M$ $\exists$ open neighbourhood $U_p \subseteq M$ homeomorphic to $\mathbb{R}^n$

(equivalently: to an open ball, or any open set in $\mathbb{R}^n$)



One also requires M **second countable** i.e. $\exists$ countable basis of open sets $\Longleftrightarrow$ M is covered by countably many such U_{p_i}

exercise

A **submanifold** $N \subseteq M$ is a mfd s.t. inclusion $N \to M$ is an embedding (i.e. a homeomorphism onto its image)

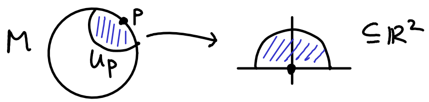
- M **n-mfd with boundary** if also allow $U_p \cong$ **upper half space $\mathbb{H}^n$**

$\{x \in \mathbb{R}^n : x_n \geq 0\} = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$

$p \longmapsto 0$

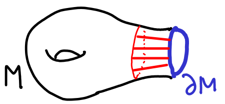
equivalently: any open nbhd of $0 \in \mathbb{H}^n$

Such p are called **boundary points**; they form the **boundary ∂M** which is an $(n-1)$-mfd without boundary.



FACT (collar nbhd thm) $\partial M \subseteq M$ has an open neighbourhood $\cong \partial M \times (0,1]$
$\partial M \longrightarrow \partial M \times 1$

M is **closed** if compact without boundary.

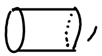
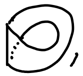
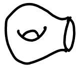


Examples

closed mfds : S^n, $\mathbb{R}P^n$, $T^n = S^1 \times \cdots \times S^1$, $\mathbb{C}P^n$, $O(n)$, $SU(n)$

n-torus

non-compact mfds: $\mathbb{R}^n$, $Mat_{m \times n} \cong \mathbb{R}^{mn}$, $GL(n,\mathbb{R})$

mfds with bdry: $\mathbb{D}^n$, $\mathbb{D}^1 \times S^1 =$ , Möbius band $=$ , $T^2 \setminus$ open disc $=$ 

FACT (Milnor 1959) Any mfd is homotopy equivalent to a CW-cx

fact If M is a compact manifold then $H_*(M)$ are finitely generated

Rmk M **triangulable** if $M \simeq$ simplicial cx.
Not all mfds are triangulable, but most of those we encounter are.

Compact manifolds have f.g. homology

Non-examinable proof

① X space is a **Euclidean neighbourhood retract** if $\exists$ **embedding** $j : X \longrightarrow \mathbb{R}^N$ some N, s.t. $i(X)$ is a retract of a nbhd $V \subseteq \mathbb{R}^N$.
(homeo onto image)

② X is **weakly locally contractible** if $\forall$ nbhd $x \in U \subseteq X$, $\exists$ nbhd $x \in V \subseteq U$ s.t. V is contractible inside U.

FACT compact $X \subseteq \mathbb{R}^n$ is ① $\Longleftrightarrow$ X is ②

Rmk If we find nbhd V as in ① with retraction $V \xrightarrow{f} X$ then any smaller nbhd V' also retracts using $f|_{V'} : V' \to X$. Similarly in ② $V' \subseteq V$ is contractible: restrict the hpy.

Lemma A X compact & ① $\Rightarrow$ X is the retract of a finite simplicial cx

pf $i(X) \subseteq \mathbb{R}^n$ compact $\Rightarrow$ lies inside some large n-simplex $\Delta^n \to \mathbb{R}^n$
Apply barycentric subdivision until simplices have diameter $<$ dist$(X, \partial V)$.
Simpl.cx. $= \bigcup\{$subsimplices which intersect $X\}$ using the restriction of retraction $V \to X$. □

Rmk Also deduce X has f.g. homology since retractions are surjective on H_*.
$(\oplus \mathbb{Z} \twoheadrightarrow H_*(\text{finite simpl. cx}) \xrightarrow{\text{retract}} H_*(X)$ so get surjection from free $\mathbb{Z}$-mod, so f.g.)

Lemma B M compact mfd $\Rightarrow$ M embeds into $\mathbb{R}^N$, some N.

pf "Just do it proof":
$\forall p \in M$, $\exists$ homeo $\mathbb{D}^n \xrightarrow{\varphi_p}$ nbhd$(p \in M)$
Pick finite subcover of φ_{p_i} of $M = \bigcup_{p \in M} \varphi_p(\mathbb{D}^n)$. Say $i = 1, \ldots, k$
$\psi_{p_i} : M \xrightarrow{\varphi_{p_i}^{-1}} \mathbb{D}^n \to \mathbb{D}^n/_{\partial \mathbb{D}^n} \cong S^n \subseteq \mathbb{R}^{n+1}$ define embedding $(\psi_{p_1}, \ldots, \psi_{p_k}) : M \to \mathbb{R}^{k \cdot (n+1)}$
Finally use: a continuous bijection from a compact space to a Hausdorff space is $\cong$ □

Rmk Same works if M has boundary, just consider its **double** $\underset{\text{identify along } \partial M}{M \cup M}$ and apply the Lemma to the double.

