C3.1 Algebraic Topology

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Please be aware there are likely typos in these notes: comments/corrections are welcome!

Course Book

- Hatcher, Algebraic Topology — Chp. 2 & 3

  This is also freely available from the author's website. You are expected to read chapters 2 & 3.

Other references

- Ulrike Tillmann's C3.1 notes — see course page
- The discussion of orientations in my B3.2 Geometry of Surfaces notes may be helpful.

Other books

Massey, A basic course in Algebraic Topology

MORE BASIC but full of ideas:

Fulton, Algebraic Topology: a first course

MORE ADVANCED:

May, A concise course in Algebraic Topology

Davis & Kirk, Lecture notes in Algebraic Topology

Bredon, Topology and Geometry

Bott & Tu, Differential forms in Algebraic Topology

Classics by Spanier, Dold, also see references in May's book

CONTENTS

6. MAYER-VIETORIS SEQUENCE

MV LES, $H_*(S^n)$

wedge sum $X \vee Y$, cone CX, suspension ΣX, connected sum $X \# Y$

7. DEGREE OF MAPS OF SPHERES

degree, application to tangent vector fields on sphere, hairy ball theorem

local degree, proof of fundamental thm of algebra

8. CELLULAR HOMOLOGY

CW complexes, cellular complex, rank $H_n^{CW} \leqslant \#$n-cells

$H_*^{CW}(\mathbb{D}^1 \times \mathbb{D}^1)$, $H_*^{CW}(\mathbb{R}P^n)$, $H_*^{CW}(S^n)$, $H_*^{CW}(\Sigma_g)$

Δ-cx $\Rightarrow$ CW cx, $H_*^{CW}(X) \cong H_*^{\Delta}(X) \cong H_*(X)$, Axioms for homology

9. COHOMOLOGY

cochains, cohomology, $H^*(X)$, $H^*_{CW}(X)$, $H^*_{\Delta}(X)$, $H^*(\mathbb{R}P^3)$

functoriality, homotopy invariance, cochain homotopy, dual of a SES

excision, LES, Mayer-Vietoris for H^*, axioms for cohomology

10. CUP PRODUCT

Cup product, $H^*(X)$ unital graded-commutative ring, pull-back is ring hom,

examples: $H^*(T^2)$, $H^*(\Sigma_2)$, remarks about intersection theory

11. KÜNNETH FORMULA AND PRODUCT SPACES

Tensor products of R-mods, tensor product of chain cxes,

algebraic Künneth thm, product spaces $X \times Y$, Euler characteristic χ

CW-cx for product space, Künneth thm, $H^*(S^n \times S^m)$, $H^*(T^n)$

12. UNIVERSAL COEFFICIENTS THEOREM

Universal coeff. thm, Background on Ext groups and free resolutions

(Co)homology with coefficients in a ring/field/module, $H^*(\mathbb{R}P^2; \mathbb{Z}/2)$

Univ. coeff. thm for PID R, Duality $H^*(X;\mathbb{F}) \simeq H_*(X;\mathbb{F})$ over fields

Structure thm for f.g. mods M over PID R, $\mathrm{Ext}^1_R(M;R)$, torsion shift H_* to H^{*+1}

13. MANIFOLDS: POINCARÉ-LEFSCHETZ DUALITY

Local orientation, orientation, fundamental class, degree, P. duality, L. duality,

Locally finite homology H_*^{lf}, cohomology with compact supports H_c^*, Cap product and P.D.,

Alexander duality, knot complements, Jordan curve thm

0. OVERVIEW OF THE COURSE

Motivation

Space X $\xrightarrow{\text{associate}}$ Algebraic object $A(X)$
like numbers, groups, rings, ...

Isomorphism of spaces $X \cong Y$ $\Longrightarrow$ Isomorphism $A(X) \cong A(Y)$

PAY-OFF Can tell spaces apart:
compute $A(X), A(Y)$ $\rightsquigarrow$ if $A(X) \not\cong A(Y)$ then $X \not\cong Y$

Examples

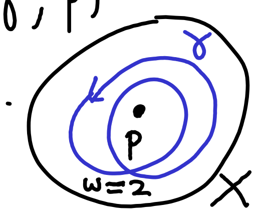
1) Set X $\longrightarrow$ $A(X) = \# X \in \mathbb{N}$
(bijection $X \to Y$) $\Longrightarrow$ same size

2) Vector space X $\longrightarrow$ $A(X) = \dim X \in \mathbb{N}$
(linear iso $X \to Y$) $\Longrightarrow$ same dim

3) Topological Space X
- $\longrightarrow$ $\# \pi_0(X) = \#$ **path components**
- $\longrightarrow$ **# connected components**
- $\longrightarrow$ $\chi(X) =$ **Euler characteristic**

(the first three $\in \mathbb{N}$)

- $\longrightarrow$ Function $X \times \mathcal{L}X \longrightarrow \mathbb{Z}$, where $\mathcal{L}X$ = loops $= C^0(S^1, X)$
$(p, \gamma) \longmapsto w(\gamma; p)$
Winding number of γ around p.



(Homeomorphism $X \to Y$) $\longrightarrow$ $A(X) = A(Y)$

CONVENTION for this course:

Space = topological space

Map of spaces = continuous map = cts map

For spaces, "$\cong$" means homeomorphism

"id" = identity map

All diagrams commute unless we say otherwise, e.g.
$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \gamma\downarrow & & \downarrow\beta \\ C & \xrightarrow{\delta} & D \end{array}$$
means $\beta \circ \alpha = \delta \circ \gamma$

Category theory is the best language to phrase all this

Algebraic topology builds <u>functors</u> such as

$$F : \left\{ \begin{array}{l} \text{Topological spaces} \\ \text{and continuous maps} \end{array} \right\} \longrightarrow \begin{cases} \left\{ \begin{array}{l} \text{Groups} \\ \text{and group homs} \end{array} \right\} \\ \left\{ \begin{array}{l} \text{Abelian groups} \\ \text{and group homs} \end{array} \right\} \\ \left\{ \text{Rings and ring homs} \right\} \end{cases}$$

We will not use much <u>category theory</u>, just basic terminology:

<u>Def</u> A **category** $\mathcal{C}$ consists of the data:

$Ob(\mathcal{C})$ = a collection of **objects**

$Hom(A,B)$ = a set of **morphisms** between any $A, B \in Ob\,\mathcal{C}$ ("**arrows**")

- with **composition** rule $Hom(B,C) \times Hom(A,B) \xrightarrow{\circ} Hom(A,C)$ which is associative.

$$A \xrightarrow{f} B \xrightarrow{g} C, \quad g \circ f : A \to C$$

- with **identity** morphs $id_A \in Hom(A,A)$ s.t. $f \circ id_A = id_B \circ f = f$ $\quad \forall (f : A \to B) \in Hom(A,B)$

<u>Example</u>

Sets = { sets with all maps between sets }

Top = { topological spaces with continuous maps }

Gps = { groups with group homs }

<u>Def</u> A (covariant) **functor** $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is the data:

- an assignment $(A \in Ob\, \mathcal{C}_1) \longmapsto (F(A) \in Ob\, \mathcal{C}_2)$
- an assignment $(A \xrightarrow{f} B) \longmapsto (F(A) \xrightarrow{F(f)} F(B))$, where $(A \xrightarrow{f} B) \in Hom_{\mathcal{C}_1}(A,B)$ and $(F(A) \xrightarrow{F(f)} F(B)) \in Hom_{\mathcal{C}_2}(F(A), F(B))$

compatible with identities and compositions:

$$F(id_A) = id_{F(A)} \qquad F(g \circ f) = F(g) \circ F(f)$$

A **contravariant functor** is defined similarly except it reverses the direction of arrows: $(F(A) \xleftarrow{F(f)} F(B)) \in Hom(F(\underline{B}), F(\underline{A}))$

(so $F(g \circ f) = F(f) \circ F(g)$ reverses order of compositions)

Examples

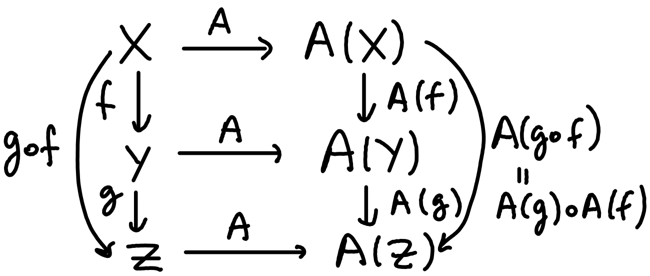
1) $F : \text{Top} \longrightarrow \text{Sets}$, $A \longmapsto A$, $f \longmapsto f$ "forget the topology and continuity"

2) $F : \text{Sets} \longrightarrow \text{Gps}$, $A \longmapsto$ free abelian group generated by A

$$\mathbb{Z}\langle A\rangle = \left\{ \sum_{\text{finite}} n_i \cdot a_i \;:\; \begin{matrix} a_i \in A \\ n_i \in \mathbb{Z} \end{matrix} \right\}$$

$$(A \xrightarrow{f} B) \longmapsto \left(\begin{matrix} F(A) : \mathbb{Z}\langle A\rangle \longrightarrow \mathbb{Z}\langle B\rangle \\ \sum n_i \cdot a_i \longmapsto \sum n_i \cdot f(a_i) \end{matrix} \right)$$

When we say a construction is **natural** we mean functorial:



A : (a category of spaces) $\longrightarrow$ (a cat. of algebraic objects)
The algebraic objects we assigned are assigned compatibly with maps of spaces, and the compatibility maps $A(f)$ are also compatible w.r.t. composition.
So we made compatible choices in constructing A.

Not to be confused with natural transformations of functors (later) which is about relating two such constructions A_1, A_2 in a compatible way

Example of a functor in algebraic topology

(see B3.5 Topology and Groups course)

$$\pi_1(X,p) = \textbf{Fundamental group} = \left\{ \begin{matrix} \gamma \in \mathcal{L}X \\ \gamma(1) = p \end{matrix} \right\} \Big/ \begin{matrix} \text{continuous deformations} \\ \text{of loops based at } p \end{matrix}$$

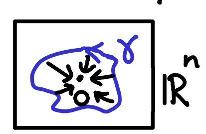
(X: topological space; $p \in X$)

Group multiplication: concatenate loops $\gamma_1 * \gamma_2$

(each travelling twice as fast)

Examples

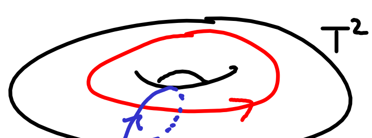
$\pi_1(\mathbb{R}^n) = 0$ ⟵ deform: $h : S^1 \times [0,1] \longrightarrow \mathbb{R}^n$, $h(t,s) = (1-s)\gamma(t)$



$\pi_1(S^1) \cong \mathbb{Z}$ ⟵ total # times wind around circle

$\pi_1(S^n) \cong 0$ $n \geq 2$ (not obvious)

$\pi_1(\text{torus}) \cong \mathbb{Z}^2$ ⟵

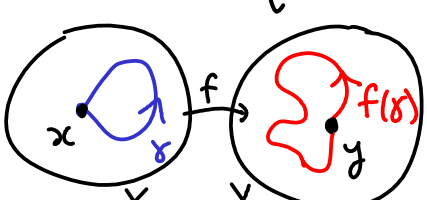


those loops generate π_1

FUNCTOR

$$\text{BasedTop} = \left\{ \begin{matrix} \text{Topological spaces with choice of basepoint,} \\ \text{and continuous basepoint-preserving maps} \end{matrix} \right\} \xrightarrow{\pi_1} \text{Gps}$$

$$(X,p) \longmapsto \pi_1(X,p)$$



$$\left((X,x) \xrightarrow[f]{\text{cts}} (Y,y)\right) \longmapsto \left(\begin{matrix} \pi_1(X,x) \xrightarrow{\text{gp. hom.}} \pi_1(Y,y) \\ \gamma \longmapsto f \circ \gamma \end{matrix} \right)$$

Lemma Functors map isomorphisms to isomorphisms (iso. means $\exists$ inverse w.r.t. composition)

Pf $A \xrightarrow{f} B \xrightarrow{g} A$ (composite $= id$) $\Rightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FA$ (composite $= F(id) = id$), similarly for $B \xrightarrow{g} A \xrightarrow{f} B$ (composite $= id$). $\square$

Def **Natural transformation** $\alpha : F \longrightarrow G$ between functors $C_1 \overset{F}{\underset{G}{\rightrightarrows}} C_2$

is an association $(A \in \mathrm{Ob}\, C_1) \longmapsto (\alpha_A : F(A) \longrightarrow G(A))$ $\in \mathrm{Hom}_{C_2}(F(A), G(A))$

such that $(A \xrightarrow{f} B)$, $f \in \mathrm{Hom}_{C_1}(A,B)$ $\Rightarrow$

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

(commutes)

It is called a **natural isomomorphism** if each α_A is an isomorphism in C_1

Example of a natural transformation in algebraic topology

Let $H_1(X,p) =$ abelianisation of $\pi_1(X,p)$ (want to identify $ab=ba$ so quotient by $\langle aba^{-1}b^{-1}\rangle$) ← **commutators**

$\Rightarrow$ natural trans. $(\text{Based Top} \xrightarrow{\pi_1} \text{Gps}) \xrightarrow{\alpha} (\text{Based Top} \xrightarrow{H_1} \text{Gps})$

which associates $(X,p) \in \text{BasedTop} \longmapsto \left(\alpha_{(X,p)} : \pi_1(X,p) \xrightarrow{\text{quotient}} H_1(X,p)\right)$

Cultural Rmk **higher homotopy groups** $\pi_n(X,p) = \left\{ \begin{array}{l} S^n \xrightarrow{\text{cts}} X \\ \text{basept} \mapsto p \end{array} \right\} \Big/ \text{cts deform}^{n}$

FACT abelian for $n \geq 2$, but hard: e.g. $\pi_k(S^n)$ not all known.

We will not study these in this course.

We will study simpler invariants called HOMOLOGY groups $H_n(X)$

FACT (Hurewicz) $\exists$ natural transformation $\pi_n \longrightarrow H_n$ which will make sense at the end of course:

$f : S^n \xrightarrow{\text{cts}} X$ gives rise to a class $f_*[S^n] \in H_n(X)$.

Exercise to practice these notions from category theory:

Summarise your undergraduate linear algebra as follows:

1) $\exists$ functor $F : \underbrace{\left\{ \begin{array}{l} \mathbb{R}^n \text{ for } n \in \mathbb{N} \\ \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) = \{m \times n \text{ matrices}\} \end{array} \right\}}_{\text{Mat}} \to \underbrace{\left\{ \begin{array}{c} \text{finite dimensional real vector spaces} \\ \text{and linear maps} \end{array} \right\}}_{\text{Vect}}$

2) A choice of basis for each vector space V determines a functor $G : \text{Vect} \to \text{Mat}$

3) Construct natural isomorphisms $G \circ F \xrightarrow{\alpha} Id_{\text{Mat}}$, $F \circ G \xrightarrow{\beta} Id_{\text{Vect}}$

When functors satisfying such natural isos exist, the categories are called **equivalent** (not isomorphic). So Mat, Vect are equivalent categories.

Aim of the course: build a functor

HOMOLOGY $H_*: \textbf{Top} \longrightarrow \textbf{Graded abelian groups}$

$$X \longmapsto H_*(X) \quad \leftarrow \text{grading } * \in \mathbb{Z}$$

$$(X \to Y) \longmapsto \begin{pmatrix} H_*(X) \longrightarrow H_*(Y) \\ \text{grading preserving hom} \end{pmatrix}$$

and a contravariant functor

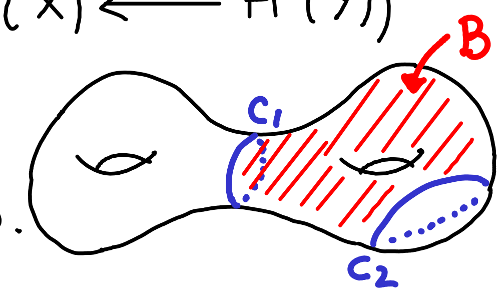
COHOMOLOGY $H^*: \textbf{Top} \longrightarrow \textbf{Graded rings}$

$$X \longmapsto H^*(X)$$

$$(X \to Y) \longmapsto (H^*(X) \longleftarrow H^*(Y))$$

Rough idea:

$H_* X$ is generated by "nice" subspaces $C \subseteq X$ which have no boundary: $\partial C = \emptyset$, modulo identify C_1, C_2 if $C_1 \cup C_2$ arises as a boundary ∂B. Call such C_1, C_2 **homologous**.



Facts

- $H_0(X) \cong \bigoplus_{\pi_0 X} \mathbb{Z}$ ← generated by a point in each path-comp. ($\pi_0 X = \{\text{path-connected components}\}$)
- $X = \sqcup X_i$ path-components $\Longrightarrow H_*(X) \cong \bigoplus H_*(X_i)$
- $\chi(X) = \sum_{d \geq 0} (-1)^d \operatorname{rank} H_d(X)$

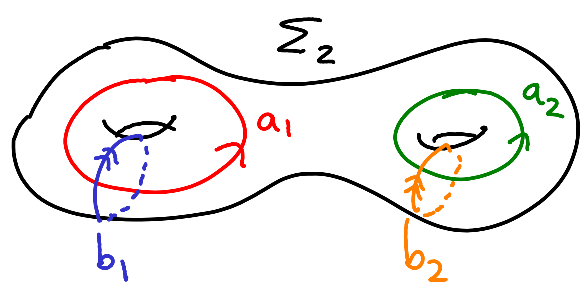
χ ↑ **Euler characteristic**; rank ↑ max # $\mathbb{Z}$-linearly independent elements

Example: compact surfaces

$$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{2g} & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$$

Σ_g ↑ orientable surface genus g

$\chi = 2 - 2g$

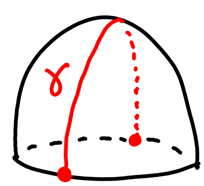


We will show that those 4 loops generate $H_1(\Sigma_2)$

$$H_*(N_h) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}^{h-1} & * = 1 \\ 0 & \text{else} \end{cases}$$

N_h ↑ non-orientable surface S^2 with h Möbius bands attached

$\chi = 2 - h$



$N_1 = \mathbb{R}P^2 = S^2 / \pm \mathrm{Id}$

Notice γ is a loop. It generates $H_1(N_1)$

Examples of homology calculations

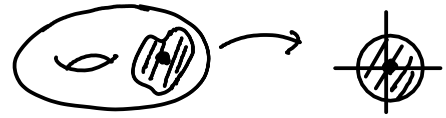
$H_*(\mathbb{R}^n) \cong H_*(\mathbb{D}^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ 0 & \text{else} \end{cases}$

n-dimensional ball
$\mathbb{D}^n = \{x : \|x\| \leq 1\} \subseteq \mathbb{R}^n$

$H_*(S^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$

$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ n-dim sphere

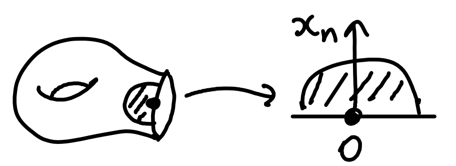
Hausdorff top. space s.t. each pt has an open neighbourhood homeo to an open ball in $\mathbb{R}^n$



$H_*(X) = \begin{cases} 0 & \text{for } * < 0 \\ 0 & \text{for } * > n \\ \mathbb{Z} & \text{for } * = n \\ 0 & \text{for } * = n \end{cases}$

- 0 for $*>n$ for **n-dimensional manifolds** X
- $\mathbb{Z}$ for $*=n$ for connected orientable compact manifold
- 0 for $*=n$ for
  - non-orientable "
  - non-compact "
  - connected **manifolds with boundary** $\neq \emptyset$

boundary point has an open nbhd homeo to open nbhd of $0 \in$ half-space: $\{x \in \mathbb{R}^n : x_n \geq 0\}$



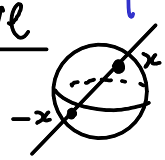
Rmk M compact connected n-mfd

$\Rightarrow H_{n-1}(M) \cong \begin{cases} \mathbb{Z}^k & \text{some } k \geq 0 \text{ if orientable} \\ \mathbb{Z}^k \oplus \mathbb{Z}/2 & \text{" " non-orientable} \end{cases}$

$H_*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/2 & \text{odd } *=1,3,5,\dots < n \\ \mathbb{Z} & *=n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$

$\mathbb{RP}^n = S^n / \pm id$

real projective space



$\mathbb{RP}^n$ orientable $\Leftrightarrow$ n odd
(e.g. $\mathbb{RP}^1 \cong S^1$)

$H_*(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \text{even } *=0,2,4,\dots,2n \\ 0 & \text{else} \end{cases}$

e.g. $\mathbb{CP}^1 \cong S^2$
stereographic projection

$\mathbb{CP}^n$ = space of complex lines through $0 \in \mathbb{C}^{n+1}$

complex projective space

$\left(\cong (\mathbb{C}^n \setminus 0) / \mathbb{C}^*\text{-rescaling} \right.$
$\left. = \{[z_0, z_1, \dots, z_n] : z_j \in \mathbb{C} \text{ not all } 0\} / [z] = [\lambda z] \text{ for } \lambda \in \mathbb{C}^* \right)$

Examples of cohomology calculations

$$H^0(X) = \prod_{\pi_0 X} \mathbb{Z} \quad \leftarrow \text{ if } \pi_0 X \text{ finite, then } \cong \bigoplus_{\pi_0 X} \mathbb{Z} \cong H_0 X$$

but if infinite then not: here $\bigoplus$ allow only finite sums

$$H^*(X) \cong \prod H^*(X_i) \quad \leftarrow X_i \text{ path-components of } X$$

FACT

> If $H_n(X)$ finitely generated abelian gp, so
> $$H_n(X) \cong \mathbb{Z}^{r_n} \oplus T_n \quad \leftarrow T_n = \text{torsion elements} = \text{elements of finite order}$$
> Then $H^n(X;R) \cong \mathbb{Z}^{r_n} \oplus T_{n-1}$ as abelian groups

$\Longrightarrow H^*(\Sigma_g), H^*(\mathbb{R}^n), H^*(\mathbb{D}^n), H^*(S^n), H^*(\mathbb{CP}^n)$ same as for H_*, but:

$$H^*(N_h) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{h-1} & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases} \qquad H^*(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases} \ (h=1)$$

$$H_*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * \text{ even} = 2, 4, \ldots \leq n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

and $H^n(\text{non-orientable compact } n\text{-mfd}) \cong \mathbb{Z}/2$.

$\Longrightarrow$ The interesting feature is the <u>ring structure</u>:

$$H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/x^{n+1}$$

grading: $|x| = 2$

$\mathbb{Z}[x]$ = polynomials in x with $\mathbb{Z}$-coefficients

$$H^*(S^n) \cong \mathbb{Z}[x]/x^2 \qquad |x| = n$$

$$H^*(T^n) \cong \Lambda[x_1, \ldots, x_n] \qquad |x_i| = 1$$

$T^n = S^1 \times \cdots \times S^1$ **n-torus**

e.g. $n=2$ (torus): $\begin{cases} \mathbb{Z} \cdot 1 & * = 0 \\ \mathbb{Z} x_1 \oplus \mathbb{Z} x_2 & * = 1 \\ \mathbb{Z}\, x_1 \wedge x_2 & * = 2 \\ 0 & \text{else} \end{cases}$

exterior algebra generated$/\mathbb{Z}$ by symbols $x_{i_1} \wedge \ldots \wedge x_{i_k}$ with $i_1 < \ldots < i_k$, product given by $\wedge$ using relations $x_i \wedge x_j = -x_j \wedge x_i$.

$$H^*(\mathbb{RP}^{2n}) \cong \mathbb{Z}/2[x]/x^{n+1} \qquad |x| = 2$$

$$H^*(\mathbb{RP}^{2n+1}) \cong \mathbb{Z}/2[x]/x^{n+1} \oplus \mathbb{Z}[-2n-1]$$

$\mathbb{Z}[-2n-1]$ means: a copy of $\mathbb{Z}$ in degree $2n+1$

$$H^*(\Sigma_g) \cong \Lambda[a_1, \ldots, a_g, b_1, \ldots, b_g] / \langle a_i b_j \text{ for } i \neq j,\ a_i b_i - a_j b_j,\ a_i a_j,\ b_i b_j \rangle$$

$|a_i| = |b_i| = 1 \quad \leftarrow$ exterior alg. instead of poly. alg since $a_i b_i = -b_i a_i$

Why more information?

$S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ have same $H_* = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & * = 2 \\ \mathbb{Z} & * = 4 \end{cases}$

(connected sum: remove a ball in each, glue along ∂ ball)

but the <u>rings</u> H^* are not iso, hence $S^2 \# S^2 \not\cong \mathbb{CP}^2 \# \mathbb{CP}^2$.

Example of why such functors are useful

Theorem Invariance of dimension ← "homeomorphisms preserve dimension"

Suppose $F_*: \text{Top} \to \text{Gps}$ functors s.t.

$F_*(\mathbb{D}^n) = 0$ for all $*$

$F_*(S^n) \neq 0$ only for $* = n$

Then: $S^n \cong S^m \iff n = m$

$\mathbb{R}^n \cong \mathbb{R}^m \iff n = m$

Non-trivial result because there are space-filling curves. e.g. Peano (1890) $\exists$ cts surjection $[0,1] \to [0,1]^2$ ($[0,1]$ = interval, $[0,1]^2$ = square). The theorem implies this is not injective. (cts. bij. compact $\to$ Hausdorff $\Rightarrow$ homeo)

Pf By Lemma, $F_n(S^n \xrightarrow{\cong} S^m)$ is an iso $F_n(S^n) \xrightarrow{\cong} F_n(S^m)$ of gps. ($F_n(S^n) = \mathbb{Z}$; $F_n(S^m) = 0$ if $n \neq m$) □

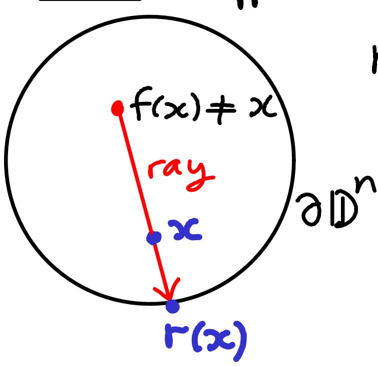
We will build such F_*:

reduced homology $\tilde{H}_*$ s.t. $\begin{cases} \tilde{H}_* = H_* & \text{for } * \neq 0 \\ \tilde{H}_0(X) \cong \mathbb{Z}^{\#(\text{path-conn. components}) - 1} \end{cases}$

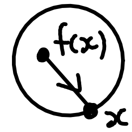
Theorem Brower fixed point thm

$f: \mathbb{D}^n \to \mathbb{D}^n$ continuous $\Rightarrow$ f has a fixed point ($f(p) = p$ some p)

Proof Suppose not. Let $r(x) = (\text{ray from } f(x) \text{ to } x) \cap \partial\mathbb{D}^n$



notice:
- $r: \mathbb{D}^n \to \partial\mathbb{D}^n = S^{n-1}$ continuous
- $r|_{\partial\mathbb{D}^n} = \text{id}_{S^{n-1}}$

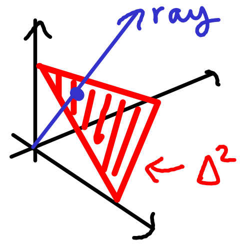


$S^{n-1} = \partial\mathbb{D}^n \xrightarrow{\text{inclusion } i} \mathbb{D}^n \xrightarrow{r} S^{n-1}$, $\quad r \circ i = \text{id}$

$\underset{\text{apply } F_{n-1}}{\Longrightarrow} F_{n-1}(r) \circ F_{n-1}(i) = \text{id} \Longrightarrow F_{n-1}(i)$ injective $F_{n-1}(S^{n-1}) \to F_{n-1}(\mathbb{D}^n)$ ($F_{n-1}(S^{n-1}) \cong \mathbb{Z}$, $F_{n-1}(\mathbb{D}^n) \cong 0$) ↯ □

Example (Brower) $A = n \times n$ matrix, $A_{ij} > 0$ real $\Rightarrow$ $\exists$ evalue $\lambda > 0$ with real evector $(v_1, \ldots, v_n)$ with $v_i \geq 0$

Hint

$\mathbb{R}^3 \supseteq$ 

$X = \{\text{rays in "positive octant"}\}$ ← $x \in \mathbb{R}^n : x_i > 0 \ \forall i$

notice $AX \subseteq X$

notice $X \cong \Delta^n = \{x \in \text{octant} : \sum x_i = 1\} \cong \mathbb{D}^n$

$\text{ray} \mapsto \text{ray} \cap \Delta^n$

1. ALGEBRAIC PRELIMINARIES : CHAIN COMPLEXES

Graded abelian groups

Def A **$\mathbb{Z}$-graded abelian group** C is an abelian group together with a direct sum decomposition

$$\boxed{C = \bigoplus_{n\in\mathbb{Z}} C_n}$$

(C_n ← abelian group)

Convention: always grade by $\mathbb{Z}$ unless say otherwise.

A graded ab. gp. A is a **graded subgp** of C if
- subgp
- $A_n \subseteq C_n$.

A **homomorphism** $h : C \to D$ of gr. ab. gps is hom of gps s.t.

$$h(C_n) \subseteq D_n$$

A **hom of degree k** is hom with

$$h(C_n) \subseteq D_{n+k}$$

Shift by k: $\mathbb{Z}$-gr. ab. gp. $C[k]$ with

$$\boxed{C[k]_n = C_{k+n}}$$

Notice: $C[k]_0 = C_k$ is now in degree zero, so shifted <u>down</u> by k

$\Rightarrow$ Can view gr. hom of deg k as a gr. hom

$$h : C \longrightarrow D[k]$$

<u>FACT</u> Finitely generated abelian groups are classified:

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/p_1^{n_1} \oplus \dots \oplus \mathbb{Z}/p_k^{n_k}$$

$n_i \in \mathbb{Z}$
p_i primes (possibly not distinct)

$\mathbb{Z}^r$: **free part**; r called **rank G**; $\mathbb{Z}/p_1^{n_1} \oplus \dots \oplus \mathbb{Z}/p_k^{n_k}$: **torsion part**

compare finite dimensional vector spaces / field $\mathbb{F}$: $V \cong \mathbb{F}^r$ ← $r = \dim V$

Chain complexes

Def A *chain complex* (C_*, ∂_*) is a gr. ab. gp. C together with a hom ∂ of degree -1 such that $\partial \circ \partial = 0$.

(∂: *differential* or *boundary homomorph*)

Thus:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

hence $\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n$, where $\operatorname{Im} \partial_{n+1} =: B_n$ (*n-boundaries*) and $\operatorname{Ker} \partial_n =: Z_n$ (*n-cycles*)

n-chains = elements of C_n

Now consider "cycles modulo boundaries":

Def The *homology* of (C_*, ∂_*) is the gr. ab. gp.

$$H_n(C_*, \partial_*) = \operatorname{Ker} \partial_n \big/ \operatorname{Im} \partial_{n+1}$$

often abbreviate by $H_*(C)$

Two cycles are called *homologous* if they differ by a boundary

Def A *chain map* $h : (C_*, \partial_*) \longrightarrow (\tilde{C}_*, \tilde{\partial}_*)$ is a graded hom such that

$$h \circ \partial_* = \tilde{\partial}_* \circ h$$

Example A chain *subcomplex* $C_* \subseteq \tilde{C}_*$ is a graded subgp with $\partial_* =$ restriction of $\tilde{\partial}_*$ to C_*.

So the inclusion $\text{incl} : (C_*, \partial_*) \longrightarrow (\tilde{C}_*, \tilde{\partial}_*)$ is a chain map.