Cor M compact mfd (possibly with bdry) $\Rightarrow$ M has f.g. homology

Pf Mfds satisfy ② since locally ball $\simeq$ pt. M embeds in $\mathbb{R}^N$ by Lemma B.
① holds by FACT. Done by Lemma A. □

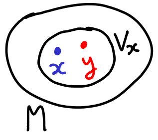
Def A **local orientation** of M at $x \in M$ is a choice of generator

excise complement of nbhd $V_x \cong \mathbb{D}^n$ — choice of homeo is not canonical!

$$\mu_x \in H_n(M, M\setminus x) \cong H_n(\mathbb{D}^n, \mathbb{D}^n\setminus 0) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}$$

Def An **orientation** of M is a **locally consistent** choice $x \mapsto \mu_x$

meaning:



$V_x \cong \mathbb{D}^n \simeq pt$

$$\begin{tikzcd} & H_n(M, M\setminus V_x) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ H_n(M,M\setminus x) \arrow[rr, "\cong"] & & H_n(M,M\setminus y) \end{tikzcd} \qquad \begin{tikzcd} & \exists\, a \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ \mu_x \arrow[rr, mapsto] & & \mu_y \end{tikzcd}$$

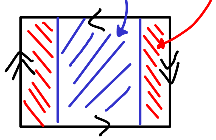
$\left(H_n(\mathbb{D}^n, \mathbb{D}^n\setminus x) \cong H_n(\mathbb{D}^n, \mathbb{D}^n\setminus y) \right)$ — $\mathbb{D}^n\setminus x \simeq \mathbb{D}^n \setminus y$

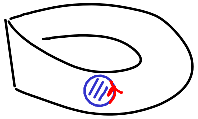
Def M **orientable** if $\exists$ orientation on M

oriented if we chose an orientation

Examples S^n, $\mathbb{R}^n$, $\mathbb{C}P^n$, orientable surfaces Σ_g, $\mathbb{R}P^n$ ← odd n

Non-example $\mathbb{R}P^2$ = Möbius band $\cup$ $\mathbb{D}^2$

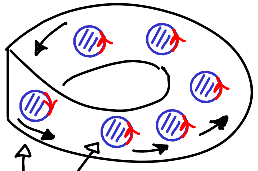




choice of μ_x is choice of orientation of boundary circle of small disc containing x

$\Rightarrow \mathbb{R}P^2$ not orientable

by local consistency can move disc continuously and preserves orientation



discs are differently oriented $\Rightarrow$ contradicts local consistency.

The fundamental class $[M]$

FACT ~~Theorem~~ For M closed n-mfd:

M orientble connected $\Rightarrow$ $H_n(M) \underset{\text{natural}}{\cong} H_n(M, M\setminus x) \underset{\text{choice}}{\cong} \mathbb{Z}$

$\underset{\substack{\uparrow \\ \text{once we choose} \\ \text{an orientation} \\ (\mu_x)_{x\in M}}}{\Rightarrow}$ $\exists\, [M] \longleftarrow \mu_x$

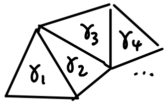
↑ called **fundamental class**

(if swap orientation: for $-\mu_x$ get $-[M]$)

M not orientable connected $\Rightarrow$ $H_n(M) = 0$

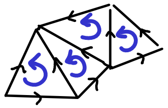
$H_n(M; \mathbb{F}_2) = \mathbb{F}_2 \cdot [M] \cong \mathbb{F}_2$ (or any field of characteristic 2)

Construction of $[M]$ if M has Δ-complex structure



M compact $\Rightarrow$ finite # n-simplices $\gamma_1, \dots, \gamma_N$

M oriented $\Rightarrow$ pick orientations of $\gamma_1, \dots, \gamma_N$ to agree with given orientation of M: ↙for $x \in \mathrm{Int}(\gamma_i)$



$$\mathbb{Z}\cdot\mu_x = H_n(M, M\setminus x) \underset{\text{exc.}}{\overset{\cong}{\leftarrow}} H_n(\gamma_i, \gamma_i\setminus x) = \mathbb{Z}\cdot\gamma_i$$
$$\mu_x \longmapsto \gamma_i$$

$\Rightarrow$ $\boxed{[M] := \sum \gamma_i}$ satisfies $\partial [M] = 0$ ✓

$$\begin{tikzcd} H_n(M) \arrow[r] & H_n(M, M\setminus x) & H_n(\gamma_i, \gamma_i\setminus x) \arrow[l, "\cong"'] \\ {[M]} \arrow[rr, bend right, mapsto] & \mu_x & \gamma_i \arrow[l, mapsto] \end{tikzcd}$$

Not difficult to see that $H_n^{CW}(X) = \mathbb{Z}\cdot[M]$, so $\Rightarrow H_n(M) \cong H_n(M, M\setminus x)$

$[M] \longmapsto \mu_x$

Also $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}\cdot[M]$ since $C_{n+1}(M) = 0$ ($\nexists (n+1)$-simplices since $\dim M = n$)

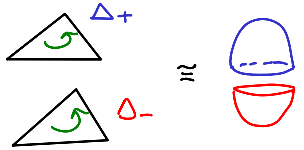
M non-orientble $\Rightarrow$ each facet of γ_i appears twice in $\partial \sum \gamma_i$

$\Rightarrow$ $\partial \sum \gamma_i = 0$ over $\mathbb{F}_2$ independently of choices of orientations of γ_i. ✓

Examples

1) $S^n = \underbrace{\overset{\Delta_+}{\Delta^n} \cup \overset{\Delta_-}{\Delta^n}}_{\text{glue bdries}}$

S^2:

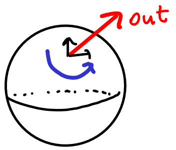


$[S^n] = \Delta_+ - \Delta_-$ if use canonical orientation we discussed

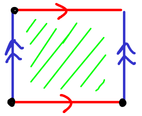
hence $\partial[S^n] = \partial\Delta - \partial\Delta = 0$

$\mathbb{D}^n \subseteq \mathbb{R}^n$ canonical orientation

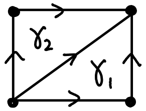
$\Rightarrow S^{n-1} = \partial \mathbb{D}^n$ " using outward normal first rule



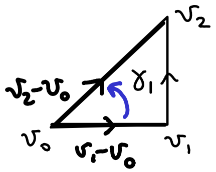
2) $T^2 =$



Δ-complex structure (compatibly with side identifications!)



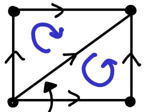
Want orientation induced by square $\subseteq \mathbb{R}^2$



$v_1 - v_0,\ v_2 - v_0$ positive $\mathbb{R}^2$-basis

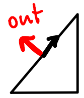
$\Rightarrow \gamma_1$ agrees with orientation

$\Longrightarrow$



$[T^2] = +\gamma_1 - \gamma_2$

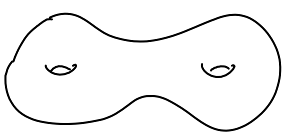
γ_2 orientation disagrees



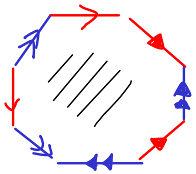
Rmk general feature: if two simplices are identified along a facet then the 2 respective outward normals are related by reflection.

So consistency $\Rightarrow$
- either simplices are compatibly oriented and the two induced orientations on facet are opposite
- or not compatibly oriented but facet orient^n is same, then need sign like in example when build $[T^2]$

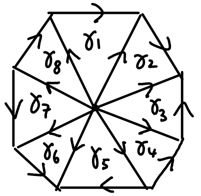
3) Recall $\Sigma_2 =$



$=$



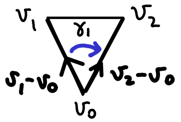
Δ-cx structure (compatible with side identifications!):



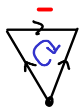
Use the orientation induced by polygon $\subseteq \mathbb{R}^2$

$\Rightarrow [\Sigma_2] = -\gamma_1 - \gamma_2 + \gamma_3 + \gamma_4 - \gamma_5 + \gamma_6 + \gamma_7 - \gamma_8$

Explanation:

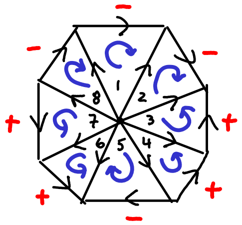


$v_1 - v_0, v_2 - v_0$ is negative $\mathbb{R}^2$-basis $\Rightarrow$



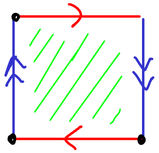
All simplices γ_i have $v_0 =$ centre of polygon

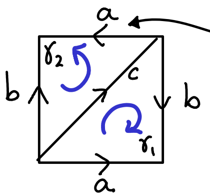
$\Rightarrow$ sign $\begin{cases} - & \text{if outer edge clockwise} \\ + & \text{if outer edge anti-clockwise} \end{cases}$



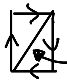
3) $\mathbb{RP}^2 =$

(non-orientable) example





won't get Δ-cx structure if you try



since get issue here

Use the orientation induced by square $\subseteq \mathbb{R}^2$

$$\Rightarrow [\mathbb{RP}^2] = -\gamma_1 + \gamma_2$$

$$\partial[\mathbb{RP}^2] = -(b - a + c) + (a - b + c)$$

$$= -2b + 2a$$

$\neq 0$ so not cycle in $C_*^{CW}(\mathbb{RP}^2)$

However, working modulo 2:

$$\partial[\mathbb{RP}^2] = 0 \in C_*^{CW}(\mathbb{RP}^2; \mathbb{F}_2) \text{ since } 2=0 \text{ in } \mathbb{F}_2$$

$$\Rightarrow [\mathbb{RP}^2] \in H^2(\mathbb{RP}^2; \mathbb{F}_2)$$

Degree

<u>Def</u> M, N oriented closed connected n-mfds, $f: M \to N$

$$f_* : H_n(M) \longrightarrow H_n(N)$$
$$[M] \mapsto \underbrace{\deg(f)}_{\in \mathbb{Z}} \cdot [N]$$