Also get *quotient complex* $\tilde{C}_* / C_*$

with $\tilde{\partial}_* [\tilde{c}] = [\tilde{\partial}_* \tilde{c}]$ (well-defined: $\tilde{\partial}_* C_* = \partial_* C_* \subseteq C_*$)

Lemma A chain map induces a hom on homology

$$H_*(h) = h_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$$
$$[x] \longmapsto [h(x)]$$

Proof $\operatorname{Ker} \partial_n \xrightarrow{\text{hom}} \operatorname{Ker} \tilde{\partial}_n$

$x \longmapsto h(x)$ since $\tilde{\partial}(h(x)) = h(\underbrace{\partial x}_{=0}) = 0$

Need $\operatorname{Im} \partial_n \longrightarrow \operatorname{Im} \tilde{\partial}_n$ to get well-defined hom

$$H_n(C) = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_n \longrightarrow \operatorname{Ker}\tilde{\partial}_n / \operatorname{Im}\tilde{\partial}_n = H_n(\tilde{C})$$

$\Rightarrow$ Need $h(b) = \tilde{\partial}(\text{something})$.
↑ boundary $b = \partial c$

Proof by "diagram chasing":

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow \cdots \\
& \downarrow & & \downarrow h_n & & \downarrow h_{n-1} & \\
\cdots \longrightarrow & \tilde{C}_{n+1} & \xrightarrow[\tilde{\partial}_{n+1}]{} & \tilde{C}_n & \xrightarrow[\tilde{\partial}_n]{} & \tilde{C}_{n-1} & \longrightarrow \cdots
\end{array}$$

$$\begin{array}{ccc}
c & \xrightarrow{\partial} & \partial c = b \\
h \downarrow & & \downarrow h \\
hc & \xrightarrow{\tilde{\partial}} & \tilde{\partial}(hc) = h\partial c = h(b)
\end{array}$$

□

Def (C_*, ∂_*) is ***exact*** (or ***acyclic***) if $H_*(C) = 0$

so $\operatorname{Im}\partial_{n+1} = \operatorname{Ker}\partial_n$

In general ***exact sequence*** means "Im(previous map) = Ker (next map)"

A ***short exact sequence (SES)*** is an exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

Easy exercise

$$\left(0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0\right) \text{ exact} \iff \begin{cases} i & \text{injective} \\ \pi & \text{surjective} \\ B/i(A) \cong C & \text{via } [b] \longmapsto \pi(b) \end{cases}$$

Examples

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2 \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\text{inclusion}} \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{\text{project}} \mathbb{Z}/2 \longrightarrow 0$$

Note A, C do not determine B.

Snake Lemma A SES of chain complexes and chain maps yields a **long exact sequence (LES) on homology**:

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{\pi_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*[-1]} \cdots$$

δ: degree -1 map $H_*(C) \to H_*(A)[-1]$ called **connecting map**

(So **exact triangle**:

$$\begin{array}{ccc} H_*(A) & \longrightarrow & H_*(B) \\ & {\scriptstyle [-1]}\nwarrow \quad \swarrow & \\ & H_*(C) & \end{array}$$
)

Pf simplify notation by identifying A with $i(A) \subseteq B$: $a \equiv i(a)$ ($a \in A \xhookrightarrow{i} B \ni i(a)$), $\partial a \equiv i \partial a = \partial i a$

$\implies$ now $A_* \subseteq B_*$ inclusion of subcomplex:

$$0 \longrightarrow (A_*, \partial_*) \xrightarrow{\text{incl}} (B_*, \partial_*) \xrightarrow{\pi} (C_*, \tilde{\partial}_*) \longrightarrow 0$$

$$\begin{array}{ccccccccc} 0 & \to & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccc} & & \exists\, b & \xrightarrow{\text{surj.}} & \text{cycle } c = \pi(b) \\ & & \downarrow & & \downarrow \\ \partial b & \longrightarrow & \partial b & \longrightarrow & \tilde{\partial} c = 0 \end{array}$$

$\nwarrow$ lifts to A by exactness

Define $\delta : H_*(C) \longrightarrow H_*(A)[-1]$ (typically b is not in A,

$c \longmapsto \partial b$ so ∂b need not be a bdry in A)

$\nwarrow$ where $b \in \pi^{-1}(c)$

Well-defined? • $\pi^{-1}(c) = \{b + a : a \in A\}$ and $\partial(b+a) = \partial b + \underbrace{\partial a}_{\text{boundary in } A}$ ✓

• cycle $\to$ cycle : $\partial(\partial b) = 0$ ✓

• boundary $\to$ boundary : $\exists \beta \xrightarrow{\text{surj.}} x \in C_{n+1}$

$$\begin{array}{ccc} \beta & \xrightarrow{\text{surj.}} & x \in C_{n+1} \\ \downarrow & & \downarrow \\ \partial\beta & \longrightarrow & \text{boundary } c = \tilde{\partial} x \\ & & \downarrow \\ & & 0 \end{array}$$

$\Rightarrow$ can pick $b = \partial \beta$

$\Rightarrow \partial b = \partial\partial\beta = 0$ ✓

Exactness at $H_n(C)$ (exercise: check exactness at H_*A, H_*B):

Need $\operatorname{Im} \pi_* = \operatorname{Ker} \delta$:

$\subseteq$: $\delta(\pi_* b) = \partial b = 0$ ✓ (b cycle)

$\supseteq$:

$$\begin{array}{ccccc} \exists a & & b & \longrightarrow & c = \pi_* b \;\; (c \in \operatorname{Ker}\delta) \\ \downarrow & & \downarrow & & \downarrow \\ \partial a = \delta c = \partial b & \longrightarrow & \partial b & \longrightarrow & 0 \end{array}$$

(b: not necessarily cycle!)

$\partial a = \delta c$: assumption $\delta c = 0 \in H_*A$

$\pi_*(b-a) = c$ (since $\pi_* A = 0$)

$\partial(b-a) = \partial b - \partial a = 0$, thus $b-a$ cycle!

$\Rightarrow c = \pi_*(b-a) \in \operatorname{Im} \pi_*$ □

Rmk $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ SES $\Rightarrow$ the connecting map of LES is

$$\boxed{\begin{array}{l} \delta : H_*(C) \longrightarrow H_*(A)[1] \\ \qquad c \longmapsto i^{-1}(\partial b) \end{array}}$$

$\forall\, b \in B$ with $\pi(b) = c$.

Lemma The construction of δ is natural (i.e. functorial)

Pf

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{i} & B & \xrightarrow{\pi} & C & \to & 0 \\ & & f\downarrow & & g\downarrow & & h\downarrow & & \\ 0 & \to & \tilde{A} & \xrightarrow{\tilde{i}} & \tilde{B} & \xrightarrow{\tilde{\pi}} & \tilde{C} & \to & 0 \end{array} \qquad \begin{array}{ccc} a = \delta c & \to & \partial b \\ f\downarrow & & \downarrow g \\ fa & \to & g\partial b \end{array} \qquad \begin{array}{ccc} b & \to & c \\ g\downarrow & & \downarrow h \\ gb & \to & hc \end{array}$$

$$\Rightarrow \delta h c = \tilde{i}^{-1} \tilde{\partial} g b = \tilde{i}^{-1} g \partial b = f a = f \delta c$$ □

Exercise Deduce the LES is natural, so

$$\begin{array}{ccccccccc} \cdots \to & H_*A & \xrightarrow{i_*} & H_*B & \xrightarrow{\pi_*} & H_*C & \xrightarrow{\delta} & H_{*-1}(A) & \to \cdots \\ & f_*\downarrow & & g_*\downarrow & & h_*\downarrow & & f_*\downarrow & \\ \cdots \to & H_*\tilde{A} & \longrightarrow & H_*\tilde{B} & \xrightarrow{\tilde{\pi}_*} & H_*\tilde{C} & \xrightarrow{\delta} & H_{*-1}(\tilde{C}) & \to \cdots \end{array}$$

5-Lemma

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \cong\downarrow\alpha & & \cong\downarrow\beta & & \downarrow\gamma & & \cong\downarrow\delta & & \cong\downarrow\varepsilon \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

exact rows $\Longrightarrow$ γ also iso.

Pf exercise (diagram chase) $\square$

Splitting Lemma

Cor $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ SES of abelian gps

If $B \xrightarrow{\beta} C$ s.t. $\beta \circ \gamma = id_C$ (with $\exists\, \gamma : C \to B$) then the SES **splits**: $B \cong A \oplus C$

(converse is obvious)

Pf

$$\begin{array}{ccccccccc} 0 & \to & A & \to & A\oplus C & \to & C & \to & 0 \\ \| & & \| & & \downarrow \alpha+\gamma & & \| & & \| \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

$\square$

Exercise If $A \xrightarrow{\alpha} B$ s.t. $\mu \circ \alpha = id_A$ (with $\exists\, \mu : B \to A$) then it splits: $B \xrightarrow[\mu\oplus\beta]{\cong} A \oplus C$

Exercise If C is a free abelian group ($C \cong \bigoplus_{i\in I} \mathbb{Z}$) then the SES splits.

Rmk A free $\not\Rightarrow$ splits, e.g. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \longrightarrow 0$

Cultural Rmk Splitting Lemma generalises the **rank-nullity theorem** from linear algebra: $V \xrightarrow{\beta} W$ linear map of vector spaces $\Rightarrow \operatorname{Im}\beta \oplus \operatorname{Ker}\beta \cong V$

Pf $0 \to \operatorname{Ker}\beta \xrightarrow{incl.} V \xrightarrow{\beta} \operatorname{Im}\beta \longrightarrow 0$ is SES, and splits since $\operatorname{Im}\beta$ free.

2. Δ-COMPLEXES AND SIMPLICIAL HOMOLOGY

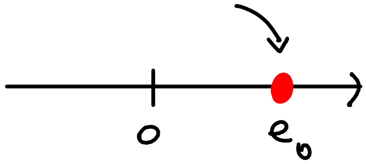
standard n-simplex

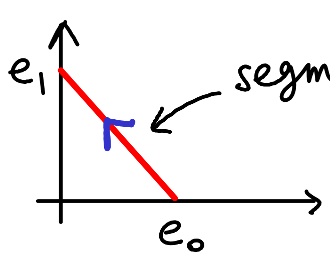
$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \begin{array}{l} t_i \geq 0 \\ \sum t_i = 1 \end{array} \right\}$$

where $(t_0, \dots, t_n) = \sum t_i e_i$ ← standard basis of $\mathbb{R}^{n+1}$: $e_0, \dots, e_n$ $(e_0 = (1,0,\dots,0), \dots$

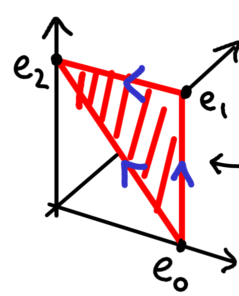
Examples

point $\Delta^0 \subseteq \mathbb{R}$





segment $\Delta^1 \subseteq \mathbb{R}^2$



triangle $\Delta^2 \subseteq \mathbb{R}^3$

Def For $\{v_0, \dots, v_n\} \subseteq \mathbb{R}^{n+k}$ ← any $k \geq 0$ s.t.

$v_1 - v_0, \dots, v_n - v_0$ $\mathbb{R}$-linearly independent

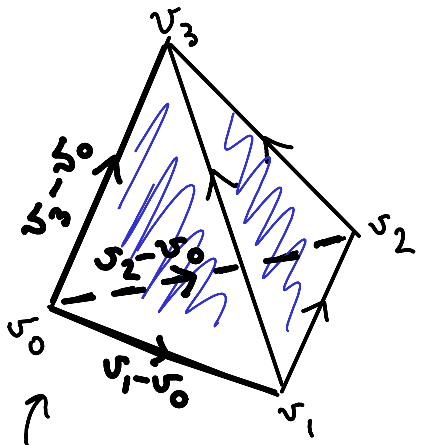
$[v_0, \dots, v_n]$ = **n-Simplex** spanned by $v_0, \dots, v_n$

= convex hull of $v_0, \dots, v_n$

$= \{ \sum t_i v_i \; : \; t_i \geq 0 \text{ and } \sum t_i = 1 \}$

= Image of linear homeo $\sigma : \Delta^n \longrightarrow \mathbb{R}^{n+k}$, $\sigma(e_i) = v_i$

(σ is the **canonical homeomorphism**)



Will often blur the distinction between map σ and its image,

$$\sigma \equiv [\sigma e_0, \dots, \sigma e_n]$$

but the ordering of the v_j will be important (so the map σ is more precise)

We encode this extra data by orienting the edges $v_i \longrightarrow v_j$ if $i < j$

Def **d-dimensional faces** $[v_{i_0}, \dots, v_{i_d}]$ for $i_0 < \dots < i_d$

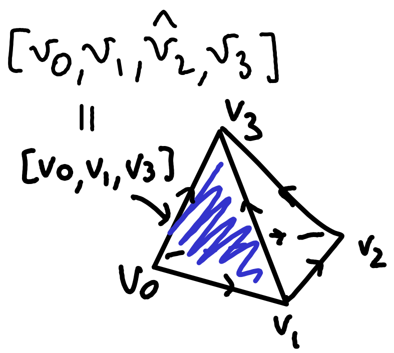
Example: 0-dim faces are the vertices $v_0, \dots, v_n$

facets = $(n-1)$-dimensional faces

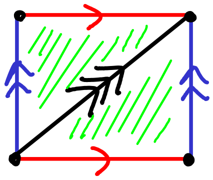
$= [v_0, \dots, \hat{v}_k, \dots, v_n]$ where we omit v_k

$= \{ \sum t_i v_i \in [v_0, \dots, v_n] \; : \; t_k = 0 \}$

$= \text{Image } \sigma|_{\Delta^{n-1}_k} : \Delta^{n-1}_k \longrightarrow \mathbb{R}^{n+k}$, where $\Delta^{n-1}_k = \{ t \in \Delta^n : t_k = 0 \}$



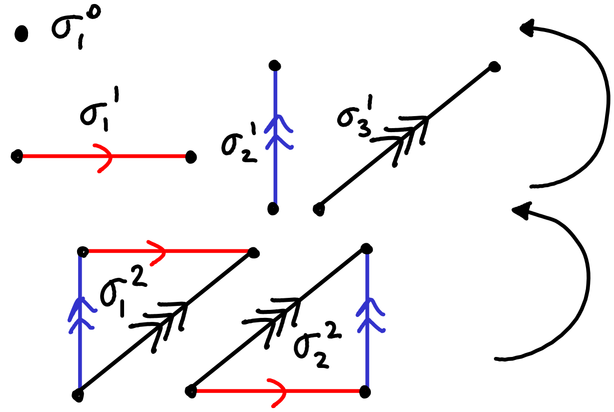
Example Can build a torus out of simplices:

$T^2 =$ 

1 0-simplex $\bullet\ \sigma_1^0$

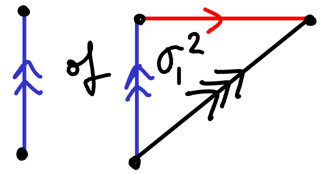
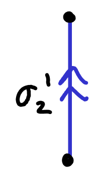
3 1-simplices

2 2-simplices



each facet is associated to another simplex, and we identify them linearly

$T^2 =$ quotient space of $\bigsqcup \sigma_i^n$ / canonical homeos associated to the facets

for example identify facet  with  via linear homeo (orientation-preserving)

Def Δ**-complex** is determined by data

- indexing set I_n, for each $n \in \mathbb{N}$
- choice of n-simplex σ_α^n (not necessarily standard) for each $\alpha \in I_n$
- gluing data: for each $\alpha \in I_n$, $0 \leq i \leq n$ associate some $\beta(\alpha, i) \in I_{n-1}$

The Δ-complex is the quotient space:

$$\bigsqcup_{\substack{\alpha \in I_n \\ n \in \mathbb{N}}} \sigma_\alpha^n \Big/ \begin{array}{l} i\text{-th facet of } \sigma_\alpha^n \text{ is identified with } \sigma^{n-1}_{\beta(\alpha,i)} \\ \text{via the order-preserving canonical linear homeo} \end{array}$$

A **Δ-complex structure** on a top. space X is a homeo from a Δ-complex to X.

Explicit description of the facet identification

$$\{\textstyle\sum s_i w_i\} = [w_0, \dots, w_{n-1}] \longrightarrow [v_0, \dots, v_n] = \{\textstyle\sum t_i v_i\}$$

$$\cup\!\!\!\mid$$

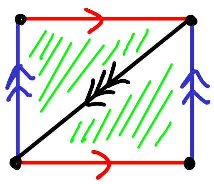
$$\{s_0 v_0 + \dots + s_{i-1} v_{i-1} + s_i v_{i+1} + \dots + s_{n-1} v_n\} = [v_0, \dots, \hat{v}_i, \dots, v_n]$$

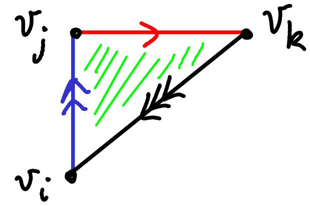
$\sigma^{n-1}_{\beta(\alpha,i)} \uparrow$ $\quad \uparrow \sigma_\alpha^n \quad$ $\sigma_\alpha^n|_{\Delta_i^{n-1}} \nearrow$

$$\Delta^{n-1} \longrightarrow \Delta_i^{n-1} \subseteq \Delta^n$$

$$(s_0, \dots, s_{n-1}) \longmapsto (s_0, \dots, s_{i-1}, 0, s_i, \dots, s_{n-1})$$

Non-example





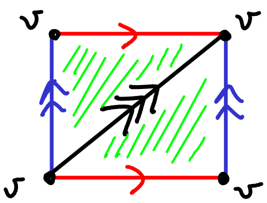
vertices are not totally ordered: $i < j < k < i$

Compare Part A Topology course: (more precisely, the "topological realisation" of a simpl. complex)

<u>Remark</u> A **simplicial complex** is a Δ-complex in which each d-dim face is uniquely determined by d distinct vertices. A homeo from such a complex to X is a **triangulation** of X.

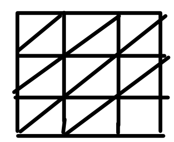
<u>Non-example</u>

$T^2 =$



both 2-simplices have vertices v, v, v

whereas $T^2 =$



is a triangulation.

Simplicial chain complex

<u>Def</u> For a Δ-complex X,

$$C_n^{\Delta}(X) = \text{free abelian group generated by the set } X_n \text{ of } n\text{-simplices of } X$$

$$= \left\{ \sum_{\alpha \in I_n} c_\alpha \cdot \sigma_\alpha^n \; : \; c_\alpha \in \mathbb{Z}, \text{ only finitely many } c_\alpha \neq 0 \right\}$$

$$\boxed{\partial_n \sigma_\alpha = \sum (-1)^i \cdot \sigma_{\beta(\alpha,i)}}$$

so: $$\boxed{\partial_n [v_0, \ldots, v_n] = \sum (-1)^i \cdot [v_0, \ldots, \hat{v}_i, \ldots, v_n]}$$

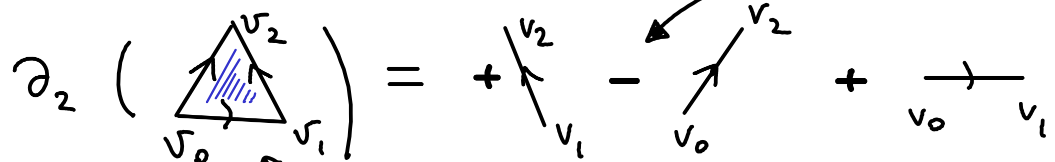
and extend linearly

We will show $\partial \circ \partial = 0$, so get **simplicial homology**:

$$\boxed{H_*^{\Delta}(X) = H_*(C_*^{\Delta}, \partial_*)}$$

Examples

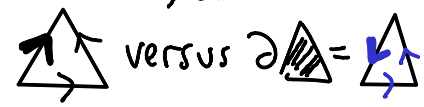
$$\partial_1\left(\overrightarrow{v_0 \quad v_1}\right) = -v_0 \quad +v_1$$



$$\partial_2\circ\partial_1\ (\text{this}) = +(v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0$$

$\partial\circ\partial = 0$ fails for  (not Δ-complex), try!

Later: The $(-1)^i$ signs keep track of whether the orientation agrees/disagrees with geometric boundary orientation, so



Lemma $\partial\circ\partial = 0$

Pf $\partial_{n-1}\left(\partial_n [v_0, \dots, v_n]\right) = \sum (-1)^i \partial_{n-1}[v_0, \dots, \hat{v}_i, \dots, v_n]$

$$= \sum_{j<i} (-1)^i (-1)^j [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$

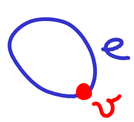
antisymmetric if swap i,j

$$+ \sum_{j>i} (-1)^i \underline{\underline{(-1)^{j-1}}} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$$

$$= 0$$

□

Example

$S^1 =$ 

Δ-complex:

X_0: 1 0-simplex • $e^0_1 = e_{\beta(1,0)} = e_{\beta(1,1)}$

X_1: 1 1-simplex → e^1_1

$$0 \to C_1^\Delta \to C_0^\Delta \to 0$$

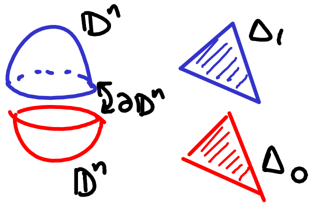
$C_1^\Delta = \mathbb{Z}e$, $C_0^\Delta = \mathbb{Z}v$

$e \longmapsto v - v = 0$

$$H_*^\Delta(S^1) = \begin{cases} \mathbb{Z} & * = 0,1 \\ 0 & \text{else} \end{cases}$$

Example

Δ-cx structure on S^n:



$S^n = \Delta^n \sqcup \Delta^n$ / glue along $\partial\Delta^n$

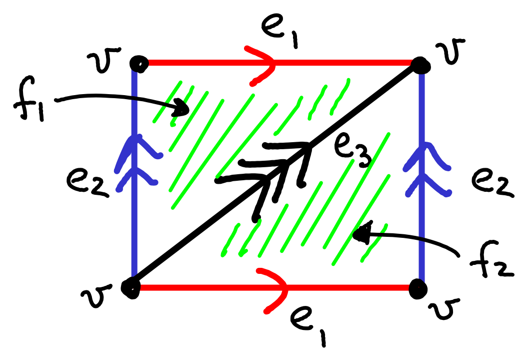
(call this Δ_1; this Δ_0)

One can deduce (but messy!):

$$H_* S^n = \begin{cases} \mathbb{Z}\langle \text{point} \rangle \cong \mathbb{Z} & * = 0 \\ \mathbb{Z}\langle \Delta_1 - \Delta_0 \rangle \cong \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$$

(pick any vertex)

Example

$X = T^2 =$ 

$$0 \to C_2^\Delta \longrightarrow C_1^\Delta \longrightarrow C_0^\Delta \to 0$$

$$C_2^\Delta = \mathbb{Z}f_1 + \mathbb{Z}f_2, \quad C_1^\Delta = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3, \quad C_0^\Delta = \mathbb{Z}v$$

$f_1 \mapsto e_1 - e_3 + e_2$
$f_2 \mapsto e_2 - e_3 + e_1$

$e_1, e_2, e_3 \mapsto v - v = 0$

$$H_*^\Delta(T^2) = \begin{cases} \mathbb{Z}v & * = 0 \\ (\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3)/\mathbb{Z}(e_1 - e_3 + e_2) & * = 1 \\ \mathbb{Z}\cdot(f_1 - f_2) & * = 2 \\ 0 & \text{else} \end{cases}$$

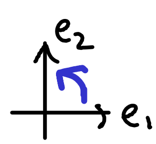
$* = 1 \leftarrow$ freely generated by e_1, e_2

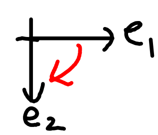
(Smith normal form of ∂_2: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} \xrightarrow{\text{row op.}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{col. op.}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
so after $\mathbb{Z}$-isos of C_2, C_1 we get $\mathbb{Z}^2 \to \mathbb{Z}^3$, $(a,b) \to (a,0,0)$)

$$\cong \begin{cases} \mathbb{Z} & * = 0, 2 \\ \mathbb{Z}^2 & * = 1 \\ 0 & \text{else} \end{cases}$$

Remark about orientations (see also my B3.2 Geometry of Surfaces notes)

For vector space an **orientation** is a choice of basis modulo linear endomorphisms of $\det > 0$

Example $\mathbb{R}^2$ 

right-hand orientation (positive) $\xrightarrow[\det < 0]{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$ 

left-hand orientation (negative)

Fact $GL(n, \mathbb{R})$ has 2 path-components $<$ $A : \det A > 0$ / $A : \det A < 0$, so can always continuously deform a basis to another within same orientation

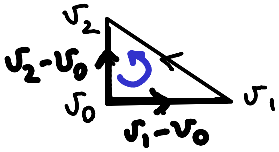
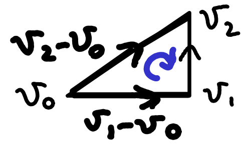
Canonical orientation on $\mathbb{R}^n$: $e_1, \ldots, e_n$ standard basis $\leftarrow$ "positive orientation"

Example $[v_0, \ldots, v_n]$ simplex $\Rightarrow$ $v_1 - v_0, \ldots, v_n - v_0$ is a basis of vector subspace

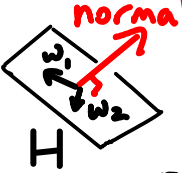
$$V = \{\Sigma a_i v_i : \Sigma a_i = 0\} \subseteq \mathbb{R}^{n+k}$$

hence a choice of orientation of V,
and each transposition of vertices $v_0, \ldots, v_n$ switches the orientation class.

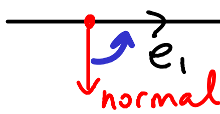
If $v_0, \dots, v_n \in \mathbb{R}^n$ then $V = \mathbb{R}^n$ so simplex's orientation can be compared with $\mathbb{R}^n$-orient^n.

Example In $\mathbb{R}^2$:  positively oriented  negatively oriented

- No canonical choice of orientation for abstract vector space. Need choose basis $v_1 \to v_n$ then declare another basis positively oriented if the change of basis matrix has det > 0.
- For hyperplane $H \subseteq \mathbb{R}^n$ with choice of normal can declare orientation of basis $w_1, \dots, w_{n-1}$ of H positive if normal, $w_1, \dots, w_{n-1}$ is positive $\mathbb{R}^n$-basis



Convention "outward normal first"

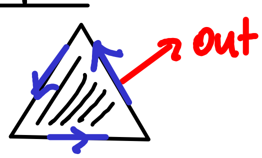
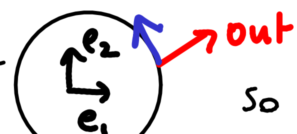
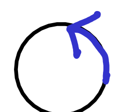
Example  $H \subseteq \mathbb{R}^2 \Rightarrow e_1$ positive basis for H

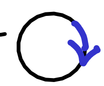
$(\text{normal}, e_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\det = +1 > 0$

Example $\Delta^n \subseteq \mathbb{R}^{n+1}$ with normal $(1,1,\dots,1)$ is positively oriented.

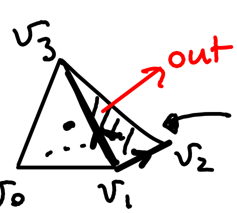
UPSHOT For an n-simplex $[v_0, \dots, v_n]$ in $\mathbb{R}^n$, each facet lies in a hyperplane and have canonical choice of normal: outward normal. Hence facets are canonically oriented.

Example

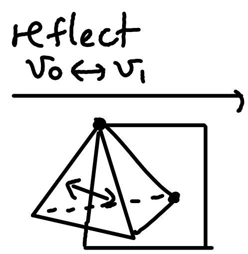
$\mathbb{R}^2 \supseteq$  in smooth world: D^2  so $\partial D^2 = S^1$  anticlockwise

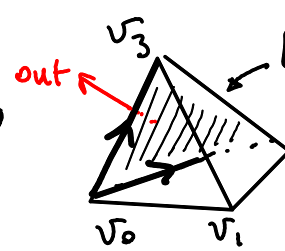
Any reflection of $\mathbb{R}^n$ will swap orientation: after $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ get  clockwise

Example



$[\hat{v}_0, v_1, v_2, v_3]$
out, $v_2 - v_1$, $v_3 - v_1$
positive $\mathbb{R}^3$-basis





$[v_0, \hat{v}_1, v_2, v_3]$
out, $v_2 - v_0$, $v_3 - v_0$
negative $\mathbb{R}^3$-basis

UPSHOT $(-1)^i$ in $(-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k]$ in definition of simplicial ∂ is there to ensure that orientations are consistent (crucial for $\partial \circ \partial = 0$)

Lemma $H_*^\Delta(X) \cong \bigoplus H_*^\Delta(X_i)$ where X_i are the path-components of X.

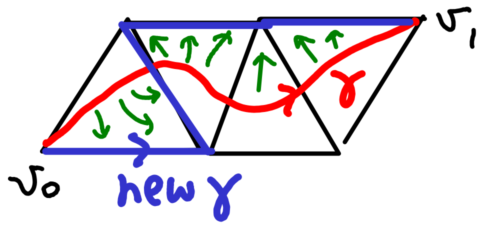
Pf $\bigoplus C_*^\Delta(X_i) \longrightarrow C_*^\Delta(X)$, $\oplus c_i \longmapsto \sum c_i$

is chain isomorphism since any simplex $\sigma: \Delta^k \to X$ has path-connected (since Δ^k path-conn.) image, so $\subseteq X_i$ some i. □

Theorem X has Δ-cx structure $\Rightarrow H_0^\Delta(X) \cong \bigoplus_{\text{path-conn. components}} \mathbb{Z}$

Pf By Lemma, WLOG X path-connected

- vertex $v \Rightarrow v \in C_0(X) \xrightarrow{\partial} C_{-1}(X) \equiv 0 \Rightarrow [v] \in H_0(X)$
- vertices $v_0, v_1 \in X \Rightarrow \exists$ path γ from v_0 to v_1



$\Rightarrow$ can homotope path so that go along edges (continuously deform)

$\Rightarrow \gamma$ is sum of 1-chains s.t. $\partial\gamma = v_1 - v_0$

$\Rightarrow [v] \in H_0(X)$ independent of choice of v

$\Rightarrow H_0(X) = \langle [v] \rangle$

- $H_0(X) = \langle [v] \rangle \leftarrow \mathbb{Z}$ is injective?

  $nv \leftarrow n$ Suppose $nv = \partial c$ some $c \in C_1(X)$

  consider the augmentation hom

$$C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

$$\sum n_i \sigma_i \longmapsto \sum n_i$$

(σ_i: 0-simplices)

notice composite is 0 since $\partial\left(\text{1-simplex } \sigma_0 \to \sigma_1\right) = \sigma_1 - \sigma_0 \xrightarrow{\varepsilon} 1 - 1 = 0$

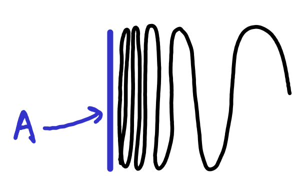
$\Rightarrow n = \varepsilon(nv) = \varepsilon\partial c = 0$. □

Rmk X top. space $\Rightarrow$ path conn. component $\subseteq$ connected component since path-conn. $\Rightarrow$ connected. For Δ-cx, these are same (since connected + locally path-conn. $\Rightarrow$ path-connected).