<u>Lemma</u> If $f^{-1}(y)$ finite,

then $\deg(f) = \sum_{x \in f^{-1}(y)} \deg(f_y)_*$

local degree

local map like in chapter 7

<u>Pf</u>

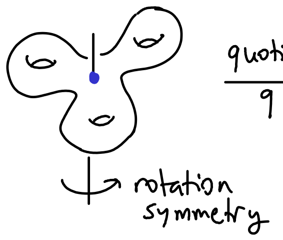
$$\begin{tikzcd}
{[M]} \arrow[dd, mapsto] & H_n(M) \arrow[r, "f_*"] \arrow[d, phantom, "\cong"] & H_n(N) \arrow[d, phantom, "\cong"] & {[N]} \\
& \bigoplus_{x \in f^{-1}(y)} H_n(M, M\setminus x) \arrow[r, "(f_x)_*"] & H_n(N, N\setminus y) & \mu_y^N \arrow[u, mapsto] \\
\sum \mu_x^M \arrow[rr, mapsto] & & \left(\sum \deg(f_x)_*\right)\cdot \mu_y^N &
\end{tikzcd}$$

□

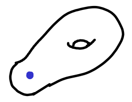
Examples

1) $S^1 \longrightarrow S^1$, $z \longmapsto z^n$, $[S^1] \longmapsto n \cdot [S^1]$ so $\deg = n$

2) $\Sigma_3 =$

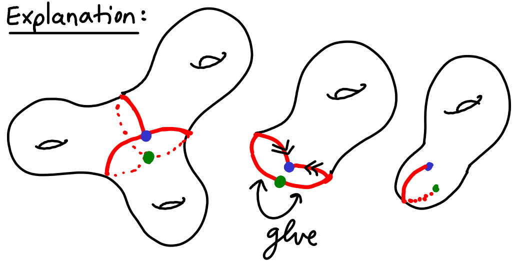


$\xrightarrow[q]{\text{quotient}}$ $\Sigma_3 / \mathbb{Z}/3$-rotation action $=$



$= \Sigma_1$ torus

Explanation:



Easy check: $\deg(q) = 3$
(e.g. use local degrees)

Cultural Rmk

For M, N, f smooth, the $\deg f = \#$(preimages of a generic point of N)

Idea: $\deg f$ tells you how many times you cover N. (almost all points work)

Poincaré duality

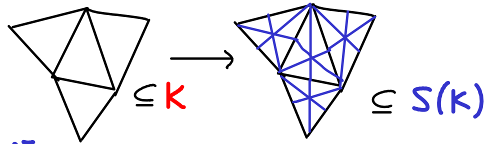
<u>FACT</u> <u>Theorem</u> For M closed n-mfd

M oriented $\Longrightarrow$ $\boxed{H^k(M) \cong H_{n-k}(M)}$ s.t. $1 \leftrightarrow [M]$, $\underset{H^0(M)}{1} \leftrightarrow \underset{H_n(M)}{[M]}$, $H^0(M) \cong H_n(M)$

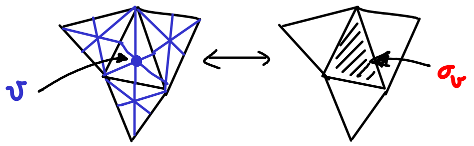
M non-oriented $\Longrightarrow$ same holds with $\mathbb{F}_2$ coefficients

<u>Sketch proof when M is a simplicial complex K</u> (Non-examinable)

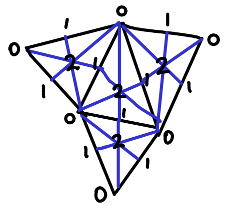
$S(K)$ = barycentric subdivision



1) simplex $\sigma = \sigma_v$ of K with $\longleftrightarrow$ $v = v_\sigma$ vertex of $S(K)$
barycentre v



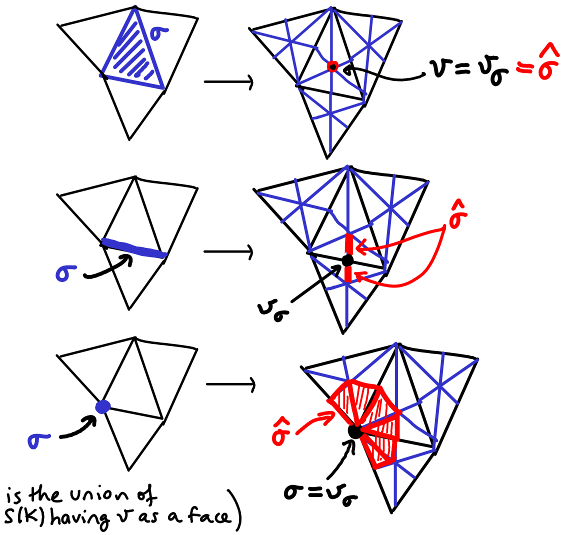
2) $ht(v) = $ (height of v) $= \dim \sigma_v$