However for top. space they need not be the same, e.g. Topologist's sine curve

$\{(x, \sin\frac{1}{x}) : x \in (0,1]\} \cup \underbrace{0 \times [0,1]}_{A} \subseteq \mathbb{R}^2$

2 path-conn. components

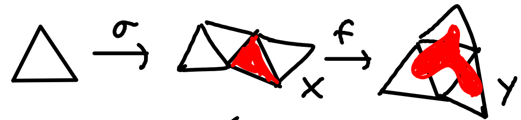


- connected
- not path-connected
- not locally path-connected

3. SINGULAR HOMOLOGY

Motivation Not obvious that H_*^Δ is functorial: $\Delta^k \xrightarrow{\sigma} X \xrightarrow{f} Y$ ← continuous map

then $f \circ \sigma$ typically not a simplex:



Solution 1: only allow **simplicial maps** $f: X \to Y$ (so $f \circ \sigma$ simplex $\forall \sigma$)

Solution 2: show that any cts map $f: X \to Y$ can be approximated arbitrarily well by a simplicial map after having performed barycentric subdivision on X, Y enough times. Also any two such approximations induce the same map $H_*^\Delta(X) \to H_*^\Delta(Y)$

Solution 3: develop new homology $H_*(X)$ which allows any cts map $\Delta^n \to X$ and prove $H_*^\Delta(X) \cong H_*(X)$ for Δ-complexes X.

↑ WILL DO THIS.

Def **Singular n-simplex** is any continuous map $\sigma: \Delta^n \longrightarrow X$ (X is any top. space)

Singular n-chains $C_n(X)$ = free abelian group generated by σ

$$= \left\{ \sum_{\substack{\text{singular} \\ n\text{-simplices } \sigma}} c_\sigma \cdot \sigma \; : \; \begin{array}{l} c_\sigma \in \mathbb{Z} \\ \text{only finitely many } c_\sigma \neq 0 \end{array} \right\}$$

$$\boxed{\partial_n \sigma = \sum (-1)^i \cdot \sigma|_{\Delta_i^n}}$$ ← i-th facet (and extend linearly)

We will show $\partial \circ \partial = 0$, so get **singular homology**:

$$\boxed{H_*(X) = H_*(C_*, \partial_*)}$$

For Δ-complex X have inclusion of subcomplex $C_*^\Delta \to C_*$

$\Longrightarrow$ induces $H_*^\Delta(X) \longrightarrow H_*(X)$ **Fact**: isomorphism (proof later, see cellular $H_*^{CW} \cong H_*$)

Corollary $H_*^\Delta(X)$ is independent of choice of Δ-cx structure on X

Lemma $\partial \circ \partial = 0$

Proof $\partial_{n+1}(\partial_n \sigma) = \partial_{n+1}\left(\sum (-1)^i \sigma|_{\Delta_i^{n-1}} \right)$ ← $[e_0, \ldots, \hat{e}_i, \ldots, e_n]$

$$= \sum_{j<i} (-1)^i (-1)^j \sigma|_{[e_0, \ldots, \hat{e}_j, \ldots, \hat{e}_i, \ldots, e_n]}$$

$$+ \sum_{j>i} (-1)^i \underline{\underline{(-1)^{j-1}}} \sigma|_{[e_0, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n]}$$

$= 0$ □

Example $X = \text{point} \implies C_n(X) = \mathbb{Z}\cdot(\sigma_n : \Delta^n \to X$ the constant map$)$

$$\partial \sigma_n = \sum (-1)^i \sigma_n|_{\Delta_i^{n-1}} = \underbrace{\sum (-1)^i}_{\begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd}\end{cases}} \sigma_{n-1} \implies \cdots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \underset{C_1}{\mathbb{Z}} \xrightarrow{0} \underset{C_0}{\mathbb{Z}} \to 0$$

$$\implies H_*(pt) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else}\end{cases}$$

Lemma $\boxed{H_*(X) \cong \bigoplus H_*(X_i)}$ where X_i are path-components of X

Pf Image of cts map $\Delta^n \to X$ is path conn. so lies in some X_i. □

Cor $\boxed{H_0(X) = \bigoplus_{X_i} \mathbb{Z}}$

Pf By Lemma, WLOG X path-connected. $\Delta^0 = pt \to X$ is cycle since $C_{-1}(X) = \phi$ (← generators of $C_0(X)$)
Given 2 points $x, y \in X$, a path $\Delta^1 = [0,1] \xrightarrow{\gamma} X$, $\gamma(0) = x$, $\gamma(1) = y$ is also a 1-chain!
So $x - y = \partial\gamma$, so x, y are homologous. Finally if $n \cdot [x] = 0 \in H_0(X)$ then $nx = \partial c$ some $c \in C_1(X) =$ generated by paths. Now run the augmentation hom. trick like we did for H_0^{Δ}: $n = \varepsilon(nx) = \varepsilon\partial c = 0$ as $\varepsilon \circ \partial = 0$. □

Naturality (i.e. functoriality)

Lemma $f : X \to Y$ continuous
$\implies H_*(f) = f_* : H_*(X) \to H_*(Y)$ induced by chain map
$f_* : C_*(X) \to C_*(Y)$

induced map (↗ $H_*(f)$)

$\boxed{f_*(\sigma) = f \circ \sigma}$ and extend linearly

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\sigma} & X \\ & \searrow^{f_*\sigma} & \downarrow f \\ & & Y \end{array}$$

Pf $\partial_n(f_*\sigma) = \sum (-1)^n f \circ \sigma|_{\Delta_i^{n-1}} = f_*\left(\sum (-1)^n \sigma|_{\Delta_i^{n-1}}\right) = f_*(\partial_n \sigma)$ □

Properties 1) $X \xrightarrow{f} Y \xrightarrow{g} Z \implies (g \circ f)_* = g_* \circ f_*$
2) $id_* = id$

Pf 1) $(g \circ f)_* \sigma = g \circ f \circ \sigma = g_*(f \circ \sigma) = g_*(f_* \sigma)$ ✓
2) $id_* \sigma = id \circ \sigma = \sigma$ ✓ □

Cor $H_* : \begin{Bmatrix} \text{topological spaces} \\ \& \\ \text{cts maps} \end{Bmatrix} \to \begin{Bmatrix} \text{graded abelian groups} \\ \& \\ \text{graded homs} \end{Bmatrix}$ is a functor

Cor $X \cong Y \implies H_*(X) \cong H_*(Y)$

4. CHAIN HOMOTOPIES AND HOMOTOPY INVARIANCE

Algebra: chain homotopies

$f_*, g_* : (C_*, \partial_*) \longrightarrow (\tilde{C}_*, \tilde{\partial}_*)$ chain maps

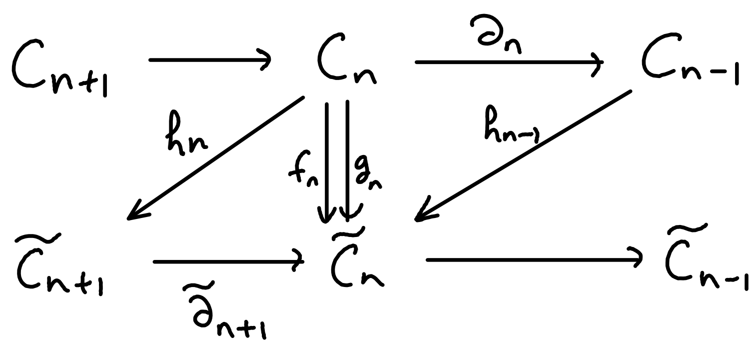
Def f_*, g_* are **chain homotopic** if $\exists$ (degree +1) hom $h : C_* \longrightarrow \tilde{C}_*[1]$ s.t.

$$\boxed{\tilde{\partial} \circ h + h \circ \partial = f - g}$$

h is called a **chain homotopy**

Consequence $f_* = g_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$ on homology

Pf



c cycle $\in C_n$

$$\Longrightarrow \quad f_n(c) - g_n(c) = \underbrace{\tilde{\partial}_{n+1} \circ h_n(c)}_{\text{boundary}} + h_{n-1} \circ \underbrace{\partial_n(c)}_{=0}$$

$$\Longrightarrow \quad f_n(c) = g_n(c) \text{ in } H_*(\tilde{C})$$ $\square$

Theorem $i_0 : X \longrightarrow X \times I$, $i_0(x) = (x, 0)$ where $I = [0,1]$

$i_1 : X \longrightarrow X \times I$, $i_1(x) = (0, x)$

$\Longrightarrow i_{0*}, i_{1*} : C_*(X) \to C_*(X \times I)$ are chain hpic.

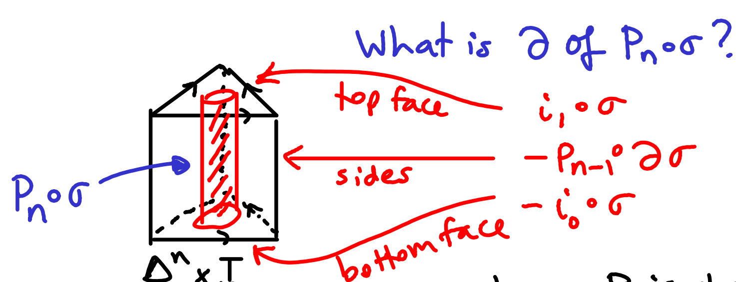
Key idea Need "prism operator" which cuts $\Delta^n \times I$ into a sum Γ_n of $(n+1)$-simplices in $\Delta^n \times I$:

$(\sigma : \Delta^n \to X) \mapsto \sigma \times id : \Delta^n \times I \to X \times I$

$\uparrow \Gamma_n$

Δ^{n+1}

prism operator P_n: $\sigma \mapsto (\sigma \times id) \circ \Gamma_n : \Delta^{n+1} \to X \times I$



hence P is chain hpy

Pf ← Non-examinable

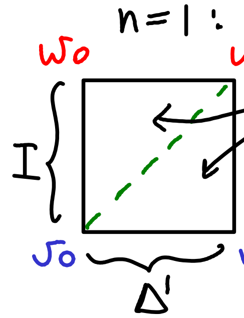
bottom facet $\Delta^n \times 0 = [v_0, \dots, v_n]$ ← $v_i = e_i \times 0$

top facet $\Delta^n \times 1 = [w_0, \dots, w_n]$ ← $w_i = e_i \times 1$

$\subseteq \Delta^n \times [0,1] \subseteq \mathbb{R}^{n+1}$

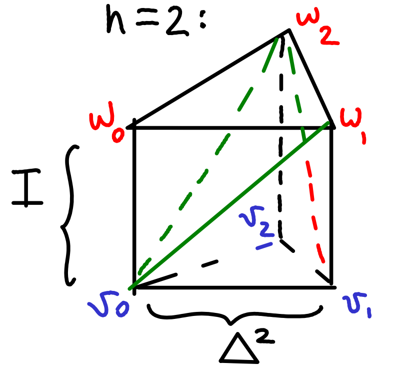
Examples

$n=1$:



$[v_0, v_1, w_1]$
$[v_0, w_0, w_1]$

$n=2$:



three 3-simplices:

$[v_0, v_1, v_2, w_2]$
$[v_0, v_1, w_1, w_2]$
$[v_0, w_0, w_1, w_2]$

Let $s_i = [v_0, \dots, v_i, w_i, \dots, w_n]$

Claim The s_i cover $\Delta \times [0,1]$ and give Δ-cx structure on $\Delta^n \times I$

Pf $\sum_{k \le i} t_k v_k + \sum_{k \ge i} s_k w_k = (t_0, \dots, t_{i-1}, \underline{t_i + s_i}, s_{i+1}, \dots, s_n, s_i + \dots + s_n)$

So given $(x_0, \dots, x_n, a) \in \Delta^n \times I$, equate and solve:

$t_0 = x_0, \dots, t_{i-1} = x_{i-1}$, $s_{i+1} = x_{i+1}, \dots, s_n = x_n$, and $\begin{cases} s_i = a - x_{i+1} - \dots - x_n \\ t_i = x_i - s_i \end{cases}$

Note $x_k \geq 0$, $\sum x_k = 1$, $a \in [0,1]$ hence $\sum t_k + \sum s_k = 1$ ✓ $\begin{cases} t_k \geq 0 \text{ for } k < i \\ s_k \geq 0 \text{ for } k > i \end{cases}$ ✓

but $\left.\begin{matrix} s_i \geq 0 \\ t_i \geq 0 \end{matrix}\right\} \Leftrightarrow \begin{cases} a \geq x_{i+1} + \dots + x_n \\ x_i + x_{i+1} + \dots + x_n \geq a \end{cases}$. Thus a solution exists if we pick $i = \min\{k : a \geq x_{i+1} + \dots + x_n\}$

There are multiple solutions if $x_{i+1} = x_{i+2} = \dots = x_j = 0$, but that is as expected: those points of $\Delta^n \times I$ belong to the faces of $s_i, s_{i+1}, \dots, s_j$. □

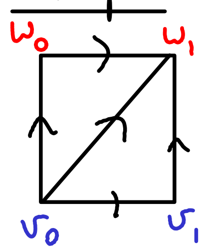
Def

$\Gamma_n = \sum (-1)^i s_i \in C_{n+1}(\Delta^n \times [0,1])$ ← geometrically this "represents" $\Delta^n \times I$ as a simplicial chain

$$\Rightarrow \partial \Gamma_n = \sum_i \sum_{j \le i} (-1)^i (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n] + \sum_i \sum_{j \ge i} (-1)^i (-1)^{j+1} [v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]$$

geometrically this "represents" $\partial(\Delta^n \times I) = (\partial\Delta^n \times I) \sqcup (\Delta^n \times \partial I)$

Example



$\Gamma_1 = [v_0, w_0, w_1] - [v_0, v_1, w_1]$ "is the square"

$\partial \Gamma_1 = [w_0, w_1] - \cancel{[v_0, w_1]} + [v_0, w_0] - [v_1, w_1] + \cancel{[v_0, w_1]} - [v_0, v_1]$

"is ∂ of square" "inside facets" cancel

Prism operator

$P : C_n(X) \longrightarrow C_n(X \times [0,1])$

$$\boxed{P(\sigma) = (\sigma \times id)_*(\Gamma_n)}$$

$\sigma : \Delta^n \to X$

$\sigma \times id : \Delta^n \times [0,1] \to X \times [0,1]$

$(\sigma \times id)(x,t) = (\sigma(x), t)$

$$\partial P(\sigma) = \partial (\sigma \times id)_*(\Gamma_n) = (\sigma \times id)_*(\partial \Gamma_n)$$

$$= \sum_i \sum_{j \le i} (-1)^i (-1)^j [i_0\sigma e_0, \ldots, \widehat{i_0\sigma e_j}, \ldots, i_0\sigma e_i, i_1\sigma e_i, \ldots, i_1\sigma e_n]$$

$$+ \sum_i \sum_{j \ge i} (-1)^i (-1)^{j+1} [i_0\sigma e_0, \ldots, i_0\sigma e_i, i_1\sigma e_i, \ldots, \widehat{i_1\sigma e_j}, \ldots, i_1\sigma e_n]$$

$$= i_{1*}\sigma - i_{0*}\sigma - P\partial\sigma$$

(★)

$(\sigma \times id)(v_i)$
$= (\sigma \times id)(e_i, 0)$
$= (\sigma(e_i), 0)$
$= i_0(\sigma)(e_i)$

$(\sigma \times id)(w_i)$
$= (\sigma(e_i), 1)$
$= i_1(\sigma)(e_i)$

$\sigma = [\sigma e_0, \ldots, \sigma e_n]$

$i_{1*}\sigma$: $i = j = 0$, 1st sum

$i_{0*}\sigma$: $i = j = n$, 2nd sum

$P\partial\sigma$: $((\partial\sigma) \times id)_* \Gamma_{n-1}$, where $\Gamma_{n-1} = \sum_i (-1)^i [v_0, \ldots, v_i, w_i, \ldots, w_{n-1}]$

now use (★) and

$\partial\sigma = \sum (-1)^k [\sigma e_0, \ldots, \widehat{\sigma e_k}, \ldots, \sigma e_n]$. □

Homotopy invariance

$f_0, f_1 : X \longrightarrow Y$

Def $f_0 \simeq f_1$ **homotopic** if $\exists$ continuous map

$$F : X \times [0,1] \longrightarrow Y$$ ← called **homotopy**

s.t. $f_0 = F \circ i_0$
$f_1 = F \circ i_1$.

Idea Think of this as a continuous family of maps

$f_t = F(\cdot, t) : X \to Y$ from f_0 to f_1.

Exercise $\simeq$ is an equivalence relation.

Homotopic **relative to $A \subseteq X$** if $F(a,t) = f_0(a) = f_1(a)$ all $a \in A$, all t.
write "$f \simeq g$ **rel A**"

Def $X \simeq Y$ **homotopy equivalent** spaces if $\exists$ maps

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \quad \text{with} \quad \begin{array}{l} g\circ f \simeq id \\ f \circ g \simeq id \end{array}$$

Rmk homeo $\Rightarrow$ hpy equivalent

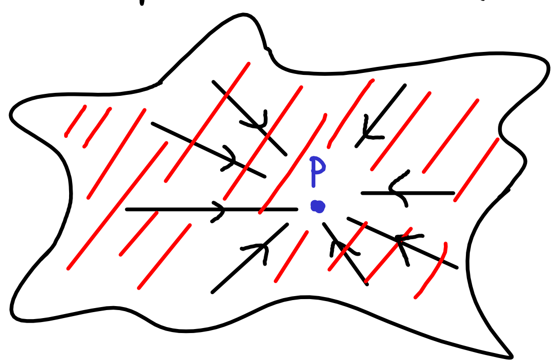
Def X **contractible** if $X \simeq pt$

equivalently $(X \xrightarrow{id} X) \simeq (X \xrightarrow{const} \text{point} \in X)$

Example • $\mathbb{R}^n \simeq pt$

$F(x,t) = tx$ then $f_0 \equiv 0$, $f_1 = id$.

• (star-shaped subsets of $\mathbb{R}^n$) $\simeq pt$



$\leftarrow$ contains line segments to a specific point p

WLOG $p=0$ & use same F
↑
translate

e.g. disc D^n

Theorem $f_0 \simeq f_1 \Longrightarrow f_{0*} = f_{1*} : H_* X \to H_* Y$

Pf

$$\begin{aligned} f_{1*} - f_{0*} &= F_* i_{1*} - F_* i_{0*} \\ &= F_* (i_{1*} - i_{0*}) \\ &= F_* (\partial P + P \partial) \\ &= \partial \circ (F_* \circ P) + (F_* \circ P) \circ \partial \end{aligned}$$

(where F = homotopy, i_0, i_1 as in previous Thm)

(the third "=" is by the previous Thm; the fourth since F_* is a chain map)

$\Longrightarrow F_* \circ P$ is chain hpy from f_{0*} to f_{1*} $\square$

Cor $X \simeq Y \Longrightarrow H_* X \cong H_* Y$

Pf $f_* g_* = id_*$, $g_* f_* = id_*$ $\square$

Example X contractible $\Longrightarrow H_* X \cong H_*(pt) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Relative homology

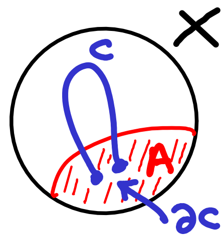
Def (X, A) **pair of spaces** if $A \subseteq X$ topological subspace

$\Rightarrow$ $i = \text{incl} : A \hookrightarrow X$ induces a subcx $i_* : C_* A \to C_* X$

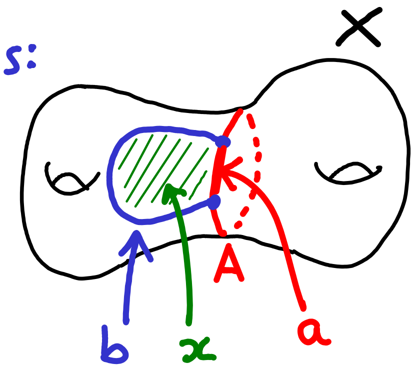
$\Rightarrow$ C_*X / C_*A quotient chain cx (recall $\partial[x] = [\partial x]$)

$$H_*(X, A) = H_*(C_*X / C_*A)$$

Idea: **relative cycles**:
$c \in C_* X$
s.t. $\partial c \in C_* A$

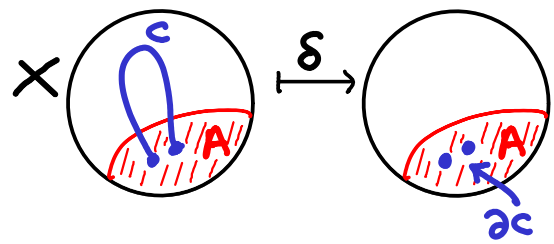


relative boundaries:
$b \in C_* X$
s.t. $\exists\, x \in C_{*+1} X$
$\partial x = b + a$ $\in C_* A$



$\Rightarrow$ $0 \to C_*A \to C_*X \to C_*X / C_*A \to 0$ SES

Cor
$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} \cdots$$
↖ **LES for the pair**



$\delta c = \partial c$
$\Uparrow$
$C_* A$
↗ Need not be ∂a some $a \in C_* A$

Reduced homology

$$\tilde{H}_* X = \text{Ker}\; H_* X \longrightarrow H_*(pt)$$ induced by map $X \to pt$

equivalently H_* of **augmented chain complex**

$$\cdots \to C_n(X) \to \cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

can view $C_{-1}(X) = \mathbb{Z} \cdot (\text{map } \emptyset \to X)$ where allow the empty simplex $\emptyset$

augmentation $\to$ $\varepsilon(\sum n_i \cdot p_i) = \sum n_i$
$n_i \in \mathbb{Z}$, p_i points $\in X$

Example $\tilde{H}_*(\mathbb{R}^n) \cong \tilde{H}_*(D^n) \cong \tilde{H}_*(pt) = 0$

Check
- $H_* \tilde{X} = H_* X$ $* \neq 0$, and $H_0 \tilde{X} \cong H_0 X \oplus \mathbb{Z}$
- $f : X \to Y \Rightarrow f_* : \tilde{H}_* X \longrightarrow \tilde{H}_* Y$

Lemma (X, A) pair $\Rightarrow \exists$ LES

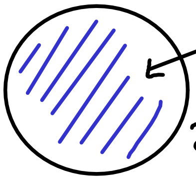
$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \to H_n(X, A) \to \tilde{H}_{n-1}(A) \xrightarrow{i_*} \cdots \to H_0(X, A) \to 0$$

Pf use augmented ch. cx. and $\mathbb{Z}/\mathbb{Z} = 0$ □

Cor $H_*(X, pt) \cong \tilde{H}_*(X)$

Pf $\tilde{H}_*(pt) = 0$. □

Example LES: $\tilde{H}_*(S^{n-1}) \longrightarrow \tilde{H}_*(\mathbb{D}^n) = 0$, $\tilde{H}_*(\mathbb{D}^n) \to H_*(\mathbb{D}^n, S^{n-1})$, $H_*(\mathbb{D}^n, S^{n-1}) \xrightarrow{[-1]} \tilde{H}_*(S^{n-1})$



$$\Longrightarrow \boxed{H_*(\mathbb{D}^n, S^{n-1}) \cong \tilde{H}_{*-1}(S^{n-1})}$$

Naturality of the LES for pairs

Def A <u>map of pairs of spaces</u> $(X,A) \xrightarrow{f} (Y,B)$ means $f: X \to Y$ and $f(A) \subseteq B$.

Lemma

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_* A & \longrightarrow & H_* X & \longrightarrow & H_*(X,A) & \longrightarrow & H_{*-1} A & \to \cdots \\
& \downarrow f_* & & \downarrow f_* & & \downarrow & & \downarrow f_* & \\
\cdots \longrightarrow & H_* B & \longrightarrow & H_* Y & \longrightarrow & H_*(Y,B) & \longrightarrow & H_{*-1} B & \to \cdots
\end{array}$$

Pf

$$\begin{array}{ccccccccc}
0 \to & C_* A & \to & C_* X & \to & C_* X / C_* A & \to 0 \\
& \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\
0 \to & C_* B & \to & C_* Y & \to & C_* Y / C_* B & \to 0
\end{array}$$

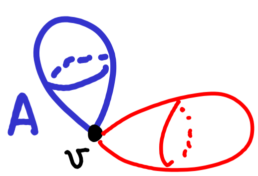
$\Longrightarrow$ claim follows by naturality of LES induced by SESs of chain cxes. □

5. EXCISION THEOREM AND QUOTIENTS

(X, A) pair

Def $r: X \to X$ <u>retraction</u> onto A if $\begin{cases} r(X) = A \\ r|_A = id_A \end{cases}$

Example



$X = \underbrace{S^2}_{=A} \vee S^2$ = two spheres glued at one point v (<u>wedge sum</u>)

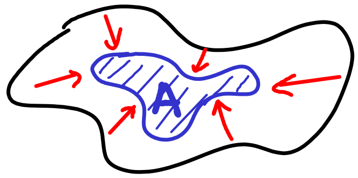
$r: X \to A$ map second sphere to v

Example In Pf of Brower fixed pt thm we built a retraction r by contradiction

Cor r retraction $\Longrightarrow$ $r_*: H_* X \to H_* A$ surjective
$incl_*: H_* A \to H_* X$ injective

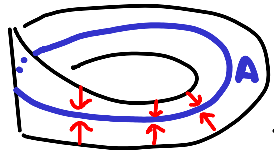
Pf $A \xrightarrow{incl} X \xrightarrow{r} A$, composite id_A; now use H_* functorial □

Def $r : X \to X$ <u>deformation retraction</u> onto A if $\begin{cases} r \text{ retraction} \\ r \simeq id_X \ \text{ rel } A \end{cases}$



X

Example X = Möbius strip
A = equator



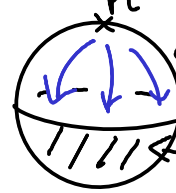
X

Lemma r def. retr. $\Rightarrow$ • $A \xrightarrow[\simeq]{incl} X$ is a homotopy equivalence.

• $incl_*$ and r_* are isos on H_*, so $H_*A \cong H_*X$

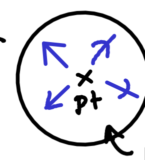
Pf $A \underset{r}{\overset{incl}{\rightleftarrows}} X$ $\quad incl \circ r = r \simeq id_X$, $\quad r \circ incl = r|_A = id_A$ $\square$

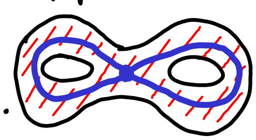
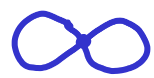
Example $S^n \setminus pt$ def. retracts to $D^n \cong$ lower hemisphere



$\Rightarrow S^n \setminus pt \simeq D^n$

$\Rightarrow S^n \setminus \{2 \text{ points}\} \simeq D^n \setminus pt \simeq S^{n-1}$



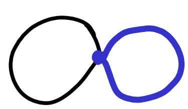
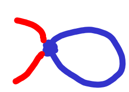
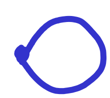
$\Rightarrow S^n \setminus \{3 \text{ points}\} \underset{\text{def. ret.}}{\simeq}$  $\underset{\text{ret.}}{\simeq}$  $S^{n-1} \vee S^{n-1}$

Excision theorem

$E \subseteq A \subseteq X$ subspaces with $\overline{E} \subseteq A^\circ$ $\Longrightarrow$ $(X \setminus E, A \setminus E) \longrightarrow (X, A)$ induces iso

$$\boxed{H_*(X \setminus E, A \setminus E) \xrightarrow{\cong} H_*(X, A)}$$

Proof Later.

Example $X = S^1 \vee S^1 =$  $\supseteq A =$  $\supseteq E =$  $\cong S^1$

$\Rightarrow H_*(X, A) \underset{\uparrow \text{exc. thm.}}{\cong} H_*($, $) \underset{\uparrow \text{hpy invce}}{\cong} H_*(D^1, \partial D^1) \cong \widetilde{H}_0(S^0) \cong \mathbb{Z}$ (S^0 = 2 points)

Rephrasing of Excision Thm

$X = A^\circ \cup B^\circ$ $\Longrightarrow$ $\boxed{H_*(X, A) \cong H_*(B, A \cap B)}$ induced by inclusion $(X, A) \longleftarrow (B, A \cap B)$

($A, B \subseteq X$ subspaces)

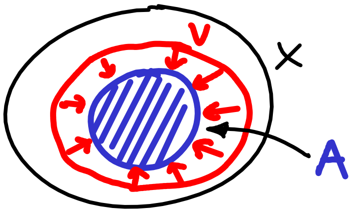
Pf Take $B = X \setminus E$ so $A \cap B = A \setminus E$. $\square$

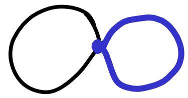
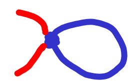
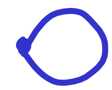
<u>Idea</u> why excision holds: $C_*(A) + C_*(B) \longrightarrow C_*(X)$ is a homotopy equivalence and $C_*(A) \cap C_*(B) = C_*(A \cap B)$. Idea: can subdivide chains in X many times, and small enough chains lie either in A or in B (or in both).

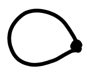
Good pairs and quotients

(X, A) pair

- **Quotient** $X/A = X/\sim$ ← equiv. relation $x \sim y \iff x = y$ or $x, y \in A$
- (X, A) **good pair** if $\begin{cases} A \neq \emptyset \\ A \text{ closed} \\ A \text{ deformation retract of nbhd } V \text{ of } A \end{cases}$

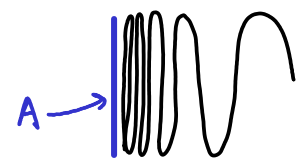


Example $X = S^1 \vee S^1 =$  $\supseteq V =$  $\supseteq A =$  $\cong S^1$

$X/A \cong$  ← all points of A are identified with the node

Non-example **Topologist's sine curve**

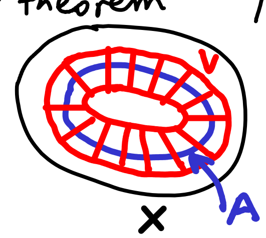
$\{(x, \sin\frac{1}{x}) : x \in (0,1]\} \cup \underbrace{0 \times [0,1]}_{A} \subseteq \mathbb{R}^2$



connected
not path-connected
not locally connected
not locally path-connected

Cultural Rmk

Smooth submanifold $\subseteq$ smooth manifold is a good pair (tubular neighbourhood theorem)



Cor (X, A) good $\Rightarrow (X, A) \longrightarrow (X/A, pt)$ induces iso

$$\boxed{H_*(X, A) \longrightarrow H_*(X/A, pt) = \widetilde{H}_*(X/A)}$$

Pf good $\Rightarrow \exists$ nbhd V of A, and $A \xrightarrow[\text{incl}]{\simeq} V$.

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{\cong} & H_n(X, V) & \xleftarrow{\cong} & H_n(X \setminus A, V \setminus A) \\ \downarrow \text{quot.} & & \downarrow \text{quot.} & & = \downarrow id_* = \text{identity} \\ H_n(X/A, \underbrace{A/A}) & \xrightarrow{\cong} & H_n(X/A, V/A) & \xleftarrow{\cong} & H_n(X/A \setminus p, V/A \setminus p) \end{array}$$

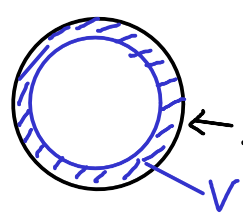
LES for pairs & 5-Lemma since $A \simeq V$, $A/A \simeq V/A$

excision

call this point p

Hence all arrows are isos. □

Example $\mathbb{D}^n \supseteq S^{n-1}$ good:



quotient ⟶

points of A identified

$$\Longrightarrow H_*(\mathbb{D}^n, S^{n-1}) \underset{\text{Cor}}{\cong} \widetilde{H}_*(\mathbb{D}^n/S^{n-1}) \cong \widetilde{H}_*(S^n)$$

$\mathbb{D}^n/S^{n-1} \cong S^n$

Recall we proved $\widetilde{H}_*(D^n, S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})$ (from LES & $\widetilde{H}_*(D^n)=0$)

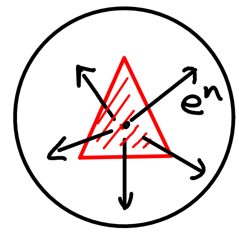
$\Longrightarrow$ inductively, using Example: $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1}) \cong \dots \cong \widetilde{H}_{k-n}(S^0) = \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{else} \end{cases}$

S^0 = 2 points; $H_0(2\text{pts}) = \mathbb{Z}\oplus\mathbb{Z}$

Generator of $H_n(S^n) \cong \widetilde{H}_n(D^n/S^{n-1}) \cong H_n(D^n, S^{n-1}) \cong \mathbb{Z}$

Observe $\exists$ homeo $e^n : \Delta^n \cong D^n$ (homework) inducing Δ-cx structure on S^{n-1}:

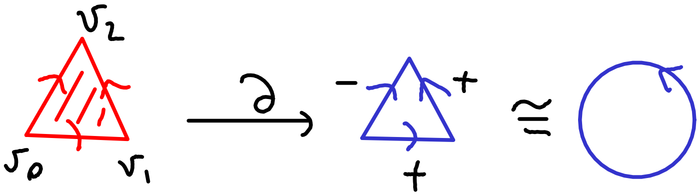
$$\partial\Delta^n \cong \partial D^n = S^{n-1}$$



stretch ctsly outwards from barycentre(Δ^n)

Example

$D^2 \cong$



S^1

Upshot ($n \geq 2$)

$$\begin{aligned} H_n(D^n, S^{n-1}) &= \mathbb{Z}\cdot e^n \\ H_{n-1}(S^{n-1}) &= \mathbb{Z}\cdot \partial e^n \\ H_n(D^n/S^{n-1}) &= \mathbb{Z}\cdot [e^n] \end{aligned}$$

LES; for $n-1 \geq 1$, so $n \geq 2$; by Cor

Exercise Recall another Δ-cx structure on S^n:

$$S^n = \Delta^n \sqcup \Delta^n / \text{glue along } \partial\Delta^n$$

call this Δ_1, this Δ_0

then $H_n S^n = \mathbb{Z}\cdot(\Delta_1 - \Delta_0)$ and $H_n(S^n, \Delta_0) \overset{\text{exc.}}{\cong} H_n(\Delta_1, \partial\Delta_1) \cong H_n(D^n, \partial D^n)$

$\Delta_1 - \Delta_0 \to \Delta_1$

Another remark about orientations

Fact {homeos $\Delta^n \to D^n$} has 2 path-components

Above we chose a path-component by constructing e^n.

If r is any reflection in $\mathbb{R}^{n+1}$ then $e^n \circ r$ is in the other path-component

(e.g. swap 2 coordinates in Δ^n)

$$H_n(S^n) \cong H_n(D^n, S^{n-1}) \xrightarrow{\cong} \mathbb{Z}$$
$$e^n \longmapsto +1$$
$$e^n \circ r \longmapsto -1$$

We will see later in the course that this corresponds to a choice of <u>orientation</u> of $\mathbb{D}^n$ and S^n.

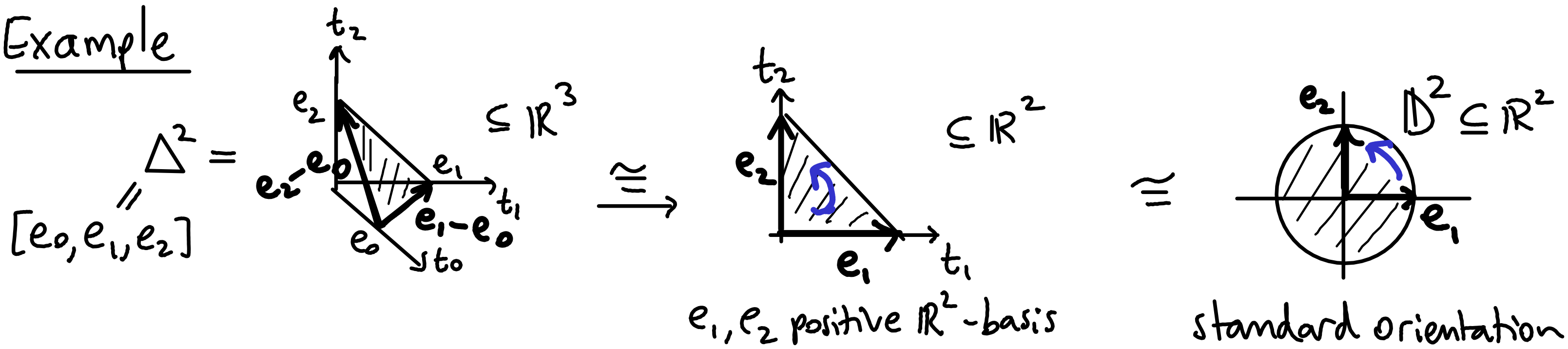
Our choice is consistent with the inclusion $\mathbb{D}^n \subseteq \mathbb{R}^n$ (with the positive (canonical) orientation of $\mathbb{R}^n$) and the inclusion

$$(\Delta^n \subseteq \underline{\underline{\mathbb{R}^{n+1}}}) \xrightarrow{\cong} \{(t_1, \dots, t_n) \in \underline{\underline{\mathbb{R}^n}} : t_i \geq 0, \sum_{i=1}^n t_i \leq 1\}$$

$$(\underline{\underline{t_0}}, \dots, t_n) \longmapsto (\underline{\underline{t_1}}, \dots, t_n)$$

$$t_i \geq 0, \sum t_i = 1$$

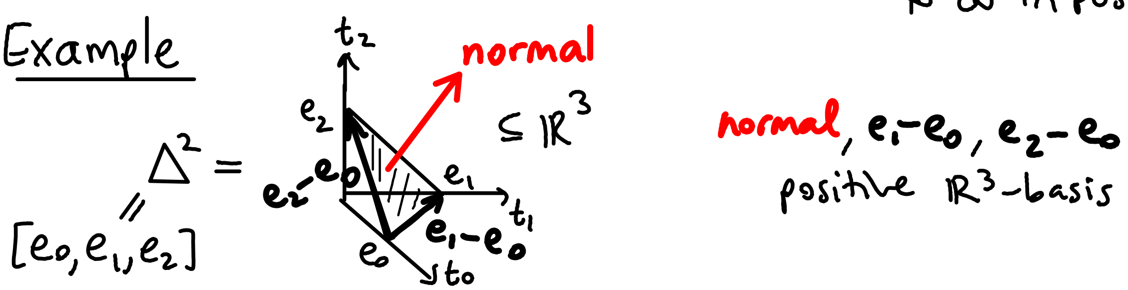
<u>Example</u>



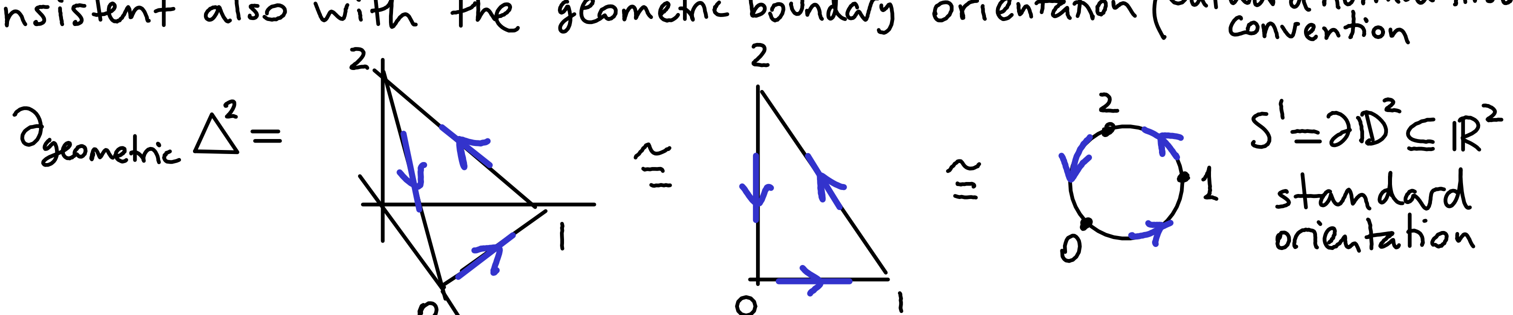
Our choice is also consistent with the "normal first" convention for orienting hyperplanes with a given choice of normal:

$\Delta^n \subseteq$ hyperplane $\{(t_0, \dots, t_n) : \sum t_j = 1\} \subseteq \underline{\underline{\mathbb{R}^{n+1}}}$ normal $(1, 1, \dots, 1)$ (so pointing to ∞ in positive quadrant)

<u>Example</u>



Consistent also with the geometric boundary orientation (outward normal first convention)



Compare $\partial \Delta = +[\hat{e}_0, e_1, e_2] - [e_0, \hat{e}_1, e_2] + [e_0, e_1, \hat{e}_2]$

This $-[e_0, e_2]$ is <u>not</u> equal to singular chain $[e_2, e_0]$ since they are different maps and we take free abelian group generated by maps.

But $[e_0, e_2] + [e_2, e_0]$ is <u>homologous</u> to 0 (Homework).

Locality (or "small simplices theorem")

$\mathcal{U} = \{\text{subspaces } U_i \subseteq X\}$ whose interiors cover X:

$$X = \bigcup U_i^\circ$$

Def $C_*^{\mathcal{U}}(X) \subseteq C_*(X)$ subcx generated by n-simplices σ with $\sigma(\Delta^n) \subseteq U_i$ some i

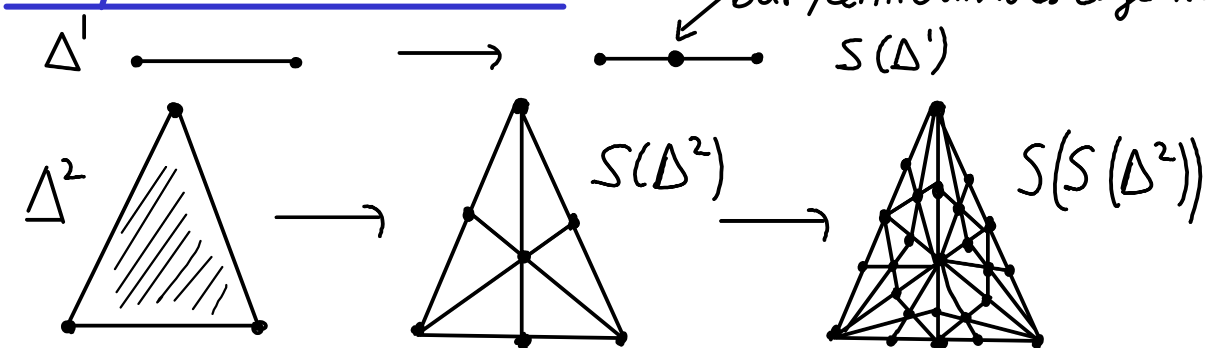
Theorem

$$\boxed{H_*(C_*^{\mathcal{U}}(X)) \cong H_*(C_*(X)) = H_* X}$$

Sketch Pf ① Barycentric subdivision

↑ Non-examinable

barycentre of $[v_0, \dots, v_n]$ is $\frac{1}{n+1}(v_0 + \dots + v_n)$



$\Rightarrow$ chain map $S: C_*(X) \longrightarrow C_*(X)$

$\sigma \longmapsto \sigma \circ S$

and $S(C_*^{\mathcal{U}}) \subseteq C_*^{\mathcal{U}}$

subdivide the boundary (inductively by dimension) then draw the new faces obtained by convex combinations involving the new vertices and the barycentre

② S chain hpic to id:

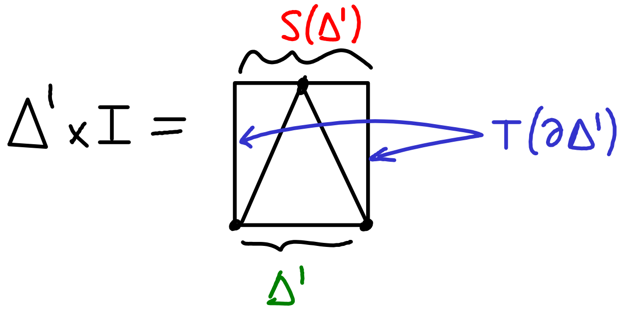
$T: C_n(X) \longrightarrow C_{n+1}(X)$

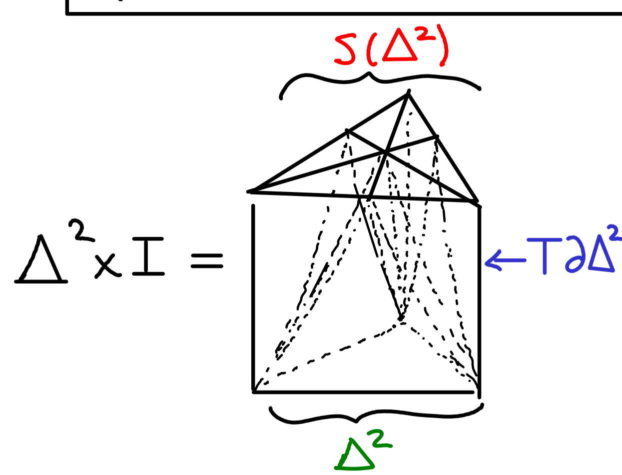
$T(\sigma): \Delta^n \times I \xrightarrow{\text{project}} \Delta^n \xrightarrow{\sigma} X$

exercise: $\partial T + T\partial = S - id$

$\Rightarrow$ $\boxed{S_*: H_*(X) \xrightarrow{id} H_*(X)}$

Idea:





③ $\forall$ n-simplex $\sigma: \Delta^n \longrightarrow X$, apply $S(\cdot)$ enough times so that $\sigma(\text{each n-simplex of subdivision}) \subseteq U_i$ some i

③ $\Rightarrow$ $\forall$ cycle c, $\exists\, n$ s.t. $S^n(c) \in C_*^{\mathcal{U}}(X)$ cycle

$$\Rightarrow H_*^{\mathcal{U}}(C) \longrightarrow H_*(X) \text{ surjective}$$

$$[S^n(c)] \longmapsto S_*^n[C] = [C] \quad \text{by ②}$$

③ $\Rightarrow$ $\forall$ bdry $c = \partial b$, $\exists\, n$ s.t. $S^n(b) \in C_*^{\mathcal{U}}(X)$

<u>claim:</u> $H_*^{\mathcal{U}}(C) \longrightarrow H_*(X)$ injective

suppose $[C] \longmapsto 0$

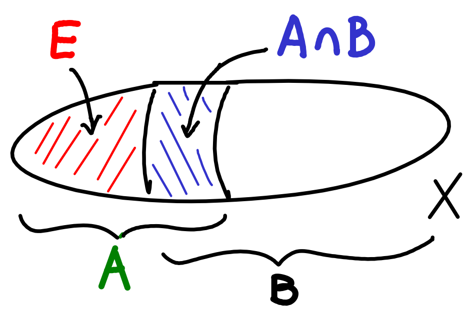
then $c = \partial b$ for $b \in C_*(X)$

now $S^n c, S^n b \in C_*^{\mathcal{U}}(X)$ for large n

$$\Rightarrow \partial S^n b = S^n \partial b = S^n c \quad \text{in } C_*^{\mathcal{U}}(X)$$

$$\Rightarrow [c] \underset{②}{=} S_*^n[c] = [S^n c] = [\partial S^n b] = 0 \quad \text{in } H_*^{\mathcal{U}}(X) \checkmark \quad \square$$

Proof of excision theorem



Let $B = X \setminus E$

use $\mathcal{U} = \{A, B\}$

so $C_*^{\mathcal{U}}(X) = C_*(A) + C_*(B) \subseteq C_*(X)$

$$\Rightarrow C_*(X\setminus E) / C_*(A \setminus E) = C_*(B) / C_*(A \cap B) \cong C_*^{\mathcal{U}}(X) / C_*(A)$$

$H_*(X\setminus E, A\setminus E)$ $\cong$ (from the quotient above)

$\Rightarrow$ Compare LES's :

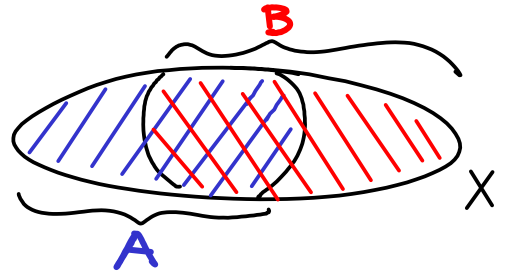
$$\begin{array}{ccccccccc}
H_*(A) & \longrightarrow & H_*(C_*^{\mathcal{U}}X) & \longrightarrow & H_*(C_*^{\mathcal{U}}X / C_*A) & \longrightarrow & H_{*-1}(A) & \longrightarrow & H_{*-1}(C^{\mathcal{U}}X) \\
\| & & \downarrow \cong \text{ locality} & & \downarrow \text{iso by 5-lemma} & & \| & & \downarrow \cong \text{ locality} \\
H_*(A) & \longrightarrow & H_*(X) & \longrightarrow & H_*(C_*X / C_*A) & \longrightarrow & H_{*-1}(A) & \longrightarrow & H_{*-1}(X)
\end{array}$$

$$H_*(C_*X / C_*A) = H_*(X, A) \qquad \square$$

6. MAYER-VIETORIS SEQUENCE ← Key computational tool

$X = A \cup B$ s.t. $X = A^\circ \cup B^\circ$

(A, B: any subspaces)



<u>MV Theorem</u> ∃ LES:

$$\cdots \longrightarrow H_*(A\cap B) \xrightarrow{i_*} H_*(A)\oplus H_*(B) \longrightarrow H_*(X) \longrightarrow H_{*-1}(A\cap B) \xrightarrow{i_*[-1]} \cdots$$

& same holds for $\widetilde{H}_*$.

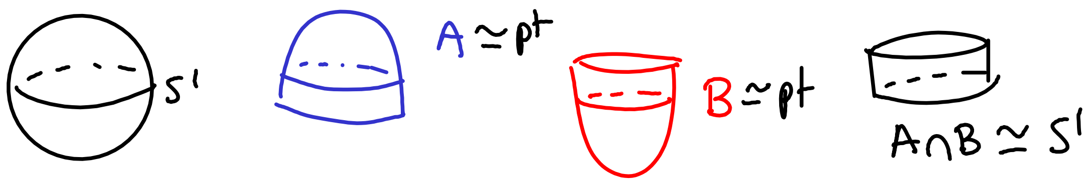
<u>Pf</u> SES $0 \longrightarrow C_*(A\cap B) \longrightarrow C_*(A)\oplus C_*(B) \longrightarrow C_*^{\mathcal{U}}(X) \longrightarrow 0$, where $\mathcal{U} = \{A, B\}$

$$\sigma \longmapsto (\sigma, -\sigma)$$
$$(\alpha, \beta) \longmapsto \alpha + \beta$$

⇒ induces the LES (using locality $H_*^{\mathcal{U}}X \cong H_*X$). ◻

<u>Exercise</u> connecting map is $\delta: H_*(X) \longrightarrow H_{*-1}(A\cap B)$
$$[\alpha+\beta] \longmapsto [\partial\alpha] = -[\partial\beta]$$

<u>Example</u> S^2



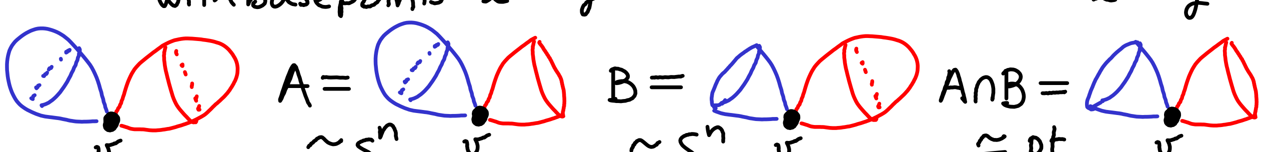
$$\cdots \to H_2(pt)\oplus H_2(pt) \to H_2S^2 \to H_1(S^1) \to H_1(pt)\oplus H_1(pt) \to \cdots$$

$H_2(pt)\oplus H_2(pt) = 0$, $H_1(S^1) \cong \mathbb{Z}$, $H_1(pt)\oplus H_1(pt) = 0$, hence $H_2S^2 \cong \mathbb{Z}$

<u>Exercise</u> Compute $H_*S^n = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$ using MV

<u>Example</u> **wedge sum** of X, Y with basepoints $x \in X$, $y \in Y$: $X \vee Y = X \times Y / x \sim y$

$X = S^n \vee S^n =$, $A =$ $\simeq S^n$, $B =$ $\simeq S^n$, $A\cap B =$ $\simeq pt$

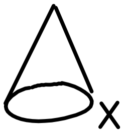


$$0 \to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\cong} H_n(X) \to 0 \to \cdots \to 0 \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to H_0(X) \to 0$$

$1 \longmapsto (1,-1)$, $H_0(X) \cong \mathbb{Z}$

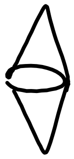
Similarly $\boxed{H_*(X\vee Y) \cong H_*(X)\oplus H_*(Y) \text{ for } * \neq 0}$ if ∃ contractible nbhds of $x \in X$, of $y \in Y$.

Cones and suspensions



$CX = (X \times [0,1]) \,/\, (x,s) \sim (y,t)$ iff equal or $s=t=1$

$\simeq pt$

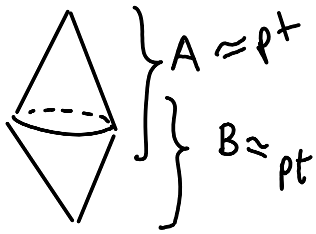


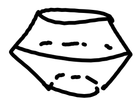
$\Sigma X = (X \times [0,1]) \,/\, (x,s) \sim (y,t)$ iff equal or $s=t=0$ or $s=t=1$

Example $CS^n \cong D^{n+1}$, $\Sigma S^n \cong S^{n+1}$.

Lemma $\boxed{H_*(\Sigma X) \cong \tilde{H}_{*-1}(X)}$

Pf

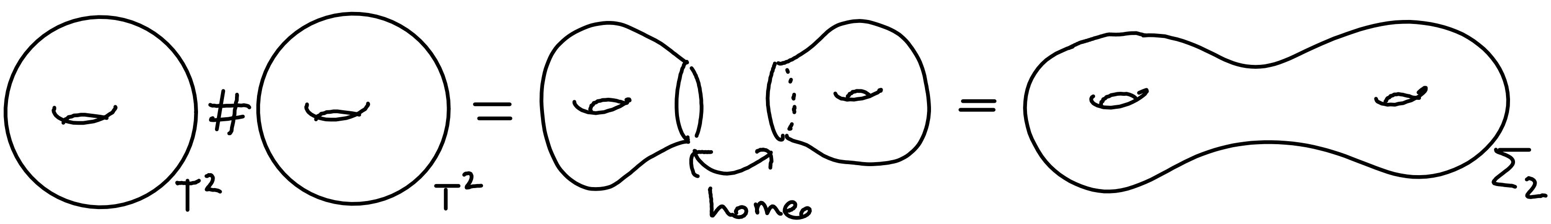




$A \cap B \simeq X$ now apply MV. □

Connected sum

M, N connected n-manifolds $\Rightarrow$ $M \# N = (M \setminus \text{open n-ball}) \cup (N \setminus \text{open n-ball})$ / identify ∂balls via a homeo



Fact compact connected orientable surfaces are homeo to S^2 or $T^2 \# \dots \# T^2$
and " " non-orientable ones: $\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2$.

(S^2: genus $g=0$; $T^2 \# \dots \# T^2$: $g = \#$ copies, called Σ_g)

Exercise (Homework) For M, N compact connected

By MV, $\boxed{H_*(M \# N) \cong H_*(M) \oplus H_*(N) \text{ for } 1 \leq * \leq n-2}$

If M or N orientable: $* = n-1$ also works

If both non-orientable: $* = n-1$ one of $\mathbb{Z}/2$ summands becomes $\mathbb{Z}$

$H_0(M \# N) \cong \mathbb{Z}$ since connected

fact: $H_n(M \# N)$ is $\mathbb{Z}$ if M, N both orientable, or 0 else (see later in course)

Cor 1) $\chi(M \# N) = \chi(M) + \chi(N) - \begin{cases} 2 & \text{for } n \text{ even} \\ 0 & \text{" " odd} \end{cases}$

2) $H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{2g} & * = 1 \\ \mathbb{Z} & * = 2 \end{cases}$ (Σ_g ← genus g)

7. DEGREE OF MAPS OF SPHERES

$$f: S^n \longrightarrow S^n \quad \Longrightarrow \quad f_n: H_nS^n \longrightarrow H_nS^n$$

$$\begin{array}{ccc} H_nS^n & \longrightarrow & H_nS^n \\ \cong & & \cong \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ 1 & \longmapsto & \mathbf{deg(f)} \in \mathbb{Z} \end{array}$$

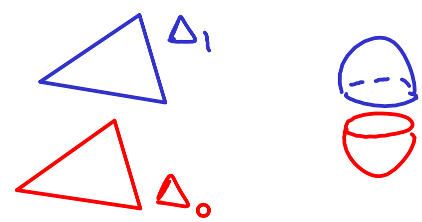
$\Rightarrow f_*: \tilde{H}_*S^n \to \tilde{H}_*S^n$ is $\deg(f)\cdot id$

Properties
1) $\deg(id) = 1$
2) $\deg(f \circ g) = \deg f \cdot \deg g$
3) $f \simeq g \Rightarrow \deg f = \deg g$
4) $f \simeq \text{const} \Rightarrow \deg f = 0$
5) f homeomorphism $\Rightarrow \deg f = \pm 1$ (sign depends on whether f is orientation-preserving or reversing)

Pf
$id_* = id$, $(f\circ g)_* = f_* \circ g_*$, $f \simeq g \Rightarrow f_* = g_*$, $\text{const}_* = 0$, f homeo $\Rightarrow f_n$ iso. □

Examples

1) $S^n = \Delta^n \times 1 \cup \Delta^n \times 0$ (call this Δ_1; call this Δ_0) $/ (b,1) \sim (b,0)$ if $b \in \partial\Delta$



recall $H_nS^n = \mathbb{Z}\cdot(\Delta_1 - \Delta_0)$

reflection: $r: S^n \longrightarrow S^n$, $r(x,t) = (x, 1-t)$

so $\Delta_0 \leftrightarrow \Delta_1$ swapped by r, so $r_*(\Delta_1 - \Delta_0) = -(\Delta_1 - \Delta_0)$

$\Rightarrow \deg(r) = -1$

2) antipodal map $-id: S^n \to S^n$ viewing $S^n \subseteq \mathbb{R}^{n+1}$

$$\Rightarrow \boxed{\deg(-id) = (-1)^{n+1}}$$

Pf $-id = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} \circ \begin{pmatrix} 1 & & \\ & -1 & \\ & & \ddots \end{pmatrix} \circ \cdots \circ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ composition of $n+1$ reflections each homotopic to r. □

3) $A \in O(n) \Rightarrow A: S^n \to S^n \Rightarrow \deg A = \det A \in \{\pm 1\}$

Pf **fact** $SO(n)$ is path-connected so $A \in SO(n)$ is $\simeq id$ so $\deg A = \det A = +1$
The other path-component of $O(n)$ is $r \circ O(n)$ where r is any reflection. □

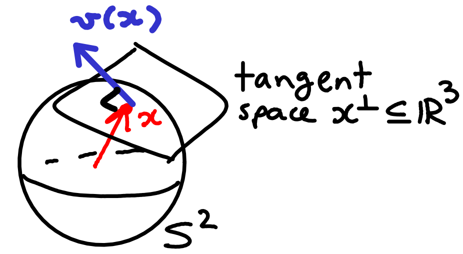
4) $\boxed{f \text{ not surjective} \Longrightarrow \deg f = 0}$

Pf If $y \notin \operatorname{Im} f \Rightarrow$

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{\ f_*\ } & H_n(S^n) \\ & {\scriptstyle f_*}\searrow \quad \nearrow & \\ & H_n(S^n \setminus y) \cong H_n(\mathbb{R}^n) = 0 & \end{array}$$

□

Application to vector fields on S^n



$v: S^n \longrightarrow \mathbb{R}^{n+1}$ tangent vector field on S^n

so $v(x) \perp x$

Cor **Hairy ball theorem** $\exists$ nowhere zero v.f. on $S^n \iff n$ odd

(case $n=2$: "you cannot comb a ball of hair without creating a tuft")

Pf Suppose $v(x) \neq 0 \quad \forall x$

$\Rightarrow$ hpy $F: S^n \times [0,1] \longrightarrow S^n$

$$F(x,t) = \cos(\pi t)\, x + \sin(\pi t) \frac{v(x)}{\|v(x)\|}$$

$\Rightarrow F_0 = id, \quad F_1 = -id$

$\Rightarrow 1 = \deg F_0 = \deg F_1 = (-1)^{n+1}$

$\Rightarrow n$ odd

For n even: $v(x) = (-x_2, x_1, \ldots, -x_{2k}, -x_{2k-1})$ $\square$

Cultural Remark Adams in 1962 proved using alg. topology:

(max # pointwise linearly independent vector fields on S^n) $= 2^b + 8a - 1$

where $n+1 = 2^{4a+b} \cdot$ (odd number), $0 \leq b \leq 3$, $a, b \in \mathbb{N}$, $n \geq 1$. ← get 0 if n even $\Rightarrow$ Cor ✓

Local degree

$$\begin{array}{ccc} f: S^n & \longrightarrow & S^n \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & y = f(x) \end{array}$$

(★) Suppose points $\neq x$ near x do not map to y:

$\exists$ nbhds $x \in U \cong D^n$, $y \in V \cong D^n$ s.t. $(U, U \setminus x) \xrightarrow{f} (V, V \setminus y)$ ← call this $f|_x$ **local map at x**

$$\Rightarrow (f|_x)_*: H_n(U, U \setminus x) \xrightarrow{f_*} H_n(V, V \setminus y)$$

$H_n(U, U\setminus x) \cong H_n(S^n, S^n \setminus x)$ (excise $S^n \setminus U$), where $S^n \setminus x \cong pt$

$H_n(S^n, S^n \setminus x) \cong \tilde{H}_n S^n \cong \mathbb{Z}$, and $H_n(V, V \setminus y) = \mathbb{Z}$

$$\mathbb{Z} \longrightarrow \mathbb{Z}$$

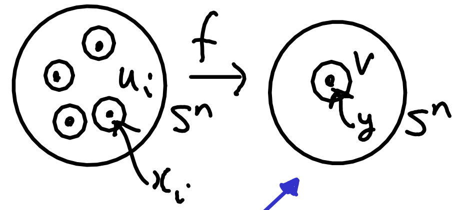
$$1 \longmapsto \deg_x f$$

Lemma $f: S^n \to S^n$, $f^{-1}(y) = \{x_1, \dots, x_k\}$

$$\Rightarrow \boxed{\deg f = \sum \deg_{x_i} f}$$

Pf

$$\begin{array}{ccc} \tilde H_n S^n & \xrightarrow{f_*} & \tilde H_n(S^n) \\ \text{quotient} \downarrow & \circlearrowright & \cong \text{ exc.} \\ H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) & \xrightarrow{f_*} & H_n(V, V \setminus y) \\ \text{exc. } S^n \setminus \cup U_i \downarrow \cong & \nearrow \sum (f|_{x_i})_* & \\ \bigoplus H_n(U_i, U_i \setminus x_i) & & \end{array}$$



Rmk: can use same V for all i by taking $\tilde V = \cap V_i$, $\tilde U_i = f^{-1}(V) \cap U_i$

is:
$$\begin{array}{ccccc} 1 & \in & \mathbb{Z} & \xrightarrow{\deg f} & \mathbb{Z} \\ \downarrow & & \downarrow & & \cong \\ & & & & \mathbb{Z} \\ (1, \dots, 1) & \in & \bigoplus_i \mathbb{Z} & \nearrow \sum \deg_{x_i} f & \end{array} \quad . \square$$

map to each summand is exc. of $S^n \setminus U_i$ so iso.