3) σ k-simplex of K

dual simplex

$$\hat{\sigma} = \bigcup_{\substack{\tau \in S(K) \\ ht(v_\sigma) \text{ is min} \\ \text{of heights of} \\ \text{vertices of } \tau}} \tau$$



Rmk: $\cup \tau$ with $ht(v_\sigma)$ max will give back σ.

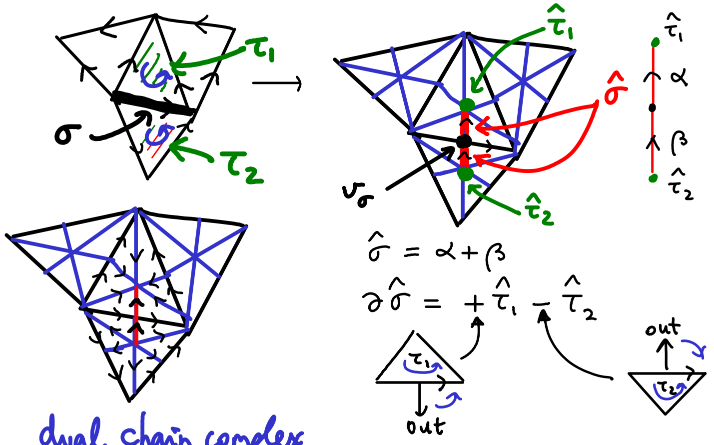
Thus $\hat{\sigma}, \sigma$ intersect transversely at v_σ.

One can also describe $\hat{\sigma}$ as

$\hat{\sigma} = \bigcap_{\substack{\text{vertices} \\ v \in \sigma}} \underline{\mathrm{Star}_{S(K)}(v)}$ (closed star is the union of simplices of S(K) having v as a face)

FACTS
- dim $\hat{\sigma}$ = $n - \dim \sigma$
- dual cells $\hat{\sigma}$ give a cell decomposition of M ("polygonal" complex rather than Δ-cx)

(★) • $\partial \hat{\sigma} = \sum_{\substack{\text{facet} \\ \sigma \subsetneq \tau \\ \tau \in K}} \pm \hat{\tau}$ need compare orientations of σ, τ (+ if σ as a facet of τ has boundary orientation)



$\hat{\sigma} = \alpha + \beta$

$\partial \hat{\sigma} = +\hat{\tau}_1 - \hat{\tau}_2$

4) **dual chain complex**

D_{n-k} = free abelian group on dual chains $\hat{\sigma}$

$H_*(M) \cong H_*(D_*, \partial_*)$ (since $\hat{\sigma}$ give a cell decomp. of M)

5) $\varphi: D_{n-k} \longrightarrow C^k(M)$
$\hat{\sigma} \longmapsto \sigma^*$ where $\sigma^*(\alpha) = \begin{cases} 0 & \text{for } k\text{-cells } \alpha \neq \sigma \\ 1 & \text{for } \alpha = \sigma \end{cases}$

- φ linear bijection ✓
- chain map:

$\varphi(\partial\hat{\sigma}) = \varphi \sum \pm \hat{\tau} = \sum \pm \tau^*$ (see ★)

$\partial^* \varphi(\hat{\sigma}) = \partial^* \sigma^* = \left(\sigma^* \circ \partial : \tau \xmapsto{\partial} \sum_{\substack{\text{facet } \sigma_i \\ \text{of } \tau}} \pm \sigma_i \xmapsto{\sigma^*} \begin{cases} \pm 1 & \text{if one of } \sigma_i = \sigma \\ 0 & \text{else} \end{cases} \right)$
$= \sum \pm \tau^* = \varphi(\partial\hat{\sigma})$ ✓

UPSHOT φ is chain iso so get iso:
$H_*(M) \underset{④}{\cong} H_*(D_*, \partial_*) \xrightarrow[\varphi]{\cong} H^{n-*}(M)$

Cor χ(odd dimensional closed orientable mfd) = 0

Pf b_i = rank $H_i(M)$ (**Betti numbers**)

$\chi(M) = b_0 - b_1 + \dots + b_{\dim M - 1} - b_{\dim M}$

equal by Poincaré duality □

(Poincaré–)Lefschetz duality

Theorem
M compact oriented n-mfd
n-mfd with boundary

$\boxed{H^k(M) \cong H_{n-k}(M, \partial M)}$

$1 \in H^0(M) \longleftarrow [M, \partial M] \in H_n(M, \partial M)$ relative fundamental class

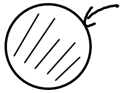
$\boxed{H_k(M) \cong H^{n-k}(M, \partial M)}$

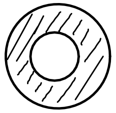
Non-oriented ⟹ same holds with $\mathbb{F}_2$ coefficients.

Pf basically same as Poincaré duality. □

Cor M compact, connected, $\partial M \neq \emptyset$ ⟹ $H^n(M) = H_0(M, \partial M) = 0$

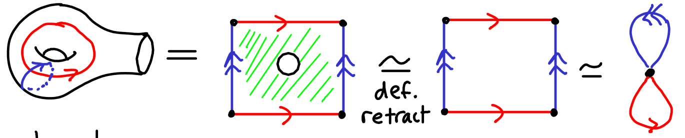
Examples

1) $\mathbb{D}^n$  $\partial\mathbb{D}^n = S^{n-1}$ $\mathbb{Z} \cong H^0 \mathbb{D}^n \cong H_n(\mathbb{D}^n, S^{n-1})$

2)  A = annulus $\subseteq \mathbb{R}^2$
$\simeq S^1$

$\mathbb{Z} \cong H^0 A \cong H_2(A, \partial A)$
$\mathbb{Z} \cong H^1 A \cong H_1(A, \partial A)$
$0 \cong H^2 A \cong H_0(A, \partial A)$

3) $M = T^2 \setminus$ open ball =



$\simeq S^1 \vee S^1$

$\Rightarrow H_*(M, \partial M) \cong H^{2-*}(S^1 \vee S^1) = \begin{cases} \mathbb{Z}^2 & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$ ← gen. by 2 loops
← gen. by $[M, \partial M]$

What happens in the non-compact case?

Locally finite homology (Borel-Moore)

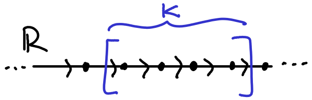
$C^{lf}_*(X)$ allow infinite sums $\sum n_i \sigma_i$, $n_i \in \mathbb{Z}$, σ_i ← generators of $C_*(X)$

s.t. given any compact subset $K \subseteq X$,

$\#\{n_i \neq 0 : K \cap \mathrm{Im}\, \sigma_i \neq \emptyset\} < \infty$.

Examples

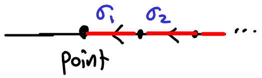
- $C^{lf}_1(\mathbb{R}) \ni \sum_{m \in \mathbb{Z}} \sigma_m$



$\sigma_m : I \cong [m, m+1] \subseteq \mathbb{R}$

$\Rightarrow$ get cycle $[\mathbb{R}] \in H^{lf}_1(\mathbb{R})$

- $C^{lf}_0(\mathbb{R}) \ni [\text{point}] = \partial(\sigma_1 + \sigma_2 + \cdots)$ is a boundary:



exercise $H^{lf}_*(\mathbb{R}) = \begin{cases} \mathbb{Z} & *=1 \\ 0 & \text{else} \end{cases}$ $(\cong H^{1-*}(\mathbb{R}))$

FACT Theorem M orientable n-mfd (possibly not compact) $\Rightarrow$ $\boxed{H^*(M) \cong H^{lf}_{n-*}(M)}$

cohomology with compact supports $H^*_c(X)$

$C^*_c(X)$: only allow cochains $\phi : C_*X \to \mathbb{Z}$ s.t. $\exists$ compact $K \subseteq X$ with (← depends on ϕ)

$\phi(C_*(X \setminus K)) = 0$ (vanish on chains in $X \setminus K$)

Example $c \in C_*(X)$ cycle $\Rightarrow$ $\phi(\alpha) =$ signed # intersections of c with α (geometric intersection #)

$\Rightarrow \phi \in C^*_c(X)$ since $\phi(\alpha) = 0$ if $\alpha \subseteq X \setminus \mathrm{Im}(c)$

Thm M orientable n-mfd (possibly not compact) $\Rightarrow$ $\boxed{H_*(M) \cong H^{n-*}_c(M)}$

Warning H^{lf}_*, H^*_c are not homotopy invariant (indeed non-trivial for $\mathbb{R}^n$)

Caused because they are not functorial. They are however functorial for **proper maps**

Mayer-Vietoris holds for H^*_c but not for H^{lf}_*. (preimages of compact sets are compact)

Fact $H^*_c(X) \cong \varinjlim H^*(X, X \setminus K)$ where compacts $K_1 \subseteq K_2$ give $H^*(M, M \setminus K_1) \to H^*(M, M \setminus K_2)$

Direct limit $\varinjlim G_i$ via maps $G_i \to G_j$ means $\sqcup G_i /$ identify $g \in G_i$ with its images under those maps

(The indices are partially ordered & directed: $\forall i,j, \exists k \geq i,j$ so can compare G_i, G_j inside G_k (via $G_i \to G_k$, $G_j \to G_k$))

Fact $\varinjlim$ is an exact functor.