Example $p: \mathbb{C} \to \mathbb{C}$ polynomial $a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, $a_i \in \mathbb{C}$ ($a_n \neq 0$)

$\Rightarrow f: S^2 = \mathbb{CP}^1 \to \mathbb{CP}^1 = S^2$ (where view $\mathbb{CP}^1 = \mathbb{C} \cup \infty \cong S^2$ ← stereographic projection)

$z \mapsto p(z)$

$\infty \mapsto \infty$

$\Rightarrow$ hpy $F(z,t) = a_n z^n + t(a_{n-1} z^{n-1} + \dots + a_0)$

$F_0 = a_n z^n$ and $F_1 = f$

$\Rightarrow \deg f = \deg(a_n z^n) \overset{\text{Lemma}}{=\!=} \sum_{k=0}^{n-1} \underbrace{\deg_{w_k} \underbrace{a_n z^n}}_{=1}$ ← $w_k = e^{\frac{2\pi i k}{n}}$

$= n$

$=$ degree of the poly p.

(orient^n preserving homeo near w_k ← holomorphic maps are always orientation preserving)

Cor (Fundamental Thm of Algebra) $n \geq 1 \Rightarrow p$ has a root

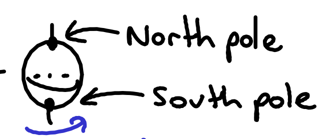
Pf $p^{-1}(0) = \emptyset \Rightarrow f^{-1}(0) = \emptyset \Rightarrow \deg f = 0$ ↯ $\square$

Cultural Rmk For smooth $f: S^n \to S^n$

$\deg f = \left(\begin{array}{l}\text{the number of preimages} \\ \text{of a generic point.}\end{array}\right)$

(i.e. almost any point works)

Example $S^2 \to S^2$



rotate by $\frac{2\pi}{d}$ about vertical axis

$\Rightarrow \deg = d = \#$ preimages of a point except if pick North/South pole

8. CELLULAR HOMOLOGY

Def **CW complex** X is sequence $\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots$

s.t.

- X^0 is any set
- $$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I_n} \mathbb{D}^n_\alpha \Big/ x \sim \varphi_\alpha(x)$$

X^n: **n-skeleton**

$\mathbb{D}^n_\alpha$: n-discs labelled by some index set I_n

$\varphi_\alpha : \partial\mathbb{D}^n \to X^{n-1}$ **attaching map** (any continuous map, often not injective)

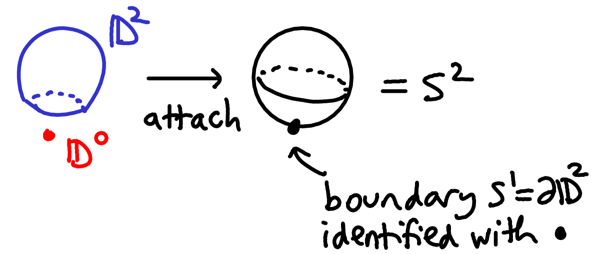
$\Rightarrow X = \bigcup_{n \geq 0} X^n$ top. space with **weak topology**:

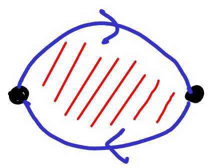
$U \subseteq X$ open $\Leftrightarrow U \cap X^n \subseteq X^n$ open $\forall n$.

$(\Leftrightarrow \varphi_\alpha^{-1}(U) \subseteq \mathbb{D}^n$ open $\forall \varphi_\alpha)$

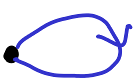
Call X **n-dimensional** if $X = X^n$ and this is the least such n.

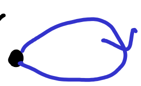
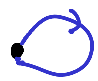
Example $S^n = (\mathbb{D}^0 \sqcup \mathbb{D}^n) / (\mathbb{D}^0 \sim \partial\mathbb{D}^n)$



Example $X = \mathbb{RP}^2 =$ 

$X^0 = \bullet = \mathbb{D}^0$

$X^1 =$  $= S^1 = (\mathbb{D}^0 \sqcup \mathbb{D}^1) / (\bullet \sim \varphi_1(x))$, $\partial\mathbb{D}^1 = S^0 = \{0,1\} \xrightarrow{\varphi_1} X^0 = \bullet$

$X^2 =$ () / (wrap ∂ of  twice around )

$= (X^1 \sqcup \mathbb{D}^2) \Big/ \left(\underset{z}{\partial\mathbb{D}^2} = S^1 \ni z \sim z^2 \in X^1 = S^1 \right)$ $\quad \partial\mathbb{D}^2 = S^1 \xrightarrow{\varphi_2} S^1$, $\varphi_2(z) = z^2$

Fact If we homotope φ_α, we get a homotopy equivalent space

Example If use another degree 2 map φ_2 above, get $X \simeq \mathbb{RP}^2$.

X is partitioned as a set by interiors of **n-cells** $\boxed{e^n_\alpha = \mathrm{Image}(\mathbb{D}^n_\alpha \to X)}$

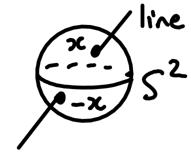
$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I_\alpha} \mathring{e}^n_\alpha$$

$$= \Big(\bigsqcup_{\alpha \in I_0} e^0_\alpha\Big) \sqcup \Big(\bigsqcup_{\alpha \in I_1} \mathring{e}^1_\alpha\Big) \sqcup \Big(\bigsqcup_{\alpha \in I_2} \mathring{e}^2_\alpha\Big) \sqcup \cdots$$

← **Rmk** interior $\mathbb{D}^0 = \mathbb{D}^0$ so $\mathring{e}^0_\alpha = e^0_\alpha$

Examples

real projective space $\mathbb{R}P^n = S^n / (\mathbb{Z}/2\text{-action by } \pm id)$ ← $x \sim -x$



$X^k = \mathbb{R}P^k$ inductively

$X^n = X^{n-1} \cup e^n$ with $\varphi : S^{n-1} \to X^{n-1} = \mathbb{R}P^{n-1} = S^{n-1}/\pm id$

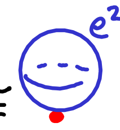
$x \longmapsto [x] = [-x]$

complex projective space

$\mathbb{C}P^n = (S^{2n+1} \subseteq \mathbb{C}^n) / (S^1\text{-action by } \lambda \cdot Id)$ ← $x \sim \lambda x$ for $\lambda \in S^1 \subseteq \mathbb{C}^*$

$X^0 = X^1 = pt = \mathbb{C}P^0$

$X^2 = X^3 = X^0 \cup e^2 = \mathbb{C}P^1$, $\varphi : S^1 \to pt$ ← $\mathbb{C}P^1 \cong$ (sphere) $\cong S^2$



$X^4 = X^5 = X^2 \cup e^4 = \mathbb{C}P^2$, $\varphi : S^3 \to \mathbb{C}P^1 = S^3/(S^1\text{-action})$

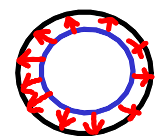
$x \longmapsto [x] = [\lambda x], \forall \lambda \in S^1$

$X^{2n} = X^{2n-2} \cup e^{2n} = \mathbb{C}P^n$, $\varphi : S^{2n-1} \to X^{2n-2} = \mathbb{C}P^{n-1}$

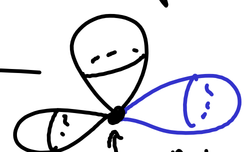
$x \longmapsto [x]$

<u>Observe</u>: For X CW complex, for $n \geq 1$: ← (For $n = 1$ $(X^0, X^{-1}) = (X^0, \emptyset)$; $X^0/X^{-1} = X^0$)

- (X^n, X^{n-1}) is a good pair ← (since $\exists$ nbhd of $\partial \mathbb{D}^n$ that deformation retracts to $\partial \mathbb{D}^n$)



- $$\boxed{X^n / X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n}$$ ←



$S^n_\alpha = \mathbb{D}^n_\alpha / \partial \mathbb{D}^n_\alpha$

X^{n-1} identified to a point

<u>Def</u> **Cellular complex** for X a CW cx,

$$\boxed{C^{CW}_n(X) = H_n(X^n, X^{n-1}) \cong H_n(X^n/X^{n-1}) \cong H_n(\bigvee_{\alpha \in I_n} S^n) = \text{free abelian gp gen. by the } n\text{-cells } e^n_\alpha}$$

since $\Delta^n \cong \mathbb{D}^n \longrightarrow (e^n \subseteq X^n) \longrightarrow \mathbb{D}^n_\alpha / \partial \mathbb{D}^n_\alpha = S^n_\alpha$ generate

↖ as usual we use the standard orientations of $\Delta^n, \mathbb{D}^n, S^n$.

Will build **cellular differential** d, prove $d \circ d = 0$,

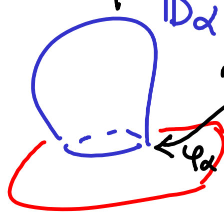
$\Rightarrow$ get $$\boxed{H^{CW}_*(X) = H_*(C^{CW}_*(X), d)}$$

$$\boxed{d\, e^n_\alpha = \sum_{\beta \in I_{n-1}} d^n_{\alpha\beta} \cdot e^{n-1}_\beta}$$

now describe the coefficients $d^n_{\alpha\beta} \in \mathbb{Z}$ and why that is a finite sum.

$$d : C_n^{CW} \longrightarrow C_{n-1}^{CW}$$

$$\begin{array}{ccccc} C_n^{CW} & & & & C_{n-1}^{CW} \\ \| & & & & \| \\ H_n(X^n, X^{n-1}) & \xrightarrow[\text{connecting map of LES}]{\delta^n_*} & H_{n-1}(X^{n-1}) & \xrightarrow[\text{quotient map}]{q^{n-1}_*} & H_{n-1}(X^{n-1}, X^{n-2}) \\ e^n_\alpha & \longmapsto & \left(\partial \mathbb{D}^m_\alpha \to \varphi_\alpha(S^{m-1})\right) & \longmapsto & \sum d_{\alpha\beta}\, e^{n-1}_\beta \end{array}$$



$\mathbb{D}^n_\alpha \to e^n_\alpha \subseteq X^n$

$\partial \mathbb{D}^n_\alpha \subseteq X^{n-1}$

φ_α

X^{n-1}

Recall LES

$$H_*(X^{n-1}) \longrightarrow H_*(X^n)$$

$$H_*(X^n) \xrightarrow{q^n_*} H_*(X^n, X^{n-1}) \xrightarrow[{[-1]}]{\delta^n_*} H_*(X^{n-1})$$

here it is important that we chose identifications $\Delta^n \cong \mathbb{D}^n$, $S^n \cong \mathbb{D}^n/\partial\mathbb{D}^n$ compatibly with orientations.

Therefore:

$$\boxed{d^n_{\alpha\beta} = \deg\Big(\underset{\substack{\| \\ \partial \mathbb{D}^n_\alpha}}{S^{n-1}} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \cong \bigvee_{I_{n-1}} S^{n-1} \longrightarrow \underset{\substack{\| \\ \mathbb{D}^{n-1}_\beta/\partial\mathbb{D}^{n-1}_\beta}}{S^{n-1}}\Big)}$$

(the last map: Quotient by $\bigvee_{I_{n-1}\setminus\beta} S^{n-1}$)

<u>Rmk</u> Only finitely many $d^n_{\alpha\beta} \neq 0$ (for fixed α) because φ_α, q are continuous and S^{n-1} compact, so get a compact image in $\bigvee_\beta S^{n-1}$, therefore cannot surject onto ∞ many S^{n-1}_β.

<u>Lemma</u> $d \circ d = 0$

<u>Pf</u> $d_n = q^{n-1}_{n-1} \circ \delta^n_n$

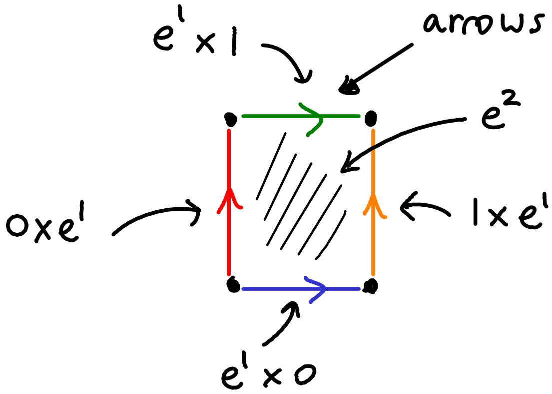
$$d_{n-1} \circ d_n = q^{n-2}_{n-2} \circ \overbrace{\delta^{n-1}_{n-1} \circ q^{n-1}_{n-1}}^{=0 \text{ by LES}} \circ \delta^n_n \qquad \square$$

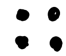
<u>Cor</u> $\boxed{\operatorname{rank} H_n^{CW}(X) \leq \#\, n\text{-cells}}$

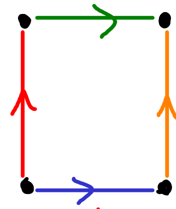
<u>Pf</u> $\#\, n\text{-cells} = \operatorname{rank} C_n^{CW}(X) \geq \operatorname{rank} \operatorname{Ker} d_n^{CW} \geq \operatorname{rank} H_n^{CW}(X)$ $\square$

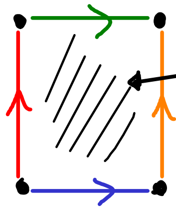
Example $I \times I$ $\qquad I=[0,1]$ $\qquad \mathbb{D}^1=[-1,1]$

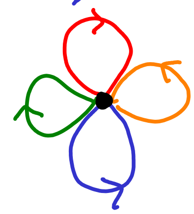
arrows here tell us how we map $[-1,1] \to$ edge (so orientation)

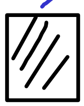


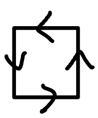
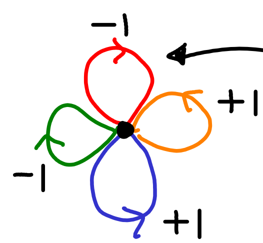
$X^0 = $  $= 4$ 0-cells

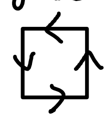
$X^1 = $  $\leftarrow$ 4 1-cells $e^1\times 0$, $1\times e^1$, $e^1 \times 1$, $0 \times e^1$

$X^2 = $  $\leftarrow$ 2-cell e^2

$X^1/X^0 = $  orientations of cells tell us how to orient the circles

$e^2 : \mathbb{D}^2 \cong$  $\longrightarrow X^1$

$\partial e^2 : S^1 \cong$  $\longrightarrow X^1/X^0 = $ 

degree -1 because top edge of  maps to  by an orientation-reversing homeomorphism.

$\Rightarrow \partial e^2 = +e^1\times 0 + 1\times e^1 - e^1 \times 1 - 0 \times e^1$

$\left(= (\partial e^1) \times e^1 - e^1 \times (\partial e^1) \right.$ $\leftarrow$ we come back to this later$)$

Example $\mathbb{R}P^n$ recall: 1 cell in each dim, $\varphi : S^k \to X^k = \mathbb{R}P^k$, $x \longmapsto [\pm x]$

$S^{k-1} = $ 

$\Delta_1 \xrightarrow{\varphi} X^{k-1}/X^{k-2} = \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \cong S^{k-1}$ $\qquad \deg = +1$

$\Delta_2 \longrightarrow -\mathrm{id}(\Delta_1)$ $\qquad \deg = (-1)^k$

$$\Rightarrow d^k_{\alpha\beta} = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

$C_*^{CW}(\mathbb{R}P^n)$ $\quad 0 \to \underset{k=n}{\mathbb{Z}} \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow{2} \underset{k=1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \longrightarrow \underset{-1}{0}$

(first map: 2 if n even, 0 " odd; second map: 0, 2 resp.)

$$H_*^{CW}(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & *=0, \text{ also } *=n \text{ if odd} \\ \mathbb{Z}/2 & *=1,3,5,\dots < n \\ 0 & \text{else} \end{cases}$$

Example S^n:

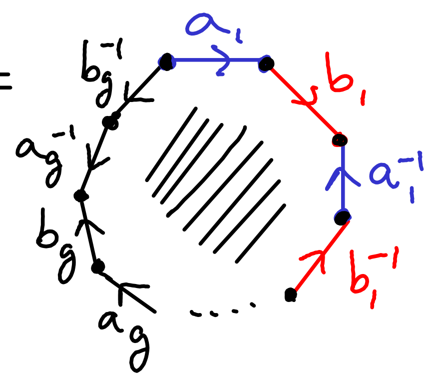
$\deg \varphi = 0$

$C_*^{CW}(S^n)$: $n \geq 2$: $0 \to \mathbb{Z}\cdot D^n \xrightarrow{0} 0 \to \cdots \to 0 \to \mathbb{Z}\cdot D^0 \to 0$

$n=1$: $0 \to \mathbb{Z}\cdot D^1 \xrightarrow{0} \mathbb{Z} D^0 \to 0$

$$\Rightarrow H_*^{CW}(S^n) \cong \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{else} \end{cases}$$

Example $\Sigma_g =$ (genus g surface)



boundary identifications

$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

← Notice all vertices are identified, call vertex v

$\partial a_i = v - v = 0$
$\partial b_i = v - v = 0$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}^{2g} \xrightarrow{\partial = 0} \mathbb{Z} \longrightarrow 0$$

$$\mathbb{Z}\cong\mathbb{Z}\cdot D \qquad \mathbb{Z}^{2g}\cong\mathbb{Z}\langle a_1, b_1, \dots, a_g, b_g\rangle \qquad \mathbb{Z}\cong\mathbb{Z}\cdot v$$

$$D \longmapsto -a_1 - b_1 + a_1 + b_1 + \cdots - a_g - b_g + a_g + b_g = 0$$

← signs: compare edge orientation with anticlockwise orientation of ∂D

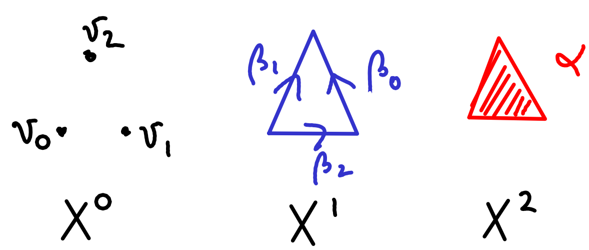
$$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0, 2 \\ \mathbb{Z}^{2g} & * = 1 \\ 0 & \text{else} \end{cases}$$

Lemma X Δ-cx structure $\Longrightarrow$ induces CW-cx structure on X and $(C_*^{CW}(X), d^{CW}) \cong (C_*^{\Delta}(X), d^{\Delta})$

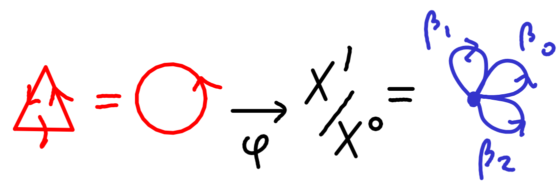
$$\Longrightarrow \boxed{H_*^{CW}(X) \cong H_*^{\Delta}(X)}$$

Pf $X^n = \bigcup$ n-simplices of X (n-simplices $\cong D^n$) and degrees are ± 1 depending on orient^n so can identify d^{CW} and d^{Δ}. □

Example $X = \text{triangle} = \Delta^2$



$\Rightarrow d^{\Delta}\alpha = \beta_0 - \beta_1 + \beta_2$



$d_{\alpha\beta_2} = d_{\alpha\beta_0} = +1$, $d_{\alpha\beta_1} = -1$

$\Rightarrow d^{CW}\alpha = d^{\Delta}\alpha$ ✓ □

Theorem X CW cx (or Δ-cx) $\Longrightarrow$ $H_*^\Delta(X) \cong H_*(X)$

$\Longrightarrow$ H_*^Δ, H_*^{CW} independent of choice of CW-cx / Δ-cx structure.

Pf ① $H_*(X^n, X^{n-1}) \cong \widetilde{H}_*(X^n/X^{n-1}) \cong \widetilde{H}_*(\bigvee_\alpha S^n) \cong \bigoplus_\alpha \widetilde{H}_* S^n$

$= 0 \iff * = n$ — lives in degree n

LES for $(X^n, X^{n-1}) \Rightarrow H_*(X^{n-1}) \to H_*(X^n)$ iso for $* < n-1$, $* > n$

② for $* < n$: $H_*(X^n) \cong H_*(X^{n+1}) \cong H_*(X^{n+2}) \cong \dots \cong H_*(X)$

(first iso: by ①; last iso: by compactness each sing. chain lands in X^N, some N)

③ for $* > n$: $H_*(X^n) \cong H_*(X^{n-1}) \cong H_*(X^{n-2}) \cong \dots \cong H_*(X^{-1}) = 0$

(first iso: by ①)

④ LES: $\dots \to H_n(X^{n-1}) \longrightarrow H_n(X^n) \xrightarrow{q_n^n} H_n(X^n, X^{n-1}) \to \dots$

$H_n(X^{n-1}) = 0$ by ②

$\Rightarrow q_n^n$ injective $\forall n$

⑤ LES: $\dots \xrightarrow{q_{n+1}^{n+1}} H_{n+1}(X^{n+1}, X^n) \xrightarrow{\delta_{n+1}^{n+1}} H_n(X^n) \longrightarrow H_n(X^{n+1}) \to 0$ (the 0 by ①)

UPSHOT

$$H_n(X) \overset{②}{\cong} H_n(X^{n+1}) \overset{⑤}{\cong} H_n(X^n) / \operatorname{im} \delta_{n+1}^{n+1} \overset{④}{\cong} \underbrace{(q_n^n H_n(X^n))}_{=\operatorname{im} q_n^n} / \operatorname{im} \underbrace{q_n^n \circ \delta_{n+1}^{n+1}}_{= d_{n+1}^{CW}} \cong H_n^{CW}(X)$$

(⑤: 1st iso thm)

$$\operatorname{im} q_n^n \overset{\text{exactness LES}}{=} \operatorname{Ker} \delta_n^n \overset{④}{=} \operatorname{Ker} \underbrace{q_{n-1}^{n-1} \circ \delta_n^n}_{= d_n^{CW}}$$

□

Rmk by ① H_k not affected if attach $(k+2)$-cells or higher

by ② Inclusion $X^n \to X$ induces iso $H_*(X^n) \to H_*(X)$ for $* < n$

Cor X n-dimensional cell cx $\Longrightarrow$ $H_*(X) = 0$ for $* > n$

Axioms for homology: Eilenberg–Steenrod axioms

It was no accident that H_*^{Δ}, H_*^{CW}, H_* all agreed.

Def A **generalised homology theory (GHT)** is a **functor** $F : \text{TopPairs} = \left(\begin{array}{l}\text{Category of pairs}\\ \text{of spaces, and}\\ \text{maps of pairs}\end{array}\right) \longrightarrow \text{Graded Abelian Gps}$

with a **natural transformation** $\delta : F_*(X,A) \longrightarrow F_{*-1}(X,\emptyset)$ satisfying:

($F_{*-1}(X,\emptyset)$ abbreviated: $F_{*-1}(X)$)

1) **homotopy invariance**: $f \simeq g \Rightarrow F(f) = F(g)$

2) **exactness**: $\exists$ LES $\cdots \to F_*(A) \to F_*(X) \to F_*(X,A) \xrightarrow{\delta} F_{*-1}(A) \to \cdots$

(the maps being $F(\text{incl}: A \to X)$ and $F(\text{incl}: (X,\emptyset) \to (X,A))$)

3) **additivity**: $(X,A) = \bigsqcup (X_i, A_i)$, $\text{incl}_i : (X_i, A_i) \longrightarrow (X,A)$

then $\sum F(\text{incl}) : \bigoplus F(X_i, A_i) \xrightarrow{\cong} F(X,A)$

4) **excision**: $\overline{E} \subseteq A^\circ \subseteq X \implies F(X \setminus E, A \setminus E) \xrightarrow{\cong} F(X,A)$, the map being $F(\text{incl})$

Remark (4) $\iff$ $X = A^\circ \cup B^\circ$, $\text{incl} : (A, A\cap B) \longrightarrow (X,B)$

then $F(\text{incl}) : F(A, A\cap B) \xrightarrow{\cong} F(X,B)$

Pf $A = X\setminus E$, $B = A$ noticing that $(X\setminus E)^\circ \cup A^\circ = X$

$E = B \setminus A$ noticing that $\overline{E} \subseteq \overline{B} \setminus A^\circ \subseteq B^\circ \setminus A^\circ \subseteq B^\circ = A^\circ$ ($X = A^\circ \cup B^\circ$ so $\partial B \subseteq A^\circ$)

Rmk In (3), the topology on the disjoint union $\bigsqcup (X_i, A_i)$ is defined by: $U \subseteq \bigsqcup (X_i, A_i)$ open $\iff$ $U \cap X_i \subseteq X_i$ open $\forall i$

FACT Theorem

a) $(F, \delta_F), (G, \delta_G)$ GHTs, $\alpha : F \to G$ a natural transformation commuting with δ_F, δ_G such that $\alpha_{\text{point}} : F(\text{point}) \to G(\text{point})$ is an iso, then α is an iso.

b) If (F, δ_F) GHT satisfies (5) **dimension**: $F_*(\text{point}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Then $\exists$ natural iso $F \cong H_*$ (such an F is called a **homology theory**)

Rmk In (b) if require $F_0(\text{point}) = G$ an abelian group (instead of $\mathbb{Z}$)

$\implies F(X,A) \cong H_*(X,A;G) =$ (homology with coefficients in G) $\leftarrow$ later in course

9. COHOMOLOGY

(C_*, ∂_*) chain cx s.t. C_* free $\mathbb{Z}$-module $\leftarrow C_* \cong \bigoplus_\alpha \mathbb{Z}$

Def n-cochains $\boxed{C^n = \mathrm{Hom}(C_n, \mathbb{Z})}$

coboundary map $\partial^n : C^n \longrightarrow C^{n+1}$

$$\partial^n(\phi) = \phi \circ \partial_{n+1}$$

$$\begin{tikzcd} C^n \arrow[d, "\phi"'] & C^{n+1} \arrow[l, "\partial_{n+1}"'] \arrow[dl, "\partial\phi"] \\ \mathbb{Z} & \end{tikzcd}$$

Notice ∂^* is degree $+1$ map (not -1)

$$\boxed{H^m((C_*, \partial_*)) = \frac{\operatorname{Ker} \partial^m}{\operatorname{Im} \partial^{m-1}}}$$

cocycles

coboundaries

Rmk If use negative grading, (C^{-*}, ∂^{-*}) is a chain complex with homology so many results from H_* carry over to H^*. So abusively write

$$\boxed{H^*(C_*, \partial_*) = H_*(\mathrm{Hom}(C_*, \mathbb{Z}), \partial^*)}$$

Remarks about dualisation

$\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$ generated by projection maps $\pi_i(x_1, \dots, x_n) = x_i$

$\alpha : \mathbb{Z}^n \to \mathbb{Z}^m$, $x \mapsto A \cdot x$ (A: $m \times n$ matrix) $\Longrightarrow$

$$\begin{tikzcd} \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z}) \arrow[l, "\text{dual}"', "\alpha^*"] \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z}^n & \mathbb{Z}^m \arrow[l, "\text{transpose}(A)"] \end{tikzcd}$$

$\alpha^* \phi = \phi \circ \alpha$

Def X space $\Longrightarrow$ singular cohomology $\boxed{H^*(X) = H^*(C^*(X), \partial^*)}$

dualise $C_* = C_*(X)$

similarly define H^*_Δ, H^*_{CW}

Example $\mathbb{RP}^3$: $C_*^{CW}(\mathbb{RP}^3)$: $0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

dualise: $C^*_{CW}(\mathbb{RP}^3)$: $0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \to 0$

$$H^*(\mathbb{RP}^3) \cong H^*_{CW}(\mathbb{RP}^3) = \begin{cases} \mathbb{Z} & * = 0, 3 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

$\leftarrow$ notice $H_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$ has moved to grading 2.

Functoriality

$f : X \longrightarrow Y \quad \Rightarrow \quad f_* : C_* X \longrightarrow C_* Y$ ↙ called **pull-back**

$\Rightarrow \quad C^* X \xleftarrow{f^*} C^* Y$ is dual: $\boxed{f^* \phi = \phi \circ f_*}$

Lemma f^* is a cochain map

$\Rightarrow \quad \boxed{f^* : H^* Y \longrightarrow H^* X}$

Pf

$$
\begin{aligned}
(\partial^* \circ f^*)(\phi) &= \partial^*(\phi \circ f_*) \\
&= (\phi \circ f_*) \circ \partial \\
&= (\phi \circ \partial) \circ f_* && \text{as } f_* \text{ chain map} \\
&= f^* \circ (\phi \circ \partial) \\
&= f^* \circ (\partial^* \phi) \\
&= (f^* \circ \partial^*)(\phi)
\end{aligned}
$$

Properties
- $id^* = id$
- $(f \circ g)^* = g^* \circ f^*$ notice order!

$\Longrightarrow \quad \boxed{H^* : \text{Top} \longrightarrow \text{GradedAbGps}}$ contravariant functor

Exercise $H^0(X) = \prod_{\pi_0 X} \mathbb{Z}$ where $\pi_0 X = \{\text{path-components of } X\}$

Homotopy invariance

Lemma $f_*, g_* : C_* \longrightarrow \tilde{C}_*$ (free) chain hpic $\Rightarrow f^* = g^* : H^* \tilde{C} \to H^* C$

Pf $f_* - g_* = \tilde{\partial} \circ h + h \circ \partial$ some $h : C_* \longrightarrow \tilde{C}_*[1]$

$f^* - g^* = h^* \circ \tilde{\partial}^* + \partial^* \circ h^*$ for dual $h^* : \tilde{C}^* \longrightarrow C^*[-1]$.