Cap product and Poincaré duality revisited

X space, $k \geq l$

$\cap : C_k(X) \times C^l(X) \longrightarrow C_{k-l}(X)$ **cap product**

$$(\sigma : \Delta^k \to X) \cap (\phi : C_l(X) \to \mathbb{Z}) = \underbrace{\phi(\overbrace{\sigma|_{[e_0,\ldots,e_l]}}^{\text{"bottom face"}})}_{\in \mathbb{Z}} \cdot \underbrace{\overbrace{\sigma|_{[e_l,\ldots,e_k]}}^{\text{"top face"} \cong \Delta^{k-l}}}_{\in C_{k-l}(X)}$$

(easy) Properties

- $\cap$ bilinear
- $\partial(\sigma \cap \phi) = (-1)^l (\partial\sigma \cap \phi - \sigma \cap \partial^* \phi)$
- cycle $\cap$ cycle is cycle
- boundary $\cap$ cycle / cycle $\cap$ boundary are boundaries

$\Rightarrow$ $\boxed{\cap : H_k(X) \times H^l(X) \longrightarrow H_{k-l}(X)}$ bilinear

Theorem (Poincaré duality) The map $\phi \longmapsto [M] \cap \phi$ gives following isos

① For M closed oriented n-mfd

$$[M] \cap \cdot : H^*(M) \xrightarrow{\cong} H_{n-*}(M)$$

② For M non-compact oriented n-mfd,

$$[M] \cap \cdot : H^*_c(M) \xrightarrow{\cong} H_{n-*}(M) \qquad ⊛$$

$$[M] \cap \cdot : H^*(M) \xrightarrow{\cong} H^{lf}_{n-*}(M)$$

Sketch Pf of ① for smooth mfds (Non-examinable)

If M smooth $\Rightarrow \exists$ "good cover" U_i of M meaning open cover s.t.

↑ FACT from Riemannian geometry ("convex neighbourhoods")

$U_i \cong \mathbb{R}^n$

$U_{i_1} \cap \cdots \cap U_{i_k} \cong \mathbb{R}^n$ or $\emptyset$

Then compute $H^*_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$ and ⊛ holds for $\mathbb{R}^n$.

$\Rightarrow$ ⊛ holds $\forall U_i$

$\Rightarrow$ by naturality of ⊛ and of Mayer-Vietoris get ⊛ for $\cup U_i$ finite (↖ use 5-lemma)

$\Rightarrow$ ⊛ for M, which is ①. □

General Pf of Poincaré duality ← Non-examinable

Step 1 : holds for $\mathbb{R}^n$

Pf $H_c^k(\mathbb{R}^n) \underset{\uparrow}{\cong} H^k(D^n, S^{n-1}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z} & k = n \end{cases} \cong H_{n-k}(\mathbb{R}^n)$

can make K larger by picking K = large ball
then $H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong H^k(K, \partial K)$

Pick Δ-cx structure for $\mathbb{R}^n$. So $[\mathbb{R}^n] = \Sigma \pm \sigma_i$ ← sum over n-simplices.
Say ∃ simplex $\sigma_0 : \Delta^n \to \mathbb{R}^n$. Define $\phi : C_c^{cw}(\mathbb{R}^n) \to \mathbb{Z}$, $\phi(\sigma_0) = \pm 1$ ⊛
$\phi(\text{other simplices}) = 0$
$\Rightarrow \delta\phi = 0$ for dim reasons

$[\mathbb{R}^n] \cap \phi = \Sigma \phi(\pm\sigma_i) = \phi(\pm\sigma_0) = 1$ (pick sign in ⊛)

Step 2 holds for $A, B, A \cap C \Rightarrow$ holds for $A \cup B$

Pf Mayer-Vietoris for H_c^*, naturality, 5-lemma ✓

Step 3 holds for A_i, and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \Rightarrow$ holds for $\cup A_i$

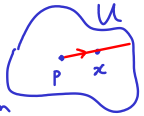
Pf By applying $\underrightarrow{\lim}$: both sides of P.D. iso commute with limits ✓

Step 4 holds for open subsets in $\mathbb{R}^n$

Pf Every such set is a union of convex open sets (e.g. balls)

By Step 3 enough to consider case of finite union.