(notice degree -1, not $+1$) □

Def h^* called **cochain homotopy**

Cor $f \simeq g : X \longrightarrow Y \implies f^* = g^* : H^* Y \longrightarrow H^* X$ □

Algebra: dual of SES

Lemma $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ exact, A, B, C free

$\Longrightarrow \quad 0 \leftarrow A^* \xleftarrow{i^*} B^* \xleftarrow{j^*} C^* \leftarrow 0$ ″ , A^*, B^*, C^* free

Pf C free $\Rightarrow \exists$ splitting $B \underset{s}{\overset{j}{\rightleftarrows}} C$ $\quad j \circ s = \mathrm{id}$

↑ Pick preimages b_i for basis e_i of C, then $s(e_i) = b_i$

$\Rightarrow \quad A \oplus C \xrightarrow[i \oplus s]{\cong} B$

(**Rmk** inverse is $B \cong A \oplus C$, $b \longmapsto i^{-1}(b - s(b)) \oplus j(b)$)

dual $\Rightarrow \quad A^* \oplus C^* \xleftarrow[i^* \oplus s^*]{\cong} B^*$ and $s^* \circ j^* = \mathrm{id}$

→ so i^* surj $\qquad$ → so j^* inj

$\Rightarrow \quad 0 \leftarrow A^* \xleftarrow{i^*} B^* \underset{s^*}{\overset{j^*}{\leftrightarrows}} C^* \leftarrow 0$

where $0 = (j \circ i)^* = i^* \circ j^*$ so $\mathrm{Im}\, j^* \subseteq \mathrm{Ker}\, i^*$

prove $\supseteq$: $i^* b = 0 \Rightarrow b - j^* s^* b \in \mathrm{Ker}\, i^* \cap \mathrm{Ker}\, s^* = \{0\}$ (↑ since $s^* j^* = \mathrm{id}$)

$\Rightarrow b = j^* s^* b \in \mathrm{Im}\, j^*$

$\Rightarrow \mathrm{Ker}\, i^* = \mathrm{Im}\, j^* \quad \square$

Excision, LES, Mayer–Vietoris

By previous Lemma get dual results:

Excision $\bar{A} \subseteq V^\circ \subseteq X \Rightarrow H^*(X \setminus A, V \setminus A) \xleftarrow[\cong]{i^*} H^*(X, V)$

LES for pair (X, A) $\quad \cdots \xleftarrow{q^*[+1]} H^{*+1}(X, A) \xleftarrow{\delta} H^*(A) \xleftarrow{i^*} H^*(X) \xleftarrow{q^*} H^*(X, A) \leftarrow \cdots$

M.V. $X = A^\circ \cup B^\circ \Rightarrow \cdots \leftarrow H^{*+1}(X) \leftarrow H^*(A \cap B) \xleftarrow{i_A^* \oplus -i_B^*} H^*(A) \oplus H^*(B) \xleftarrow{j_A^* \oplus j_B^*} H^*(X) \leftarrow \cdots$

where $A \cap B \xrightarrow{i_A} A \xrightarrow{j_A} X$, $A \cap B \xrightarrow{i_B} B \xrightarrow{j_B} X$ are the obvious maps

Axioms for cohomology These are analogous to the axioms for homology except we reverse all arrows, and we change axiom (3): $\prod$ instead of $\oplus$

additivity: $(X, A) = \bigsqcup (X_i, A_i)$, $\mathrm{incl}_i : (X_i, A_i) \longrightarrow (X, A)$

then $\boxed{\prod F(\mathrm{incl}) : \prod F(X_i, A_i) \xleftarrow{\cong} F(X, A)}$

10. CUP PRODUCT

Theorem $H^*(X)$ (space) is ① unital ② graded-commutative ring via

$\cup : H^k(X) \times H^\ell(X) \longrightarrow H^{k+\ell}(X)$ determined by

$$\cup : C^k(X) \times C^\ell(X) \longrightarrow C^{k+\ell}(X)$$
$$(\phi \cup \psi)(\sigma) = \phi(\sigma|_{[e_0,\dots,e_k]}) \cdot \psi(\sigma|_{[e_k,\dots,e_{k+\ell}]})$$

① $1 \in C^0(X)$ constant function $\Rightarrow 1 \cup \phi = \phi \cup 1$

② $\phi \cup \psi = (-1)^{\deg\phi \cdot \deg\psi} \psi \cup \phi$

Useful trick

If X is CW-cx, then $C_*^{CW}(X) \xrightarrow[\simeq]{\text{inclusion}} C_*(X)$, so $C^*_{cw}(X) \xleftarrow[\simeq]{\text{restriction}} C^*(X)$. So to define/determine a class in $H^*(X)$ it is enough to define its values on CW chains (provided it is a CW-cycle). So doing: $H^k_{cw} \times H^\ell_{cw} \xrightarrow{\cong} H^k \times H^\ell \xrightarrow{\cup} H^{k+\ell}$.

Proof of Theorem

$$\partial^*(\phi\cup\psi)(\sigma) = (\phi\cup\psi)(\partial\sigma)$$
$$= (\phi\cup\psi) \sum(-1)^i \sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_n]} \quad (n = k+\ell)$$
$$= \sum_{i\le k} (-1)^i \phi(\sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_{k+1}]}) \cdot \psi(\sigma|_{[e_{k+1},\dots,e_n]})$$
$$+ \sum_{i>k} (-1)^i \phi(\sigma|_{[e_0,\dots,e_k]}) \cdot \psi(\sigma|_{[e_k,\dots,\hat{e}_i,\dots,e_n]}) \cdot \underbrace{(-1)^{i-k}(-1)^{k-i}}_{1}$$
$$= ((\partial^*\phi)\cup\psi(\sigma)) + (-1)^k \phi\cup\partial^*\psi$$

induces $[\phi]\cup[\psi] = [\phi\cup\psi]$:

Well-defined:
- cycles → cycle: $\partial(\phi\cup\psi) = \overbrace{(\partial\phi)}^{=0}\cup\psi \pm \phi\cup\overbrace{(\partial\psi)}^{=0} = 0$ ✓
- $[\phi] = [\phi + \partial\alpha]$ so need $[(\partial\alpha)\cup\psi] = 0$

  $(\partial\alpha)\cup\psi \underset{\partial\psi=0}{=} \partial(\alpha\cup\psi)$ ✓
- Similarly $[\phi]\cup[\partial\beta] = 0$

bilinear, associative, distributive: true at chain level unital: $(\partial 1)(\sigma) = 1(\sigma|_{[e_0]}) - 1(\sigma|_{[e_1]}) = 1-1=0$

$(1\cup\psi)(\sigma) = 1(\sigma|_{[e_0]})\cdot\psi(\sigma|_{[e_0,\ldots,e_n]}) = \psi(\sigma)$ ($\phi\cup 1=\phi$ similar)

graded-comm. sketch proof:

Let $r: C_n(X)\to C_n(X)$, $r(\sigma)=\varepsilon_n\,\overline{\sigma}$ where: $\varepsilon_n = (-1)^{\frac{n(n+1)}{2}}$

and $\overline{\sigma}|_{[v_0,\ldots,v_n]} = \sigma|_{[v_n,\ldots,v_0]}$ ← reverse order of vertices: is product of $\underbrace{n+(n-1)+\cdots+1}_{n(n+1)/2}$ transpositions

(idea: each transposition is a reflection in hyperplane, so reverses orientation of simplex, so insert ε_n to compensate)

one checks:
- r chain map
- $\underline{r^*\varphi\cup r^*\psi}_{\varepsilon_k\varepsilon_\ell} = \underline{r^*(\psi\cup\varphi)}_{\varepsilon_{k+\ell}}$ — differ by $(-1)^{k\ell}$
- $r\simeq \mathrm{id}$ so can drop $r^*=\mathrm{id}$ on cohomology

($r-\mathrm{id} = P\partial+\partial P$ with

$P\sigma = \sum(-1)^i\varepsilon_{n-i}\,(\sigma\circ\pi)|_{[v_0,\ldots,v_i,w_n,\ldots,w_i]}$)

v_i, w_i like for prism operator; projection $\Delta^n\times I\xrightarrow{\pi}\Delta^n$ □

Naturality of cup product

Lemma $f:X\to Y \implies f^*:H^*Y\to H^*X$ hom of unital rings

Pf
$$\begin{aligned}f^*(\varphi\cup\psi)(\sigma) &= (\varphi\cup\psi)(f_*\sigma)\\ &= \varphi(f_*\sigma|_{[e_0,\ldots,e_k]})\cdot\psi(f_*\sigma|_{[e_k,\ldots,e_n]})\\ &= ((\varphi\circ f_*)\cup(\psi\circ f_*))(\sigma)\\ &= (f^*\varphi\cup f^*\psi)(\sigma)\end{aligned}$$

unital: $f^*(1) = 1\circ f_* = 1$ □

UPSHOT

H^*: Top $\longrightarrow$ {Graded-commutative unital rings with graded unital ring homs} contravariant functor.

Warning An (iso)morphism $H^*(Y)\to H^*(X)$ of groups will also preserve the ring structure if f^* is induced by a map of spaces $X\to Y$ (by above Lemma).

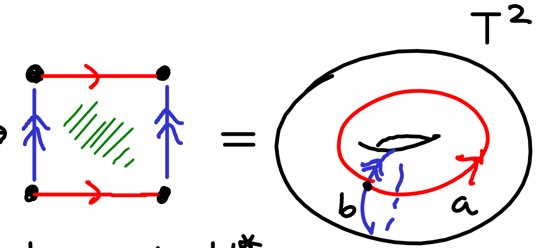
$\implies$ Cor The excision theorem iso on cohomology is an iso of rings.

However the connecting hom in M.V. or LES cannot possibly be a ring hom since it drops gradings by 1 ($\Rightarrow \delta(a\cup b)$ and $\delta(a)\cup\delta(b)$ have different grading!)

Example $H^1(T^2)\times H^1(T^2)\xrightarrow{\cup} H^2(T^2)$ bilinear form $\mathbb{Z}^2\times\mathbb{Z}^2\to\mathbb{Z}$ with matrix $\begin{pmatrix}0&+1\\-1&0\end{pmatrix}$

Pf recall:

$*$		$H_*(T^2)$	$H^*(T^2)$
0	$\mathbb{Z}$	$\mathbb{Z}\cdot pt$	$\mathbb{Z}\cdot 1$
1	$\mathbb{Z}^2$	$\mathbb{Z}a\oplus\mathbb{Z}b$	$\mathbb{Z}a^*+\mathbb{Z}b^*$
2	$\mathbb{Z}$	$\mathbb{Z}\cdot D$	$\mathbb{Z}\cdot D^*$

where $D:\Delta^2\cong$ square $\to$ 

$1, a^*, b^*, D^*$ are dual basis in H^*

Identify $H^*(T^2)\cong H^*_\Delta(T^2)$ so at chain level:

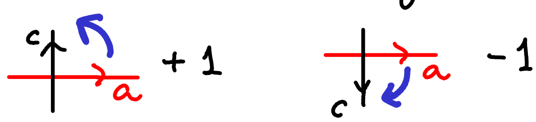
$a^*: C_1^{cw}(X)\to\mathbb{Z}$, $a\mapsto 1$, $b\mapsto 0$

$b^*: C_1^{cw}(X)\to\mathbb{Z}$, $a\mapsto 0$, $b\mapsto 1$

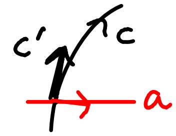
$D^*: C_2^{cw}(X)\to\mathbb{Z}$, $D\mapsto 1$

$\Rightarrow$ $b^*(c)$ ($c\in C_1^{cw}$) $=$ # a intersects c counted with orientation signs

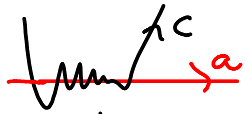
$a^*(c)$ ($c\in C_1^{cw}$) $= -$ # b intersects c counted with signs.



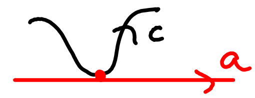
Fact Same holds for smooth singular 1-chains $c:\Delta^1\cong I\to T^2$ which intersect a transversely: velocity vectors a', c' span $\mathbb{R}^2$



Otherwise ill-defined:



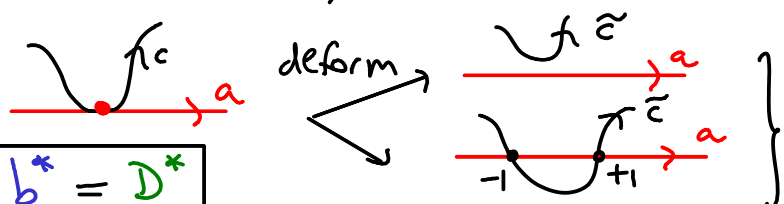
and



are bad.

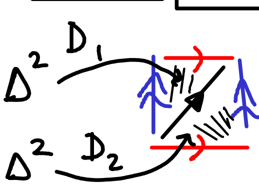
trick need first pick homologous representative which is smooth & transverse, by continuously deforming the chain (continuous map $\simeq$ id so id on H_*)

Example



both cases: $a^*(\tilde{c})=0$

claim $\boxed{a^*\cup b^* = D^*}$



$(a^*\cup b^*)(D_1+D_2) = a^*(D_1|_{[e_0,e_1]})\cdot b^*(D_1|_{[e_1,e_2]}) +$ same for D_2

(D_1+D_2 homologous to T; $D_1|_{[e_0,e_1]}$ = a, $D_1|_{[e_1,e_2]}$ = b)

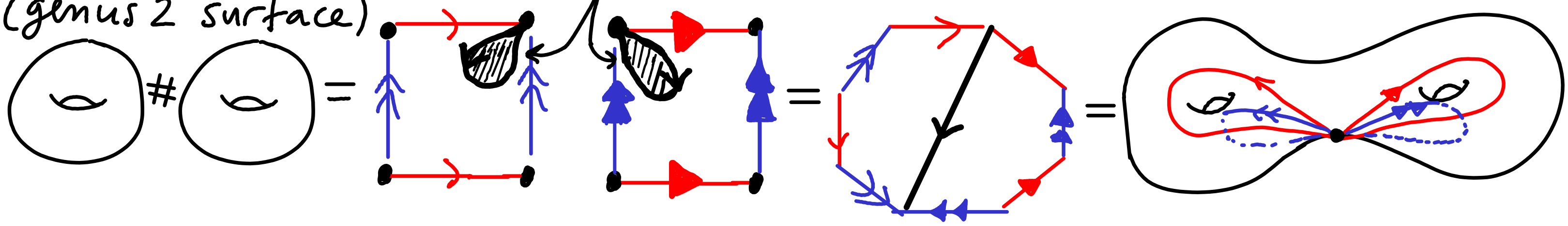
$= a^*(a)\,b^*(b) + a^*(b)\,b^*(a)$

$= 1$

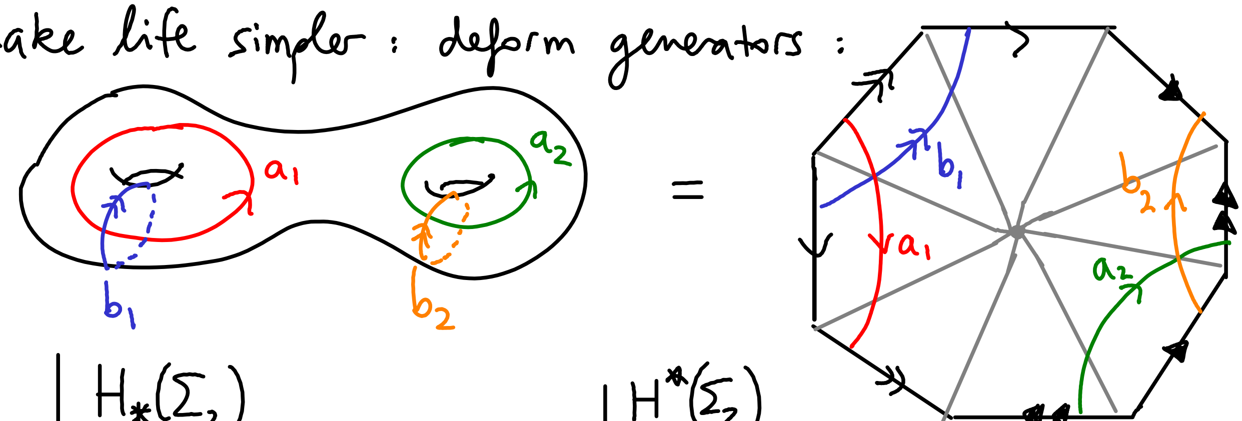
Graded-comm. $\Rightarrow b^*\cup a^* = -D^*$, $a^*\cup a^* = (-1)^{1\cdot 1}a^*\cup a^*$ so $=0$, sim'ly $b^*\cup b^*=0$. ∎

Idea $\cup$ just counts (signed) geometric intersection # of corresponding curves. Why "$a\cap a = 0$"? Can deform a to make it disjoint from a: 

Exercise Σ_2
(genus 2 surface)



Make life simpler: deform generators:



		$H_*(\Sigma_2)$	$H^*(\Sigma_2)$
0	$\mathbb{Z}$	$\mathbb{Z}\cdot pt$	$\mathbb{Z}\cdot 1$
1	$\mathbb{Z}^4$	$\mathbb{Z}a_1+\mathbb{Z}b_1+\mathbb{Z}a_2+\mathbb{Z}b_2$	$\mathbb{Z}\langle a_1^*, b_1^*, a_2^*, b_2^*\rangle$ ← dual basis
2	$\mathbb{Z}$	$\mathbb{Z}\cdot D$	$\mathbb{Z}\cdot D^*$

Notice on $C_1^{CW}(\Sigma_2)$: $a_i^*(c) = -\#(b_i \text{ intersects } c)$ (↖ signed count)
$b_i^*(c) = \#(a_i \text{ '' } \text{ '' })$

Exercise $a_i^* \cup b_j^* = \delta_{ij}\cdot D^* = -b_j^* \cup a_i^*$ (where $\delta_{ij} = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$)

$a_i^* \cup a_i^* = b_i^* \cup b_i^* = 0$

} so same as geometric intersection numbers of corresponding curves.

Cultural Rmk on general theory (Intersection Theory / Differential Topology)

M^m oriented m-mfd, $N^n \subseteq M^m$ oriented compact n-dim submfd $\Rightarrow$ $H_n(N) \xrightarrow{incl_*} H_n(M)$ ← see later in course
$\exists\ [N] \longmapsto [N]$

N, M also smooth (see Differential Geometry course) $\Rightarrow$ $\omega_N \in H^{m-n}(M)$ counts # intersecns with N (↖ with signs)

$N_1^{n_1}, N_2^{n_2} \subseteq M$ compact oriented smooth submfds, $n_1+n_2=m$ (so complementary dimensions) $\Rightarrow$ $\omega_{N_1} \cup \omega_{N_2} = \#(N_1\cap N_2)\cdot [M]^*$

$\#(N_1\cap N_2)$: geometric intersection # → may require deforming unless N_1, N_2 are already transverse

Fact (Thom 1954)
Not all $a\in H^j(M)$ arise as ω_N for connected compact oriented codim$=j$ smooth submfd N
But $\exists\ N\in\mathbb{N}$ s.t. $N\cdot a$ does arise. They do arise for $H^*(M;\mathbb{Q}), H^*(M,\mathbb{R}), H^*(M;\mathbb{Z}/2)$

11. KÜNNETH FORMULA AND PRODUCT SPACES

Algebra: tensor products

R ring (comm. with 1) — e.g. abelian groups = $\mathbb{Z}$-mods; vector spaces$/\mathbb{F}$ = $\mathbb{F}$-mods

Def A, B R-modules $\Rightarrow$ **Tensor product** is R-module

$$A \otimes_R B = \langle (a,b) : a \in A,\ b \in B \rangle \,/\, \text{relations of bilinearity \& rescaling}$$

(or $A \otimes B$) — $\langle\ \rangle$: R-mod generated — write $a \otimes b$ for its class

bilinearity: $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$
$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$

rescaling: $r \cdot (a \otimes b) = (ra) \otimes b = a \otimes (rb)$ $\quad \forall r \in R$

- So general element looks like $\sum a_k \otimes b_k$ (finite sum) ← NOT UNIQUELY!
- Don't confuse with $A \times B$: e.g. $0 \otimes b = 0$ $\forall b$

Rmk Can define $A \otimes_R B$ also by a universal property: for all R-mods C,

$$\mathrm{Hom}_R(A \otimes_R B ; C) \xrightarrow[\text{natural}]{\cong} \{R\text{-bilinear maps } A \times B \to C\}$$

Using above description of $A \otimes B$: $\varphi \longmapsto (A \times B \to C,\ (a,b) \mapsto \varphi(a \otimes b))$

Example ($R = \mathbb{F}$) V, W v.s.$/\mathbb{F}$ (basis v_i, basis w_j) $\Rightarrow$ $V \otimes W$ v.s.$/\mathbb{F}$ basis $v_i \otimes w_j$
$\dim_{\mathbb{F}} V \otimes W = \dim V \cdot \dim W$

Exercise V, W finite $\dim/\mathbb{F}$ $\Rightarrow$ $V^* \otimes W \cong \mathrm{Hom}_{\mathbb{F}}(V, W)$
Hint $f: V \to \mathbb{F}$, $w \in W$, $f \otimes w \mapsto (V \to W,\ v \mapsto f(v) \cdot w)$

Examples ($R = \mathbb{Z}$)
- $\mathbb{Z}^n \otimes \mathbb{Z}^m \cong \mathbb{Z}^{n \cdot m}$
- $\mathbb{Z}/n \otimes \mathbb{Z} \cong \mathbb{Z}/n$ ← $1 \otimes x = x \otimes 1$
- $\mathbb{Z}/2 \otimes \mathbb{Z}/3 = 0$ ← $1 \otimes x = 3 \otimes x = 1 \otimes 3x = 0$
- $\mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong \mathbb{Z}/2$ ← $1 \otimes 1 = 3 \otimes 1 = 1 \otimes 3$; $1 \otimes 2 = 2 \otimes 1 = 0$

Examples
- $A \otimes B \cong B \otimes A$
- $(\bigoplus_i A_i) \otimes (\bigoplus_j B_j) \cong \bigoplus_i \bigoplus_j (A_i \otimes B_j)$
- $A \otimes R \cong A$ (so "$\cdot \otimes_R R$ does nothing")
- $A \otimes R/d \cong A/d \cdot A$

} hence now know $A \otimes B$ for any f.g. R-mods A, B.

for example $\mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong (\mathbb{Z}/4)/2 \cdot \mathbb{Z}/4 \cong \mathbb{Z}/2$ ← (**Rmk** $(\mathbb{Z}/n)/m \cdot \mathbb{Z}/n \cong \mathbb{Z}/\gcd(m,n)$)

More generally: $\begin{cases} R/I \otimes_R R/J \cong R/(I+J) & \text{for ideals } I, J \subseteq R. \\ A \otimes_R R/J \cong A/J \cdot A \end{cases}$

Warning $\cdot \otimes A$ often not an exact functor, i.e. does not preserve exact sequences

indeed it can ruin injectivity: $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ now take $\cdot \otimes \mathbb{Z}/2$ get $0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$.

Fact $\boxed{\cdot \otimes_{\mathbb{Z}} \mathbb{Q} \text{ and } \cdot \otimes_{\mathbb{Z}} \mathbb{R} \text{ are exact functors on } \mathbb{Z}\text{-mods}}$ ← More generally $\cdot \otimes_R \operatorname{Frac}(R)$ is exact on R-mods where Frac R is fraction field, and R is an integral domain "localisation is an exact functor"

example A f.g. $\mathbb{Z}$-mod $\Rightarrow A \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1 \oplus \dots \oplus \mathbb{Z}/d_k$ some $d_i \neq 0$

$\Rightarrow A \otimes \mathbb{Q} \cong \mathbb{Q}^r \Rightarrow \boxed{r = \operatorname{rank} A = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})}$

Tensor product of chain cxes

$C_*, \tilde{C}_*$ chain cxes of R-mods

$$(C_* \otimes \tilde{C}_*)_n = \bigoplus_{i+j=n} C_i \otimes \tilde{C}_j$$

$$\boxed{\partial(x \otimes y) = (\partial x) \otimes y + (-1)^{\deg x}\, x \otimes \tilde{\partial} y}$$ "Leibniz rule"

Think of ∂ as an operator of deg $= -1$ acting from left

since ∂ "jumps over x" get $(-1)^{\deg \partial \cdot \deg x}$

Exercise $\partial \circ \partial = 0$ ← would fail without sign

$$Z_i \otimes \tilde{Z}_j \subseteq Z_{i+j}(C_* \otimes \tilde{C}_*) \text{ and } \left.\begin{matrix} Z_i \otimes \tilde{B}_j \\ B_i \otimes \tilde{Z}_j \end{matrix}\right\} \subseteq B_{i+j}(C_* \otimes \tilde{C}_*)$$

Cor $\exists$ natural maps

$$\boxed{\begin{matrix} H_i(C_*) \otimes H_j(\tilde{C}_*) \longrightarrow H_{i+j}(C_* \otimes \tilde{C}_*) \\ \sum [c_k] \otimes [\tilde{c}_k] \longmapsto \sum [c_k \otimes \tilde{c}_k] \end{matrix}}$$

FACT:

Algebraic Künneth Thm

$C_*, H_*(C_*)$ f.g. free R-mods (PID) (no assumption on $\tilde{C}_*$)

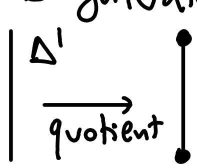
$$\Rightarrow \boxed{\bigoplus_{i+j=n} H_i(C_*) \otimes H_j(\tilde{C}_*) \cong H_n(C_* \otimes \tilde{C}_*)}$$ via

Example

$*$	$H_*(S^1)$
0	$A \cong \mathbb{Z}$
1	$B \cong \mathbb{Z}$
2	0

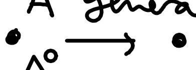
$*$	$H_*(S^1 \times S^1)$ ← torus	
0	$A \otimes A$	$\cong \mathbb{Z}$
1	$(A \otimes B) \oplus (B \otimes A)$	$\cong \mathbb{Z}^2$
2	$B \otimes B$	$\cong \mathbb{Z}$
3	0	

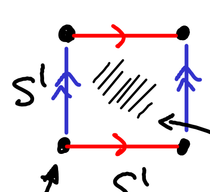
B generated by



$S^1 = \Delta^1 / \text{endpts}$

A generated by

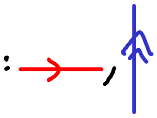


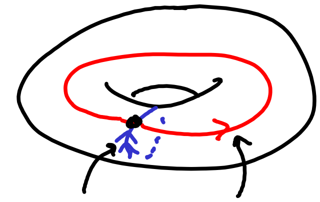


H_0 generator is •

square $\cong \Delta^2$ generator of H_2

H_1 generators:





$\in \mathbb{Z}^2$

Product spaces

X, Y CW-cxes $\Rightarrow X \times Y$ CW-cx with cells $e_\alpha \times e_\beta$ attaching maps

(e_α cells of X, e_β cells of Y)

$$\partial(\mathbb{D}^i_\alpha \times \mathbb{D}^j_\beta) = (\partial \mathbb{D}^i_\alpha) \times \mathbb{D}^j_\beta \cup \mathbb{D}^i_\alpha \times (\partial \mathbb{D}^j_\beta)$$

$$\varphi_\alpha \times id : (\partial \mathbb{D}^i_\alpha) \times \mathbb{D}^j_\beta \to X^{i-1} \times Y^j, \qquad id \times \varphi_\beta : \mathbb{D}^i_\alpha \times (\partial \mathbb{D}^j_\beta) \to X^i \times Y^{j-1}$$

$$X^{i-1} \times Y^j \to (X \times Y)^{i+j-1} \leftarrow X^i \times Y^{j-1}$$

Algebra: Euler characteristic

C finitely generated graded abelian gp (so $\mathbb{Z}$-mod)
(more generally: R-mod for PID R)

Def Euler characteristic $\chi(C) = \sum (-1)^i \operatorname{rank} C_i$

Example/Motivation X finite CW-cx then take $C = C^{CW}_*(X)$ to get

$$\boxed{\chi(X) = \#(0\text{-cells}) - \#(1\text{-cells}) + \#(2\text{-cells}) - \cdots}$$

Lemma If C_* f.g. chain cx $\Rightarrow$ $\boxed{\chi(C_*) = \chi(H_*(C_*))}$ $\left(= \sum (-1)^i \operatorname{rank} H_i(C)\right)$

Pf Observation: $\operatorname{rank} C_i = \dim_{\mathbb{Q}}(C_i \otimes_{\mathbb{Z}} \mathbb{Q})$ $\leftarrow$ for R-mods, do $\dim_{\mathbb{F}}(C_i \otimes_R \mathbb{F})$ with $\mathbb{F} = \operatorname{Frac}(R)$

$\Rightarrow$ WLOG assume C_i are vector spaces/field $\mathbb{F}$.

Abbreviate $|C_i| = \dim_{\mathbb{F}} C_i$.

$0 \to Z_i \to C_i \xrightarrow{\partial} B_{i+1} \to 0 \Rightarrow |C_i| = |Z_i| + |B_{i+1}|$ (Rank-nullity thm)

$0 \to B_i \to Z_i \to H_i \to 0 \Rightarrow |H_i| = |Z_i| - |B_i|$

$\Rightarrow |C_i| - |H_i| = |B_{i+1}| - |B_i|$

$$\Rightarrow \chi(C_*) - \chi(H_*) = \sum (-1)^i |B_{i+1}| - \sum (-1)^i |B_i| = \sum (-1)^i (-|B_i| + |B_i|) = 0. \quad \square$$

Cor X space $\Rightarrow$

$$\boxed{\begin{aligned} \chi(X) &= \sum (-1)^i \operatorname{rank} H_i(X) && \leftarrow \text{if finite rank } H_*(X) \\ &= \sum (-1)^i \operatorname{rank} C_i(X) && \leftarrow \text{if finite rank } C_*(X) \end{aligned}}$$

So $\chi(X)$ is invariant up to hpy equivalence! **Example** $\chi\left(\begin{smallmatrix}\text{platonic}\\ \text{solid}\end{smallmatrix}\right) = \chi(S^2) = 2$

Cor $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$ $\forall$ finite CW-cxes X, Y

Pf $\sum (-1)^k \operatorname{rank} H^{CW}_k(X \times Y) = \sum (-1)^k \operatorname{rank} C^{CW}_k(X \times Y)$

$$= \sum_k \sum_{i+j=k} (-1)^{i+j} \operatorname{rank} C^{CW}_i(X) \cdot \operatorname{rank} C^{CW}_j(Y). \quad \square$$

Lemma $d(e^i_\alpha \times e^j_\beta) = (d e^i_\alpha) \times e^j_\beta + (-1)^i e^i_\alpha \times (d e^j_\beta)$

hence $\boxed{C^{CW}_*(X \times Y) \cong C^{CW}_*(X) \otimes C^{CW}_*(Y)}$

(hence if $H_*(Y)$ free then by Künneth $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$)

Pf $(\partial \mathbb{D}^i_\alpha)\times \mathbb{D}^j_\gamma \xrightarrow{\varphi_\alpha \times id} X^{i-1}\times Y^j \longrightarrow X^{i-1}\times Y^j \Big/ \underbrace{(X\times Y)^{i+j-2}\cap (X^{i-1}\times Y^j)}$

$\uparrow$ This proof is Non-examinable

$\bigstar := $

$\| \leftarrow$ easy check

$X^{i-2}\times Y^j \cup X^{i-1}\times Y^{j-1}$

$X^{i-1} = X^{i-2} \cup (\mathbb{D}^{i-1}_\beta \cup \dots)/\sim$

$Y^j = Y^{j-1} \cup (\mathbb{D}^j_\gamma \cup \dots)/\sim$

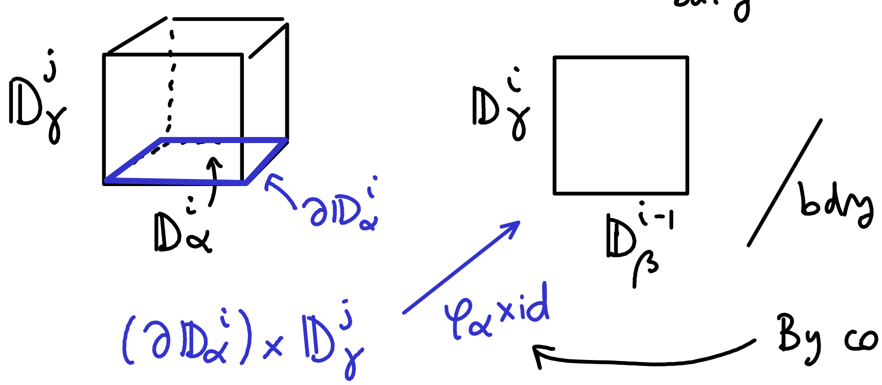
} get $\sim$ from attaching maps

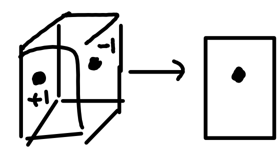
$X^{i-1}\times Y^j = X^{i-2}\times Y^j \cup X^{i-1}\times Y^{j-1} \cup (\mathbb{D}^{i-1}_\beta \times \mathbb{D}^j_\gamma \cup \dots)/\sim$

$\Rightarrow \bigstar = (\mathbb{D}^{i-1}_\beta \times \mathbb{D}^j_\gamma \cup \dots)$ / boundaries

$= \mathbb{D}^{i-1}_\beta \times \mathbb{D}^j_\gamma \Big/ \partial(\mathbb{D}^{i-1}_\beta \times \mathbb{D}^j_\gamma) \quad \vee \quad \dots$

$(\partial \mathbb{D}^i_\alpha)\times \mathbb{D}^j_\gamma \xrightarrow{\varphi_\alpha \times id} \mathbb{D}^{i-1}_\beta \times \mathbb{D}^j_\gamma \big/ \text{bdry} \quad \vee \dots$





e.g.

By considering local degrees now we see we get degree $= d_{\alpha\beta}$ for this.

$\Rightarrow$ get contribution $(de^i_\alpha)\times e^j_\beta$ ✓

similarly

$\mathbb{D}^i_\alpha \times \partial \mathbb{D}^j_\gamma \xrightarrow{id\times\varphi_\gamma} \mathbb{D}^i_\alpha \times \mathbb{D}^{j-1}_\delta \big/ \text{bdry}$

$\Rightarrow$ degree $(-1)^i d_{\gamma\delta}$

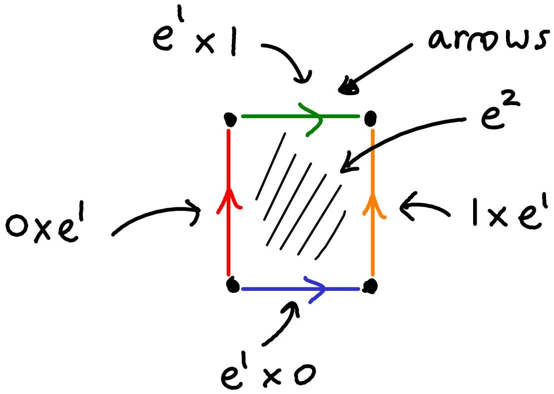
so get $(-1)^i e^i_\alpha \times de^j_\beta$

$(-1)^i$ caused by orientations:

Could reorder factors: $\mathbb{D}^i_\alpha \times \mathbb{D}^j_\gamma \cong \mathbb{D}^j_\gamma \times \mathbb{D}^i_\alpha$ by $\begin{pmatrix} 0 & Id_j \\ Id_i & 0 \end{pmatrix}$ whose det $= (-1)^{ij}$. Then $\partial\mathbb{D}^j_\gamma \times \mathbb{D}^i_\alpha \to \mathbb{D}^{j-1}_\delta \times \mathbb{D}^i_\alpha/\text{bdry}$ gives degree $d_{\gamma\delta}$.

Swap factors $\mathbb{D}^{j-1}_\delta \times \mathbb{D}^i_\alpha/\text{bdry}$ by $\begin{pmatrix} 0 & Id_i \\ Id_{j-1} & 0 \end{pmatrix}$, det $= (-1)^{i(j-1)}$. Total sign $= (-1)^i$.

Example Recall after definition of H_*^{cw} we had example $I \times I$:



$$\partial e^2 = + e^1 \times 0 + 1 \times e^1 - e^1 \times 1 - 0 \times e^1$$

$$= (\partial e^1) \times e^1 - e^1 \times (\partial e^1) \quad \checkmark$$

($-$ is $(-1)^{\dim e^1}$)

A further comment on orientation sign $(-1)^i$

$$\mathbb{D}^i \times \mathbb{D}^j \cong \Delta^i \times \Delta^j$$

$\Delta^i = [v_0, .., v_i]$, $\Delta^j = [w_0, .., w_j]$ ← viewed in $\mathbb{R}^i, \mathbb{R}^j$

project $\mathbb{R}^{i+1} \to \mathbb{R}^i$
$(t_0, .., t_i) \mapsto (t_1, .., t_i)$

$$\partial(\mathbb{D}^i \times \mathbb{D}^j) \cong \partial\Delta^i \times \Delta^j \cup \Delta^i \times \partial \Delta^j$$

$\partial\Delta^i = \sum_k (-1)^k [v_0, \dots, \hat{v}_k, \dots, v_i]$, $\quad \partial\Delta^j = \sum (-1)^k [w_0, .., \hat{w}_k, .., w_j]$

$(-1)^k$ would be correct orientation sign for basis $w_1 - w_0, .., \widehat{w_k - w_0}, .., w_j - w_0$ but actually we have $[v_0, .., v_i] \times [w_0, .., \hat{w}_k, .., w_j] \subseteq \mathbb{R}^i \times \mathbb{R}^j$

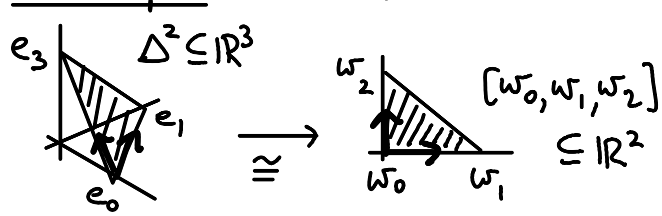
and $(-1)^{i+k}$ is the orientation sign for the basis

$$v_1 - v_0, .., v_i - v_0, w_1 - w_0, .., \widehat{w_k - w_0}, .., w_j - w_0$$

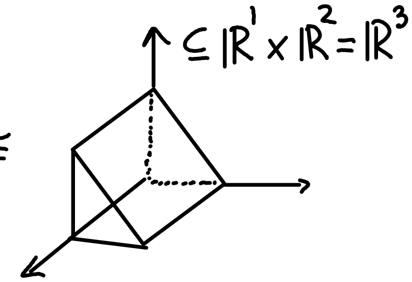
for the hyperplane in $\mathbb{R}^{i+j+1}$ containing $[v_0, .., v_i] \times [w_0, .., \hat{w}_k, .., w_j]$

$\Rightarrow$ need $(-1)^i$ to fix orientation sign.

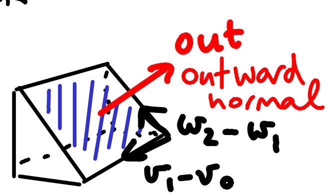
Example $\Delta^1 \times \Delta^2$



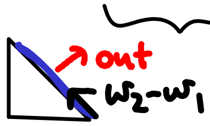
$\Delta^1 \times \Delta^2 \cong$



$[v_0, v_1] \times [\hat{w}_0, w_1, w_2]$



out, $v_1 - v_0$, $w_2 - w_1$ is negative $\mathbb{R}^3$-basis



out, $w_2 - w_1$ is positive $\mathbb{R}^2$-basis ← differ due to $(-1)^i$, $i = 1$.

Projections $X \times Y \xrightarrow{p_X} X$, $X \times Y \xrightarrow{p_Y} Y$

FACT:

Künneth Theorem If $H_n(Y)$ finitely generated, free $\forall n$

(no conditions on X; e.g. $Y \simeq$ finite CW cx; automatic if use field coefficients)

$H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y)$	$H^n(X \times Y) \cong \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y)$ $p_X^* a \cup p_Y^* b \longleftarrow a \otimes b \quad (\star)$

Recall for cellular homology this on generators is:
$e^i_\alpha \times e^j_\beta \longleftarrow e^i_\alpha \otimes e^j_\beta$

This is hom of rings if use following product
$(a \otimes b) \cdot (\tilde{a} \otimes \tilde{b}) = (-1)^{|b| \cdot |\tilde{a}|} (a \cup \tilde{a}) \otimes (b \cup \tilde{b})$
(gradings) think of it as "exchanging order of $b, \tilde{a}$"

Rmk
An indirect proof the Thm is to write down two generalised cohomology theories $F(X,A) = H^*(X,A) \otimes H^*(Y)$ and $G(X,A) = H^*(X \times Y, A \times Y)$, and consider the natural transformation $\alpha : F \to G$ given by $(\star)$, notice for $X = pt$, $A = \emptyset$ both F, G give $H^*(Y)$.

Example $X = S^n$, $Y = S^m$ $\quad n \neq m$

$H_*(S^n \times S^m) \cong \begin{cases} \mathbb{Z} & * = 0, n, m, n+m \\ 0 & \text{else} \end{cases} \cong H^*(S^n \times S^m)$

$a_n \cup a_m = a_{n+m}$ where $a_i := \text{dual}(e^i)$

$H_*(S^n \times S^n) \cong \begin{cases} \mathbb{Z} & * = 0, 2n \\ \mathbb{Z}^2 & * = n \\ 0 & \text{else} \end{cases} \cong H^*(S^n \times S^n)$

$a_n \cup a_n = a_{2n}$

Cor $H^k(T^n) \cong \Lambda^k[x_1, \ldots, x_n]$ (n-torus $S^1 \times \cdots \times S^1$; exterior algebra)

$= $ free abelian gp. on gens. $\{x_{i_1} \wedge \cdots \wedge x_{i_k} : i_1 < \cdots < i_k\}$

so rank $= \binom{n}{k}$

where $x_i = p_i^*(\text{gen. of } H^1(S^1))$

$p_i : T^n \to S^1$ projections to factors.

Pf idea Künneth & induction $(T^n = T^{n-1} \times S^1)$ $\square$

FACT cup product equals composition

$\cup : H^i(X) \otimes H^j(X) \longrightarrow H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$

$(\Delta^i \xrightarrow{\sigma_1} X) \otimes (\Delta^j \xrightarrow{\sigma_2} X) \longmapsto (\Delta^i \times \Delta^j \xrightarrow{\sigma_1 \times \sigma_2} X \times X)$, $\quad \Delta^i \times \Delta^j \simeq \Delta^{i+j}$

$\Delta =$ diagonal map
$X \longrightarrow X \times X$
$x \longmapsto (x, x)$

12. UNIVERSAL COEFFICIENTS THEOREM

↙ This proof is Non-examinable

(C_*, ∂_*) chain cx

$\Rightarrow 0 \to Z_* = \operatorname{Ker}\partial_* \xrightarrow{\text{incl}} C_* \xrightarrow{\partial_*} B_{*-1} = \operatorname{Im}\partial_{*-1} \to 0$ is SES

($\circlearrowleft \partial = 0$ on Z_*; $\partial = 0 \circlearrowleft$ on B_{*-1})

> **FACT**: Submodules of a free $\mathbb{Z}$-module are free
>
> Rmk The same holds for R-mods if R is PID

($\mathbb{Z}$-module $\equiv$ abelian gp; free means: $\bigoplus_{\text{indexing set}} \mathbb{Z}$)

(**PID** = principal ideal domain = integral domain R s.t. every ideal $= R\cdot a$ some a)

Assume C_* free $\mathbb{Z}$-mod

$\overset{\text{FACT}}{\Rightarrow}$ Z_*, B_* free (as $\operatorname{Ker}\partial^*, \operatorname{Im}\partial^*$ are submods of C_*)

$\Rightarrow$ SES splits, choose splitting $C_* \underset{s}{\overset{\partial_*}{\rightleftarrows}} B_{*-1}$ so $\partial_* \circ s = id$

$\Rightarrow$ dual SES

$$0 \leftarrow Z^* \xleftarrow{\text{incl}^*} C^* \xleftarrow{\partial^*} B^{*-1} \leftarrow 0$$

note: incl^* = restrict to Z_* since $\text{incl}^* \circ \phi : Z_* \xrightarrow{\text{incl}} B_* \xrightarrow{\phi} \mathbb{Z}$

$$\begin{array}{ccccccccc} 0 & \leftarrow & Z^n & \leftarrow & C^n & \xleftarrow{\partial^*} & B^{n-1} & \leftarrow & 0 \\ & & \uparrow \partial=0 & & \uparrow \partial & & \uparrow \partial = 0 & & \\ 0 & \leftarrow & Z^{n-1} & \leftarrow & C^{n-1} & \xleftarrow{\partial^*} & B^{n-2} & \leftarrow & 0 \end{array}$$

(Rmk Although $\partial^* = 0 : B^* \to B^{*+1}$ the map $\partial^* : B^{n-1} \to C^n$ need not $= 0$
$\varphi : B_{n-1} \to \mathbb{Z}$
$\Rightarrow \partial^*\varphi = \varphi \circ \partial : C_n \xrightarrow{\partial} B_{n-1} \xrightarrow{\varphi} \mathbb{Z}$)

Connecting map $\delta : Z^{n-1} \to B^{n-1}$ of LES:

$$\begin{array}{ccccc} 0 & & \partial^*\varphi & \xleftarrow{\partial^*} & \varphi|_{B_*} = \phi|_{B_*} \\ \uparrow & & \uparrow & & \\ \varphi|_{Z_*} = \phi & \longleftarrow & \exists\, \varphi & & \end{array}$$

($B_* \subseteq Z_*$) $\Rightarrow$ $\boxed{\delta(\phi) = \phi|_{B_*}}$

LES $\Rightarrow$ $\cdots \leftarrow Z^n \leftarrow H^nC \xleftarrow{\partial^*} B^{n-1} \xleftarrow{\delta^{n-1}} Z^{n-1}$

($\phi|_{B_{n-1}} \longleftarrow\!\shortmid \phi$)

($H^*B = B^*, H^*C = C^*$ since $\partial^* = 0$)

$\Rightarrow 0 \leftarrow \operatorname{Ker}\delta^n \leftarrow H^nC \leftarrow B^{n-1}/\operatorname{Im}\delta^{n-1} \leftarrow 0$

$\operatorname{Ker}\delta^n = \{\phi \in Z^n : \phi(B_n) = 0\} \Rightarrow$ so: $\phi : Z_n \to \mathbb{Z}$ factors through $Z_n/B_n = H_n(C_*)$

$= \operatorname{Hom}(H_n(C_*), \mathbb{Z})$

> ✪ **Universal Coefficients Thm**:
>
> $0 \to B^{n-1}/\operatorname{Im}\delta^{n-1} \to H^n(C) \to \operatorname{Hom}(H_n(C_*), \mathbb{Z}) \to 0$ is SES
>
> $B^{n-1}/\operatorname{Im}\delta^{n-1} = \operatorname{Ext}^1(H_{n-1}(C); \mathbb{Z})$ (see next Lemma)
>
> $[\varphi] \longmapsto (\varphi : H_n(C_*) \to \mathbb{Z})$

and SES splits (but not naturally): $B^{n-1}/\operatorname{Im}\delta^{n-1} \underset{s^*}{\overset{\partial^*}{\rightleftarrows}} H^n(C)$ $\quad s^* \circ \partial^* = id$

(since $\partial \circ s = id \Rightarrow id = (\partial \circ s)^* = s^* \circ \partial^*$)

$\Rightarrow H^n(C) \cong \operatorname{Hom}(H_n(C), \mathbb{Z}) \oplus \operatorname{Ext}^1(H_{n-1}(C); \mathbb{Z})$

Lemma $\mathrm{Ext}^1(H_{n-1}(C);\mathbb{Z}) \cong B^{n-1}/\mathrm{Im}\,\delta^{n-1}$ canonically

Algebra background on Extension groups $\mathrm{Ext}^i(M;\mathbb{Z})$

general case

M R-module, R ring (comm. with 1)

$\Rightarrow \exists$ **free resolution**:

$$\cdots \longrightarrow P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \longrightarrow 0 \quad \text{exact, } P_i \text{ free } R\text{-mods}$$

(pick gens x_α for M $\Rightarrow P_0 = \bigoplus_\alpha R \xrightarrow{\varphi_0} M$, $e_\alpha \mapsto x_\alpha$

" " y_β for $\mathrm{Ker}\,\varphi_0 \Rightarrow P_1 = \bigoplus_\beta R \xrightarrow{\varphi_1} \mathrm{Ker}\,\varphi_0$, $e_\beta \mapsto y_\beta$

continue inductively)

our case

$H_{n-1}(C_*)$ $\mathbb{Z}$-mod

$$0 \to B_{n-1} \hookrightarrow Z_{n-1} \to H_{n-1}(C) \to 0$$

with $B_{n-1} = P_1$, $Z_{n-1} = P_0$, $H_{n-1}(C) = M$.

Take $\mathrm{Hom}(\cdot\,;\mathbb{Z})$ and drop $\mathrm{Hom}(M;\mathbb{Z})$

$$0 \longrightarrow \mathrm{Hom}(P_0;\mathbb{Z}) \xrightarrow{\varphi_1^*} \mathrm{Hom}(P_1;\mathbb{Z}) \xrightarrow{\varphi_2^*} \cdots$$

Is cochain complex but not exact

$\Rightarrow$ take cohomology groups:

Def $\mathrm{Ext}^0(M;\mathbb{Z}) = \mathrm{Ker}\,\varphi_1^*$

$\mathrm{Ext}^1(M;\mathbb{Z}) = \mathrm{Ker}\,\varphi_2^* / \mathrm{Im}\,\varphi_2^*$

$\cdots$

Fact: independent of choices P_i, φ_i

Example 1 $\mathrm{Ext}^0(M;\mathbb{Z}) \cong \mathrm{Hom}(M,\mathbb{Z})$

$$\begin{array}{ccccc} P_1 & \xrightarrow{\varphi_1} & P_0 & \xrightarrow{\varphi_0} & M \\ & {\scriptstyle 0}\searrow & \downarrow \phi & \swarrow & \\ & & \mathbb{Z} & & \end{array}$$

descends: $m \longmapsto \phi(\varphi_0^{-1} m)$

well defined since $\phi(\mathrm{Ker}\,\varphi_0) = 0$, where $\mathrm{Ker}\,\varphi_0 = \mathrm{Im}\,\varphi_1$

Example 2 $\mathrm{Ext}^1(M;\mathbb{Z}) =$

$$\left\{\phi : \begin{array}{ccccc} P_2 & \to & P_1 & \to & P_0 \\ & {\scriptstyle 0}\searrow & \downarrow \phi & & \\ & & \mathbb{Z} & & \end{array}\right\} \Big/ \left\{\phi = \varphi \circ \varphi_1 : \begin{array}{ccc} P_1 & \xrightarrow{\varphi_1} & P_0 \\ & {\scriptstyle \phi}\searrow & \downarrow \varphi \\ & & \mathbb{Z} \end{array}\right\}$$

$$0 \to B^{n-1} \longrightarrow Z^{n-1} \longrightarrow 0$$

Proof of Lemma

By **Example 2**,

$\mathrm{Ext}^1(H_{n-1}(C_*);\mathbb{Z}) =$

$$= \left\{\begin{array}{ccc} 0 \to B_{n-1} & \to & Z_{n-1} \\ \phi \downarrow & & \\ \mathbb{Z} & & \end{array}\right\} \text{ modulo}$$

those arising from restriction

$$\left\{\begin{array}{ccc} 0 \longrightarrow B_{n-1} & \longrightarrow & Z_{n-1} \\ & {\scriptstyle \phi|_{B_{n-1}}}\searrow & \downarrow \phi \\ & & \mathbb{Z} \end{array}\right\}$$

Thus $B^{n-1}/\mathrm{Im}\,\delta^{n-1}$. $\square$

Rmk If R PID, then $\mathrm{Ker}(P_0 \to M)$ is free (since submod of free mod P_0)

$\Rightarrow$ can pick $P_1 = \mathrm{Ker}(P_0 \to M)$, $P_k = 0$ for $k \geq 2$ $\Rightarrow \mathrm{Ext}^k(M;\mathbb{Z}) = 0$ $k \geq 2$

(Co)homology with coefficients in a ring/field/module

Motivation

So far we had (C_*, ∂_*) (← group hom) chain cx of abelian groups } in graded sense

$\Rightarrow H_*(C_*) = \operatorname{Ker}\partial_* / \operatorname{Im}\partial_*$ abelian group (since Ker ∂, Im ∂ are)

We cannot use a chain cx of (non-abelian) groups, because $\operatorname{Im}\partial_*$ need not be a normal subgroup of $\operatorname{Ker}\partial_*$.

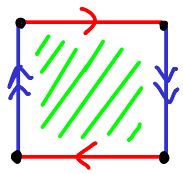
However, abelian groups can be thought of as $\mathbb{Z}$-modules, then given any **abelian group G**, define **homology with coeffs in G**

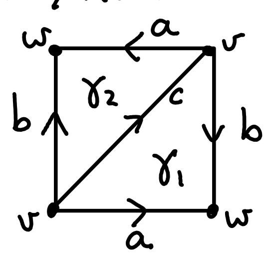
$$\boxed{H_*(C_*;G) = H_*(C_* \otimes_{\mathbb{Z}} G)}$$ ← with differential $\partial_* \otimes \mathrm{id}$

Def X space $\Longrightarrow$ $\boxed{H_*(X;G) = H_*(C_*(X) \otimes_{\mathbb{Z}} G)}$

Explanation:
$C_k(X)$ free $\mathbb{Z}$-mod $\cong \bigoplus_{I_k} \mathbb{Z} \Rightarrow C_k(X) \otimes_{\mathbb{Z}} G \cong \bigoplus_{I_k} G$: just replace $\mathbb{Z}$ by G (as $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot \cong \cdot$)

Why care? We hope to get more/new invariants of spaces

Example $X = \mathbb{RP}^2 =$ 



$*$	$C^\Delta_*(\mathbb{RP}^2; G)$
0	$Gv \oplus Gw$
1	$Ga \oplus Gb \oplus Gc$
2	$G\gamma_1 \oplus G\gamma_2$

for $G = \mathbb{Z}/2$: $0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow[\begin{pmatrix}1&1\\1&1\\1&1\end{pmatrix}]{\partial_2} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow[\begin{pmatrix}1&1&0\\1&1&0\end{pmatrix}]{\partial_1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0$

$$\Rightarrow H_*(\mathbb{RP}^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

compare: $H_*(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ 0 & \text{else} \end{cases}$ $(G = \mathbb{Z}$ case$)$

Form cochain complex using $\operatorname{Hom}_{\mathbb{Z}}(\cdot, G)$ (= group homs to G) in place of $\operatorname{Hom}(\cdot, \mathbb{Z})$

$$\boxed{\begin{aligned} H^*(C_*;G) &= H_*(\operatorname{Hom}_{\mathbb{Z}}(C_*, G)) \\ H^*(X;G) &= H^*(\operatorname{Hom}_{\mathbb{Z}}(C_*(X); G)) \end{aligned}}$$

(X space → $H^*(X;G)$; with differential ∂^*: $\partial^*\phi = \phi \circ \partial_*$; so: $H^*(C_*(X); G)$)

Universal coefficients thm (same proof using $\operatorname{Hom}_{\mathbb{Z}}(\cdot, G)$)

$$\boxed{\begin{aligned} 0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(C_*); G) \to H^n(C_*;G) &\longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_n(C_*), G) \to 0 \\ [\varphi] &\longmapsto (\varphi : H_n(C_*) \to G) \end{aligned}}$$

Example $X=\mathbb{RP}^2$, $G=\mathbb{Z}/2$, apply $\mathrm{Hom}_{\mathbb{Z}}(\cdot,G)$ to the complex in previous example

$$0 \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow[\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}]{\partial^1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow[\begin{pmatrix}1&1\\1&1\\0&0\end{pmatrix}]{\partial^0} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \leftarrow 0$$

$$\Rightarrow H^*(\mathbb{RP}^2;\mathbb{Z}/2) \cong \begin{cases}\mathbb{Z}/2 & *=0\\ \mathbb{Z}/2 & *=1\\ \mathbb{Z}/2 & *=2\\ 0 & \text{else}\end{cases}$$

compare: $H^*(\mathbb{RP}^2) \cong \begin{cases}\mathbb{Z} & *=0\\ 0 & *=1\\ \mathbb{Z}/2 & *=2\\ 0 & \text{else}\end{cases}$ ($G=\mathbb{Z}$ case)

Can generalise further:

C_* = chain cx of ...	coefficients in:	
abelian gps ($\mathbb{Z}$-mods)	abelian gp G ($\mathbb{Z}$-mod)	$H_*(C_*;G) = H_*(C_* \otimes_{\mathbb{Z}} G)$
R-modules ($\nwarrow$ ring (comm. with 1))	R-module M	$H_*(C_*;M) = H_*(C_* \otimes_R M)$

Rmk $H_*(C;M)$ will be an R-module since $\ker\partial, \mathrm{Im}\,\partial$ are (∂_* is R-linear hom by assumption)

X space $\Rightarrow C_k(X;R) = C_k(X) \otimes_{\mathbb{Z}} R \cong \bigoplus_{I_k} R$: just replace $\mathbb{Z}$ by R (as $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot \cong \cdot$)

$$\Rightarrow \boxed{H_*(X;M) = H_*(C_*(X;R);M) = H_*(C_*(X;R) \otimes_R M)}$$

so just replace each $\mathbb{Z}$ by M in $C_*(X)$

Form cochain complex using $\mathrm{Hom}_R(\cdot,M)$ (= R-linear homs to M) in place of $\mathrm{Hom}(\cdot,\mathbb{Z})$ with differential ∂^*: $\partial^*\phi = \phi \circ \partial_*$

$$\boxed{\begin{aligned} H^*(C_*;M) &= H_*(\mathrm{Hom}_R(C_*,M)) \\ H^*(X;M) &= H^*(\mathrm{Hom}_R(C_*(X;R);M)) \end{aligned}}$$

(X space) so: $H^*(C_*(X;R);M)$

Rmk These are R-mods. If we use $M=R$, then they are also rings via cup product

> **Universal Coefficients Thm** For R any PID, C_* chain cx of R-mods,
>
> $$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*);M) \to H^n(C_*;M) \to \mathrm{Hom}_R(H_n(C_*),M) \to 0 \text{ is SES}$$
>
> $\mathrm{Ext}^1_R(H_{n-1}(C_*);M) \cong B^{n-1}/\mathrm{im}\,\delta^{n-1}$ working over R using homs to M
>
> $[\varphi] \longmapsto (\varphi : H_n(C_*) \to M)$
>
> and the SES splits but the splitting is not natural.
>
> Same proof using $\mathrm{Hom}_R(\cdot,M)$

Example $R = \mathbb{F}$ field $\Rightarrow C_*, H_*, H^*$ are vector spaces $/\mathbb{F}$.

Rmk all $\mathbb{F}$-mods (i.e. vector spaces$/\mathbb{F}$) are free $\mathbb{F}$-mods $\cong \bigoplus \mathbb{F} b_i$ (basis b_i) up to iso they are determined by $\dim_{\mathbb{F}}$ = cardinality of basis.

Cor C_* = chain cx of $\mathbb{F}$-vector spaces $\Rightarrow$ $\boxed{H^n(C_*;\mathbb{F}) \cong (H_n(C_*))^*}$ ← dual v.s.: $\mathrm{Hom}_{\mathbb{F}}(H_n(C_*);\mathbb{F})$

Pf Pick any basis v_i for $\mathbb{F}$-v.s. B_{n-1}, extend it to a basis v_i, w_j of Z_{n-1} (also works in ∞ dim case).

$\Rightarrow$ can extend any $\mathbb{F}$-linear map $\psi: B_{n-1} \to \mathbb{F}$ to $\phi: Z_{n-1} \to \mathbb{F}$ just pick any values $\phi(w_j) \in \mathbb{F}$ e.g. $\phi(w_j) = 0$.

$\Rightarrow$ $B^{n-1}/\mathrm{im}\,\delta^{n-1} = 0$ so $H^n(C_*;\mathbb{F}) \longrightarrow \mathrm{Hom}(H_n(C_*);\mathbb{F})$ iso □

Cor $\boxed{H^n(X;\mathbb{F}) \cong H_n(X;\mathbb{F})^*}$ ← dual v.s. for any field $\mathbb{F}$.

Cor $\boxed{H^n(X;M) \cong H^n_{CW}(X;M) \cong H^n_\Delta(X;M)}$

(first ≅: if $X \simeq$ CW-cx; second ≅: if $X \simeq \Delta$-cx)

Pf Cor holds for homology and the isos are natural. ← i.e. functorial w.r.t. maps

The universal coeff. thm SES is natural. So result holds by 5-Lemma. □

Algebra: structure thm for f.g. abelian groups and R-mods

Fact 1 A f.g. abelian gp $\Longrightarrow$ $A \cong \mathbb{Z}^r \oplus \mathbb{Z}/p_1^{n_1} \oplus \cdots \oplus \mathbb{Z}/p_k^{n_k}$

($\mathbb{Z}^r$: free part F; $\mathbb{Z}/p_1^{n_1} \oplus \cdots \oplus \mathbb{Z}/p_k^{n_k}$: torsion part T)

where $p_i \in \mathbb{Z}$ prime (need not be distinct)

Also r, k, p_i, n_i are unique (up to reordering)

Example $\mathbb{Z}/4 = \mathbb{Z}/2^2 \not\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$

$\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$ (Chinese Remainder Thm)

Fact 2 $T \cong \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_k$ with $d_1 | d_2 | \cdots | d_k$ ($d_i \in \mathbb{N}$ unique)

Example $\mathbb{Z}/2 \oplus \mathbb{Z}/2^2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3\cdot 2^2$ $d_1 = 2$, $d_2 = 12$

Fact 3 M f.g. R-mod, R PID, then:

$$\boxed{\begin{aligned} M &\cong F \oplus T \\ F &\cong R^r \\ T &\cong R/d_1 \oplus R/d_2 \oplus \cdots \oplus R/d_k \end{aligned}}$$

$r \in \mathbb{N}$ unique, called **rank** of M

$d_1 | \cdots | d_k$ non-zero, not invertible

d_i called **invariant factors**

unique up to mult^n by invertible elements ← e.g. ± 1 if $R = \mathbb{Z}$

Rmk $T = \{m \in M : rm = 0 \text{ some } r \neq 0 \in R\}$ = **torsion elements**

$F \cong M/T$

Torsion shift

Easy Exercise

$$\mathrm{Ext}^*_R\Big(\bigoplus_i M_i ; \prod_j N_j\Big) \cong \prod_i \prod_j \mathrm{Ext}^*_R(M_i ; N_j)$$ ← any R-mods M_i, N_j

↓

Upshot To compute $\mathrm{Ext}^1_R(M;R)$ for $M = R^r \oplus R/_{d_1} \oplus \cdots$ just need:

$$\mathrm{Ext}^1_R(R;R) = 0$$ ← since $0 \to R \xrightarrow{id} R \to 0$, with $P_1 = R$, $P_0 = R$

$$\mathrm{Ext}^1_R(R/d;R) \cong R/d$$ ← since $0 \to R \xrightarrow{d} R \xrightarrow{1} R/d \to 0$, $\phi: R \to R$, so choice of $\phi(1) \in R$ modulo ϕ coming from

$$\begin{array}{ccc} R & \xrightarrow{d} & R \\ \phi \downarrow & \swarrow \varphi & \\ R & & \end{array}$$

so $\phi(1) = d \cdot \varphi(1) \in d \cdot R$

$$\Longrightarrow \boxed{\mathrm{Ext}^1_R(M;R) \cong \mathrm{Torsion}(M)}$$

Exercises

- $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n ; \mathbb{Z}/m) \cong \mathbb{Z}/\gcd(m,n)$
- G abelian gp $\Rightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z};G) = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/d;G) \cong G/_{d\cdot G}$ ($d \neq 0$)
- R any ring (comm. with 1), $x \in R$ not zero divisor $\Rightarrow$ $\mathrm{Ext}^*_R(R/_{(x)} ; N) \cong \begin{cases} \{n \in N : x\cdot n = 0\} & * = 0 \\ N/x\cdot N & * = 1 \\ 0 & \text{else} \end{cases}$ (N an R-mod)

Cor If $H_n(X;R)$ f.g. R-mod $\forall n$, R PID,

$\Longrightarrow H_n(X;R) = R^{r_n} \oplus T_n$ (free & torsion parts)

$$\Longrightarrow \boxed{H^n(X;R) \cong R^{r_n} \oplus T_{n-1}}$$ ← torsion moves up!