By induction on # convex open sets :



fix $p \in U$
$U \longrightarrow \mathbb{R}^n$
$x \longmapsto \frac{x-p}{\text{dist } p \text{ to } \partial U \text{ along ray to } x}$

1 convex set $U \cong \mathbb{R}^n$ via a proper homeomorphism,
now use Step 1 ✓

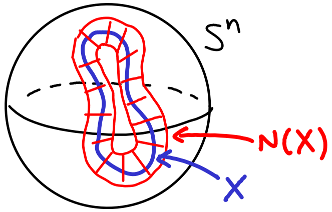
2 convex sets : KEY TRICK convex set ∩ convex set is convex in $\mathbb{R}^n$!
⇒ use Step 2 & previous case

k+1 convex sets : $A = \bigcup\{\text{first } k \text{ convex sets}\}$, $B =$ last convex set
$\Rightarrow A \cap B \subseteq B$ is a union of k convex sets
} ⇒ use Step 2 & Inductive hypothesis ✓

Step 5 holds for mfd M

Consider open sets in M for which it holds.
By a Zorn's Lemma argument we get a maximal open subset U where holds.
If $U \neq M$ pick $p \in M \setminus U$ and nbhd $V \cong \mathbb{R}^n$ of p. Then holds for $U, V, U \cap V$
(note $U \cap V \subseteq V \cong \mathbb{R}^n$ open, so Step 4 applies) so by Step 2 holds for $U \cup V$
Contradicts maximality. ✓ □

Alexander duality



$\emptyset \neq X \subsetneq S^n$ compact subset s.t. (in fact, enough to assume X is locally contractible)
∃ open neighbourhood N(X) which deformation retracts to X
such that $\overline{N(X)} \subseteq S^n$ is an n-mfd with boundary.

Theorem $\boxed{\tilde{H}_*(X) \cong \tilde{H}^{n-*-1}(S^n \setminus X)}$

Pf later

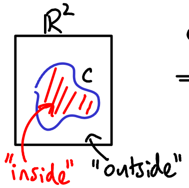
Example $X \subseteq S^3$ knot (i.e. $X = \text{image}(S^1 \xrightarrow[\text{homeomorphism onto the image}]{} S^3)$)
↖ embedding

$\Rightarrow N(X) \cong$ solid torus $\simeq S^1$

$$\Rightarrow \begin{aligned} \tilde{H}_0(X) &= 0 = \tilde{H}^2(S^3 \setminus X) \\ \tilde{H}_1(X) &= \mathbb{Z} = \tilde{H}^1 \quad " \\ \tilde{H}_2(X) &= 0 = \tilde{H}^0 \quad " \end{aligned}$$

so the homology of a knot complement does not tell knots apart (always same)

Theorem (Jordan curve theorem)



$C \simeq S^1$ closed curve in $\mathbb{R}^2 \subseteq S^2$ ← e.g. by stereographic projection $S^2 \cong \mathbb{C} \cup \infty$

$\Rightarrow \mathbb{R}^2 \setminus C$ has 2 path-components (= connected components)

Similarly for $C \simeq S^n \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}$.

Pf $C \simeq S^n \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}$ $\Rightarrow \tilde{H}_n(S^n) \cong \mathbb{Z} \cong \tilde{H}^0(S^{n+1} \setminus C)$
$\Rightarrow H^0(S^{n+1} \setminus C) \cong \mathbb{Z}^2$
$\Rightarrow S^{n+1} \setminus C$ has 2 path components. □

Proof Alexander duality

$Y := S^n \setminus N(X) \qquad (\simeq S^n \setminus X)$

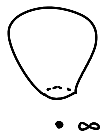
for $* < n-1$

$$\begin{aligned}\widetilde{H}^{n-*-1}(Y) &= H^{n-*-1}(Y)\\ &\underset{\text{Lefschetz}}{\cong} H_{*+1}(Y, \partial Y)\\ &\underset{\text{exc.}}{\cong} H_{*+1}(S^n, \overline{N(X)})\\ &\underset{\substack{\text{LES}\\ \text{using } * < n-1}}{\cong} \widetilde{H}_{*+1}(\underbrace{\overline{N(X)}}_{\simeq X})\end{aligned}$$

for $* = n-1$

$$\begin{aligned}\widetilde{H}^0(Y) \oplus \mathbb{Z} &\cong H^0(Y)\\ &\underset{\text{Lef.}}{\cong} H_n(Y, \partial Y)\\ &\underset{\text{exc.}}{\cong} H_n(S^n, \overline{N(X)})\end{aligned}$$

$$\begin{tikzcd}
 & & \cong \widetilde{H}_{n-1}(\underbrace{\overline{N(X)}}_{\simeq X}) \oplus \mathbb{Z} & & \\
0 \arrow[r] & \widetilde{H}_n(S^n) \arrow[r] \arrow[ur] \arrow[dr, "\cong"'] & H_n(S^n, \overline{N(X)}) \arrow[r] \arrow[d] & \widetilde{H}_{n-1}(\overline{N(X)}) \arrow[r] & 0 \\
 & & H_n(S^n, S^n \setminus \infty) \cong \mathbb{Z} & & \\
 & & S^n = \mathbb{R}^n \cup \infty \arrow[u] & &
\end{tikzcd}$$



for $* = n$

$H^{n-*-1}(Y) = H^{-1}(Y) = 0$

$H_n(X) \cong H_n(N(X)) \underset{\text{Lef.}}{\cong} 0$ n-mfd with bdry $\neq \emptyset$. $\square$