($\cong$: not natural)

Pf $0 \to T_{n-1} \to H^n(X;R) \to \mathrm{Hom}(R^{r_n} \oplus T_{n-1} ; R) \to 0$

$\mathrm{Hom}(R^{r_n} \oplus T_{n-1} ; R) \cong (\mathrm{Hom}(R;R))^{r_n} \oplus \mathrm{Hom}(T_{n-1};R)$

- $(\mathrm{Hom}(R;R))^{r_n} \cong R^{r_n}$: $R \to R$, $1 \mapsto x$; x determines the hom
- $\mathrm{Hom}(T_{n-1};R) = 0$ since $T_{n-1} \to R$, $1 \mapsto 0$ (R is integral domain, so no torsion elts $\neq 0$)

$\Longrightarrow 0 \to T_{n-1} \to H^n(X;R) \to R^{r_n} \to 0$

R^{r_n} free, so can split the SES (pick lifts of basis) — so not canonical. □

Example

$*$	$H_*(\mathbb{R}P^3)$	$H^*(\mathbb{R}P^3)$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/2$	0
2	0	$\mathbb{Z}/2$
3	$\mathbb{Z}$	$\mathbb{Z}$

($\mathbb{Z}/2$ in H_1 → $\mathbb{Z}/2$ in H^2): torsion moves up

Universal coefficients Theorem in homology

FACT **Theorem** C_* chain cx of free R-mods (R ← PID), M R-module

$\Rightarrow$ SES $0 \to H_*(C_*) \otimes_R M \to H_*(C_* \otimes_R M) \to \mathrm{Tor}_1^R(C_{*-1}; M) \to 0$ ← defined below.

$[c] \otimes m \mapsto [c \otimes m]$

The SES splits, but the splitting is not natural.

Torsion groups: A, B R-mods (R comm. ring with 1)

pick $\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} A \to 0$ free resolution (exact sequence, P_i free R-mods)

take $\cdot \otimes B$, omit $A \otimes B$ $\Longrightarrow$ $\cdots \to P_2 \otimes B \xrightarrow{\varphi_2 \otimes id} P_1 \otimes B \xrightarrow{\varphi_1 \otimes id} P_0 \otimes B \to 0$ not exact but is chain cx

$\mathrm{Tor}_k^R(A,B) = H_k(\text{this complex})$ ← fact: independent of choices of P_i, φ_i

Rmk R PID $\Rightarrow$ $\ker \varphi_0$ free $\Rightarrow$ can pick $P_1 = \ker \varphi_0$, $P_k = 0$ for $k \geq 2$

$\Rightarrow$ only $\mathrm{Tor}_0^R, \mathrm{Tor}_1^R$ can be non-zero

Example $\mathrm{Tor}_*^{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}/b) = ?$

$0 \to \mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z} \xrightarrow[\text{quotient}]{\varphi_0} \mathbb{Z}/a \to 0$ free resolution

take $\cdot \otimes \mathbb{Z}/b$, drop $\mathbb{Z}/a \otimes \mathbb{Z}/b$ $\Longrightarrow$ $0 \to \mathbb{Z}/b \xrightarrow{\cdot a} \mathbb{Z}/b \to 0$ (since $\mathbb{Z} \otimes_{\mathbb{Z}} G \cong G$ any G)

$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/a; \mathbb{Z}/b) = (\mathbb{Z}/b)/a\cdot\mathbb{Z}/b \cong \mathbb{Z}/\gcd(a,b) \cong \mathbb{Z}/a \otimes \mathbb{Z}/b$

$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/a; \mathbb{Z}/b) = \{x \in \mathbb{Z}/b : a \cdot x = 0\} \cong \mathbb{Z}/\gcd(a,b)$

Facts $\mathrm{Tor}_0^R(A,B) = P_0 \otimes B / \mathrm{Im}(\varphi_1 \otimes id) \cong A \otimes B$

$\mathrm{Tor}_*^R(A,B) \cong \mathrm{Tor}_*^R(B,A)$

Exercise $\mathrm{Tor}_*^R(\oplus A_i, \oplus B_j) \cong \bigoplus_i \bigoplus_j \mathrm{Tor}_*^R(A_i, B_j)$

$\mathrm{Tor}_*^R(A,B) = 0$ for $* \geq 1$ if A or B is free (use $M \otimes_R R \cong M$)

$\Downarrow$ deduce $\mathrm{Tor}_1^R(A,M)$ for f.g. R-mods A (R ← PID)

$$\mathrm{Tor}_*^R(R/u; M) \underset{R \text{ PID}}{\cong} \begin{cases} M/u\cdot M & *=0 \\ u\text{-torsion}(M) = \{x \in M : u\cdot x = 0\} & *=1 \\ 0 & \text{else} \end{cases}$$

Example $H_*(\mathbb{RP}^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & *=0 \\ \mathbb{Z}/2 & *=1 \\ \mathbb{Z}/2 & *=2 \end{cases}$ $\quad H_*(\mathbb{RP}^2) \otimes \mathbb{Z}/2 \cong \begin{cases} \mathbb{Z} \otimes \mathbb{Z}/2 \\ \mathbb{Z}/2 \otimes \mathbb{Z}/2 \\ 0 \end{cases} \cong \begin{cases} \mathbb{Z}/2 \\ \mathbb{Z}/2 \\ 0 \end{cases}$

← caused by $\mathrm{Tor}_1^{\mathbb{Z}}(H_1(\mathbb{RP}^2); \mathbb{Z}/2) = \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$

Künneth Thm

R PID $\Longrightarrow$ natural SES: (C_* free ch. cx. R-mods, D_* any ch. cx. R-mods)

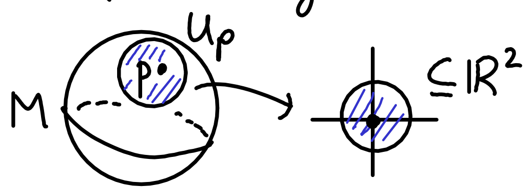
$$0 \to \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \to H_n(C_* \otimes D_*) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(H_i(C), H_j(D)) \to 0$$

and the SES splits but the splitting is not natural. **Example** R = field, then this $= 0$.

13. MANIFOLDS : POINCARÉ–LEFSCHETZ DUALITY

- M **n-mfd** is Hausdorff topological space s.t. $\forall p \in M$ $\exists$ open neighbourhood $U_p \subseteq M$ homeomorphic to $\mathbb{R}^n$

(equivalently: to an open ball, or any open set in $\mathbb{R}^n$)



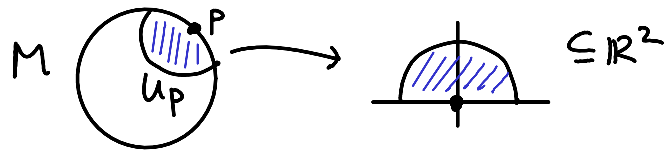
One also requires M **second countable** i.e. $\exists$ countable basis of open sets
$\Longleftrightarrow$ M is covered by countably many such U_p.
(↖ exercise)

A **submanifold** $N \subseteq M$ is a mfd s.t. inclusion $N \to M$ is an embedding (i.e. a homeomorphism onto its image)

- M **n-mfd with boundary** if also allow $U_p \cong$ **upper half space** $\mathbb{H}^n$ $= \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0} = \{x \in \mathbb{R}^n : x_n \geq 0\}$, $p \longmapsto 0$

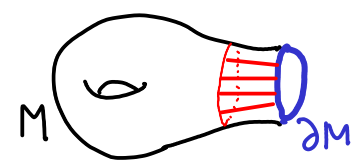
(equivalently: any open nbhd of $0 \in \mathbb{H}^n$)

Such p are called **boundary points**; they form the **boundary ∂M** which is an $(n-1)$-mfd without boundary.



FACT (collar nbhd thm) $\partial M \subseteq M$ has an open neighbourhood $\cong \partial M \times (0,1]$
$\partial M \longrightarrow \partial M \times 1$

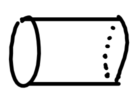
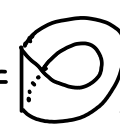
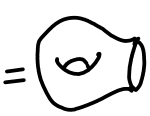
M is **closed** if compact without boundary.



Examples

closed mfds : S^n, $\mathbb{RP}^n$, $T^n = S^1 \times \cdots \times S^1$ (**n-torus**), $\mathbb{CP}^n$, $O(n)$, $SU(n)$

non-compact mfds: $\mathbb{R}^n$, $Mat_{m \times n} \cong \mathbb{R}^{mn}$, $GL(n, \mathbb{R})$

mfds with bdry: $\mathbb{D}^n$, $\mathbb{D}^1 \times S^1 =$ , Möbius band $=$ , $T^2 \setminus$ open disc $=$ 

FACT (Milnor 1959) Any mfd is homotopy equivalent to a CW–cx

fact If M is a compact manifold then $H_*(M)$ are finitely generated

Rmk M **triangulable** if $M \simeq$ simplicial cx.
Not all mfds are triangulable, but most of those we encounter are.

Compact manifolds have f.g. homology ← Non-examinable proof

① X space is a **Euclidean neighbourhood retract** if $\exists$ **embedding** $j: X \longrightarrow \mathbb{R}^N$ some N, s.t. $i(X)$ is a retract of a nbhd $V \subseteq \mathbb{R}^n$.
↖(homeo onto image)

② X is **weakly locally contractible** if $\forall$ nbhd $x \in U \subseteq X$, $\exists$ nbhd $x \in V \subseteq U$ s.t. V is contractible inside U.

FACT compact $X \subseteq \mathbb{R}^n$ is ① $\Longleftrightarrow$ X is ②

Rmk If we find nbhd V as in ① with retraction $V \xrightarrow{f} X$ then any smaller nbhd V' also retracts using $f|_{V'}: V' \to X$. Similarly in ② $V' \subseteq V$ is contractible: restrict the hpy.

Lemma A X compact & ① $\Longrightarrow$ X is the retract of a finite simplicial cx

pf $i(X) \subseteq \mathbb{R}^n$ compact $\Rightarrow$ lies inside some large n-simplex $\Delta^n \to \mathbb{R}^n$
Apply barycentric subdivision until simplices have diameter $<$ dist$(X, \partial V)$.
Simpl.cx. $= \bigcup \{$subsimplices which intersect $X\}$ using the restriction of retraction $V \to X$. □

Rmk Also deduce X has f.g. homology since retractions are surjective on H_*.
($\bigoplus \mathbb{Z} \twoheadrightarrow H_*$(finite simpl. cx) $\xrightarrow{\text{retract}} H_*(X)$ so get surjection from free $\mathbb{Z}$-mod, so f.g.)

Lemma B M compact mfd $\Longrightarrow$ M embeds into $\mathbb{R}^N$, some N.

Pf "Just do it proof":
$\forall p \in M$, $\exists$ homeo $\mathbb{D}^n \xrightarrow{\varphi_p}$ nbhd$(p \in M)$
Pick finite subcover of φ_{p_i} of $M = \bigcup_{p \in M} \varphi_p(\mathbb{D}^n)$. Say $i = 1, \ldots, k$
$\psi_{p_i}: M \xrightarrow{\varphi_{p_i}^{-1}} \mathbb{D}^n \to \mathbb{D}^n/\partial\mathbb{D}^n \cong S^n \subseteq \mathbb{R}^{n+1}$ define embedding $(\psi_{p_1}, \ldots, \psi_{p_k}): M \to \mathbb{R}^{k \cdot (n+1)}$
Finally use: a continuous bijection from a compact space to a Hausdorff space is $\cong$ □

Rmk Same works if M has boundary, just consider its **double** $M \cup M$ (identify along ∂M) and apply the Lemma to the double.

Cor M compact mfd (possibly with bdry) $\Longrightarrow$ M has f.g. homology

Pf Mfds satisfy ② since locally ball $\simeq$ pt. M embeds in $\mathbb{R}^N$ by Lemma B.
① holds by FACT. Done by Lemma A. □

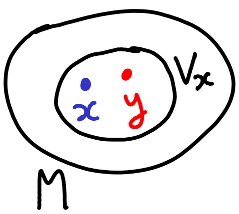
Def A **local orientation** of M at $x \in M$ is a choice of generator

$$\mu_x \in H_n(M, M\setminus x) \cong H_n(\mathbb{D}^n, \mathbb{D}^n \setminus 0) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}$$

(first $\cong$: excise complement of nbhd $V_x \cong \mathbb{D}^n$; choice of homeo is not canonical!)

Def An **orientation** of M is a **locally consistent** choice $x \mapsto \mu_x$

meaning:



$$\begin{array}{ccc} & H_n(M, M\setminus V_x) & \\ \cong \swarrow & & \searrow \cong \\ H_n(M, M\setminus x) & \xrightarrow{\cong} & H_n(M, M\setminus y) \end{array}$$

($V_x \cong \mathbb{D}^n \simeq pt$)

$$\exists\, a: \quad a \mapsto \mu_x,\ a \mapsto \mu_y, \quad \mu_x \longmapsto \mu_y$$

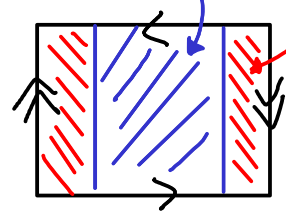
$\left(H_n(\mathbb{D}^n, \mathbb{D}^n \setminus x) \cong H_n(\mathbb{D}^n, \mathbb{D}^n \setminus y) \right.$ ← $\left. \mathbb{D}^n \setminus x \simeq \mathbb{D}^n \setminus y \right)$

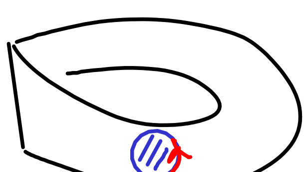
Def M **orientable** if $\exists$ orientation on M

oriented if we chose an orientation

Examples S^n, $\mathbb{R}^n$, $\mathbb{C}P^n$, orientable surfaces Σ_g, $\mathbb{R}P^n$ ← odd n

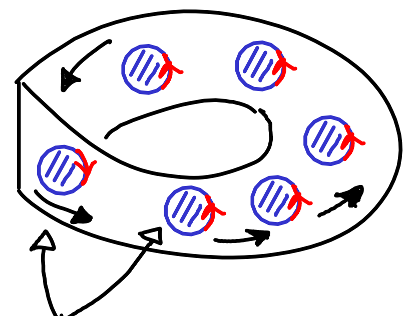
Non-example $\mathbb{R}P^2$ = Möbius band $\cup$ D^2





Choice of μ_x is choice of orientation of boundary circle of small disc containing x

by local consistency can move disc ⟶ continuously and preserves orientation



discs are differently oriented
$\Rightarrow$ contradicts local consistency.

$\Rightarrow \mathbb{R}P^2$ not orientable

The fundamental class [M]

FACT
Theorem For M closed n-mfd:

M orientble, connected $\Longrightarrow$ $H_n(M) \underset{\text{natural}}{\cong} H_n(M, M\setminus x) \underset{\text{choice}}{\cong} \mathbb{Z}$

$\Longrightarrow$ $\exists\ [M] \longleftrightarrow \mu_x$

↑ once we choose an orientation $(\mu_x)_{x\in M}$

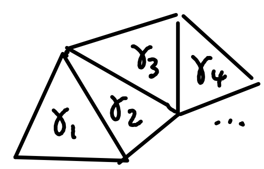
↑ called **fundamental class**

(if swap orientation: for $-\mu_x$ get $-[M]$)

M not orientable, connected $\Longrightarrow$ $H_n(M) = 0$

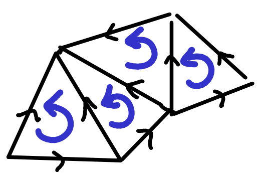
$H_n(M; \mathbb{F}_2) = \mathbb{F}_2 \cdot [M] \cong \mathbb{F}_2$ ← (or any field of characteristic 2)

Construction of [M] if M has Δ-complex structure



M compact $\Longrightarrow$ finite # n-simplices $\gamma_1, \dots, \gamma_N$

M oriented $\Longrightarrow$ pick orientations of $\gamma_1, \dots, \gamma_N$ to agree with given orientation of M: ↙ for $x \in \mathrm{Int}(\gamma_i)$



$$\mathbb{Z}\cdot\mu_x = H_n(M, M\setminus x) \xrightarrow[\text{exc.}]{\cong} H_n(\gamma_i, \gamma_i\setminus x) = \mathbb{Z}\cdot\gamma_i$$
$$\mu_x \longmapsto \gamma_i$$

$\Longrightarrow$ $\boxed{[M] := \sum \gamma_i}$ satisfies $\partial[M] = 0$ ✓

$$H_n(M) \longrightarrow H_n(M, M\setminus x) \xrightarrow{\cong} H_n(\gamma_i, \gamma_i \setminus x)$$
$$[M] \qquad \mu_x \longmapsto \gamma_i$$

Not difficult to see that $H_n^{CW}(X) = \mathbb{Z}\cdot[M]$, so $\Rightarrow H_n(M) \cong H_n(M, M\setminus x)$
$$[M] \longmapsto \mu_x$$

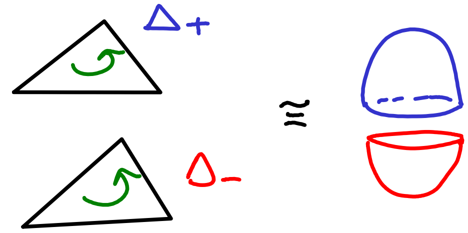
Also $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}\cdot[M]$ since $C_{n+1}(M) = 0$ ($\nexists$ $(n+1)$-simplices since $\dim M = n$)

M non-orientble $\Longrightarrow$ each facet of γ_i appears twice in $\partial \sum \gamma_i$

$\Longrightarrow$ $\partial \sum \gamma_i = 0$ over $\mathbb{F}_2$ independently of choices of orientations of γ_i. ✓

Examples

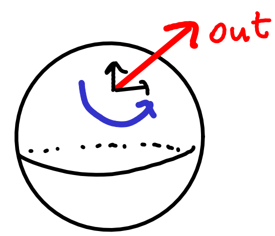
1) $S^n = \underbrace{\Delta^n \cup \Delta^n}_{\text{glue bdries}}$ (first Δ^n is Δ_+, second Δ^n is Δ_-)

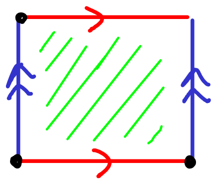
S^2: 

$[S^n] = \Delta_+ - \Delta_-$ if use canonical orientation we discussed

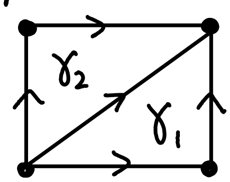
hence $\partial[S^n] = \partial\Delta - \partial\Delta = 0$

$\mathbb{D}^n \subseteq \mathbb{R}^n$ canonical orientation
$\Rightarrow S^{n-1} = \partial\mathbb{D}^n$ " using outward normal first rule

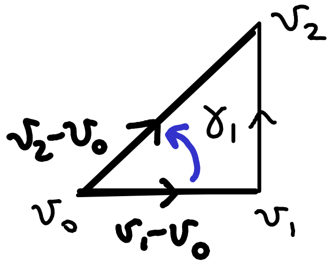


2) $T^2 =$ 

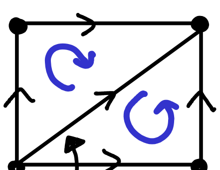
Δ-complex structure (compatibly with side identifications!)



Want orientation induced by square $\subseteq \mathbb{R}^2$

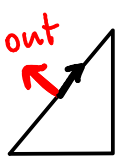


$v_1 - v_0,\ v_2 - v_0$ positive $\mathbb{R}^2$-basis
$\Rightarrow \gamma_1$ agrees with orientation

$\Longrightarrow$ 

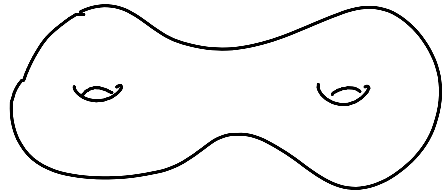
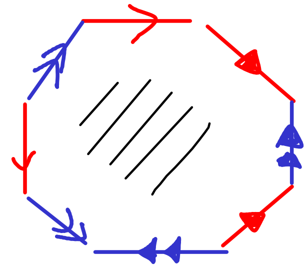
$[T^2] = +\gamma_1 - \gamma_2$

($\uparrow$ γ_2 orientation disagrees)

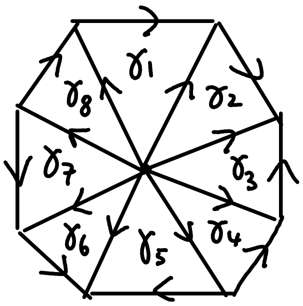


Rmk general feature: if two simplices are identified along a facet then the 2 respective outward normals are related by reflection.

So consistency $\Rightarrow$
- either simplices are compatibly oriented and the two induced orientations on facet are opposite
- or not compatibly oriented but facet orient^n is same, then need sign like in example when build $[T^2]$

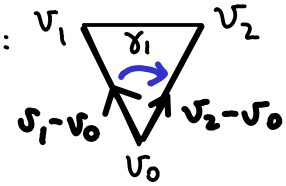
3) Recall Σ_2 =  = 

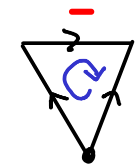
Δ-cx structure (compatible with side identifications!):



Use the orientation induced by polygon $\subseteq \mathbb{R}^2$

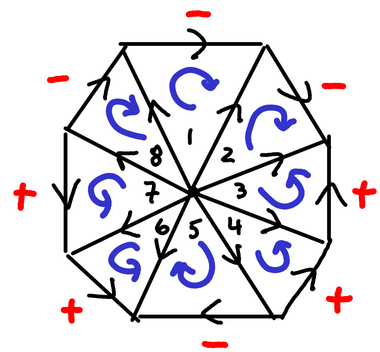
$$\Rightarrow [\Sigma_2] = -\gamma_1 - \gamma_2 + \gamma_3 + \gamma_4 - \gamma_5 + \gamma_6 + \gamma_7 - \gamma_8$$

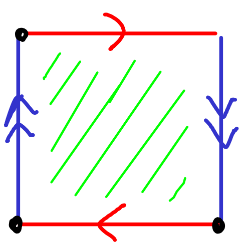
Explanation: 

$v_1 - v_0, v_2 - v_0$ is negative $\mathbb{R}^2$-basis $\Rightarrow$ 

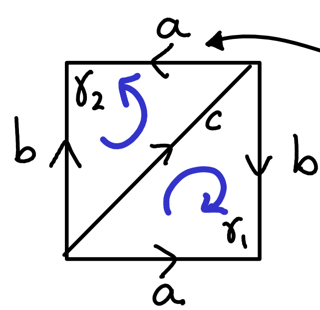
All simplices γ_i have v_0 = centre of polygon

$\Rightarrow$ sign $<$ − if outer edge clockwise; + ——— anti ———



3) $\mathbb{R}P^2$ = 

(non-orientable) example



won't get Δ-cx structure if you try  since get issue here

Use the orientation induced by square $\subseteq \mathbb{R}^2$

$$\Rightarrow [\mathbb{R}P^2] = -\gamma_1 + \gamma_2$$

$$\partial[\mathbb{R}P^2] = -(b - a + c) + (a - b + c)$$

$$= -2b + 2a$$

$$\neq 0 \quad \text{so } \underline{\text{not}} \text{ cycle in } C_*^{CW}(\mathbb{R}P^2)$$

However, working modulo 2:

$$\partial[\mathbb{R}P^2] = 0 \in C_*^{CW}(\mathbb{R}P^2; \mathbb{F}_2) \text{ since } 2 = 0 \text{ in } \mathbb{F}_2$$

$$\Rightarrow [\mathbb{R}P^2] \in H^2(\mathbb{R}P^2; \mathbb{F}_2)$$

Degree

Def M, N oriented closed connected n-mfds, $f: M \to N$

$$f_*: H_n(M) \longrightarrow H_n(N)$$
$$[M] \mapsto \underline{\deg(f)} \cdot [N]$$
$$\deg(f) \in \mathbb{Z}$$

Lemma If $f^{-1}(y)$ finite,

then $\deg(f) = \sum_{x \in f^{-1}(y)} \deg(f_y)_*$ (local degree; local map like in chapter 7)

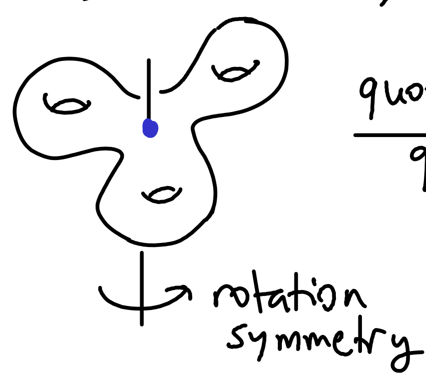
pf

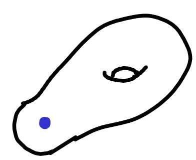
$$\begin{array}{ccccc} [M] & H_n(M) & \xrightarrow{f_*} & H_n(N) & [N] \\ \big\downarrow & \cong & & \cong & \big\uparrow \\ & \bigoplus_{x \in f^{-1}(y)} H_n(M, M \setminus x) & \xrightarrow{(f_x)_*} & H_n(N, N \setminus y) & \mu_y^N \\ \sum \mu_x^M & \longmapsto & & \left(\sum \deg(f_x)_*\right) \cdot \mu_y^N & \end{array}$$

□

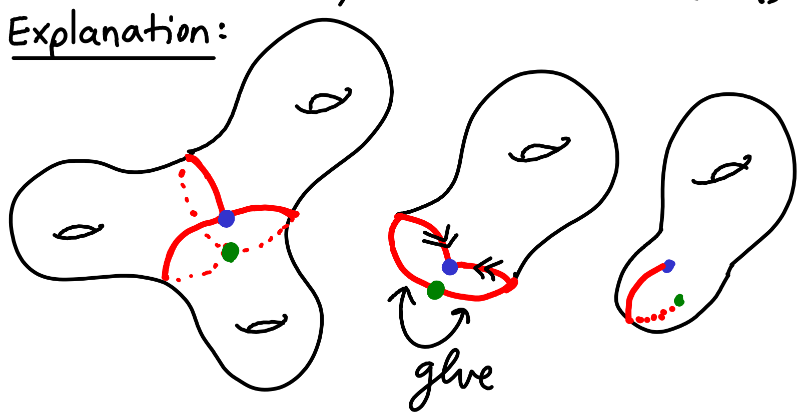
Examples

1) $S^1 \longrightarrow S^1$, $z \longmapsto z^n$, $[S^1] \longmapsto n \cdot [S^1]$ so $\deg = n$

2) $\Sigma_3 =$ 

$\xrightarrow[q]{\text{quotient}} \Sigma_3 / \mathbb{Z}/3$-rotation action $=$  $= \Sigma_1$ torus

Explanation:



Easy check: $\deg(q) = 3$
(e.g. use local degrees)

Cultural Rmk

For M, N, f smooth, the $\deg f = \#$(preimages of a generic point of N)
Idea: $\deg f$ tells you how many times you cover N. (almost all points work)

Poincaré duality

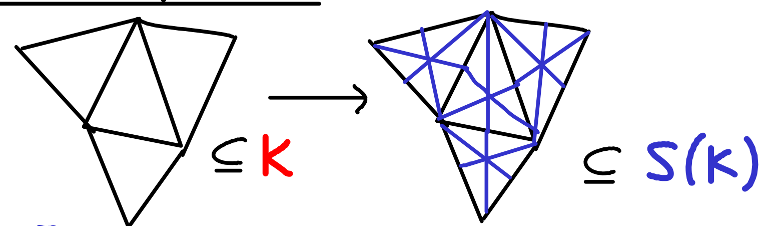
FACT **Theorem** For M closed n-mfd

M oriented $\Longrightarrow$ $\boxed{H^k(M) \cong H_{n-k}(M)}$ s.t. $1 \leftrightarrow [M]$, where $1 \in H^0(M)$ and $[M] \in H_n(M)$, $H^0(M) \cong H_n(M)$

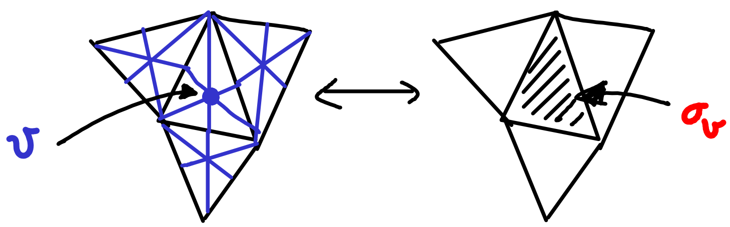
M non-oriented $\Longrightarrow$ same holds with $\mathbb{F}_2$ coefficients

Sketch proof when M is a simplicial complex K (Non-examinable)

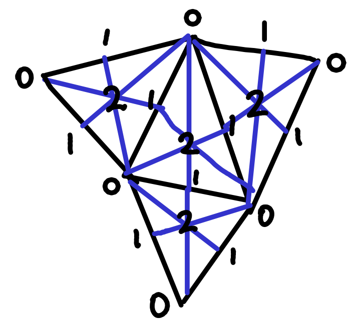
$S(K)$ = barycentric subdivision



1) simplex $\sigma = \sigma_v$ of K with barycentre v $\longleftrightarrow$ $v = v_\sigma$ vertex of $S(K)$

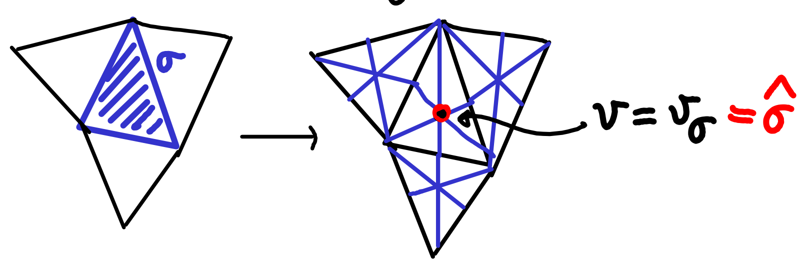


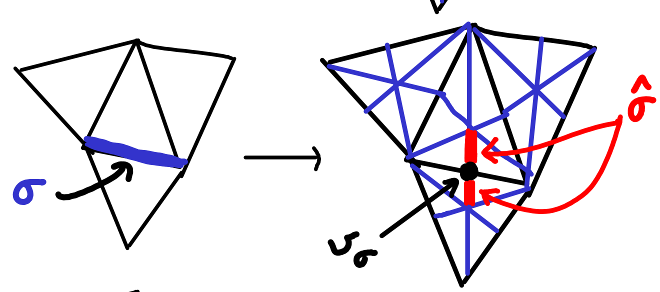
2) $ht(v) = ($**height** of $v) = \dim \sigma_v$

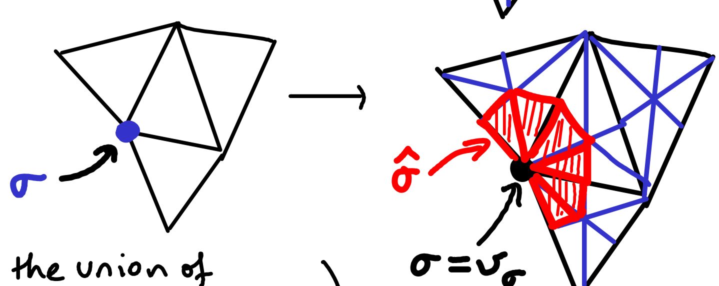


3) σ k-simplex of K $\downarrow$ **dual simplex**

$$\hat{\sigma} = \bigcup_{\substack{\tau \in S(K) \\ ht(v_\sigma) \text{ is min} \\ \text{of heights of} \\ \text{vertices of } \tau}} \tau$$







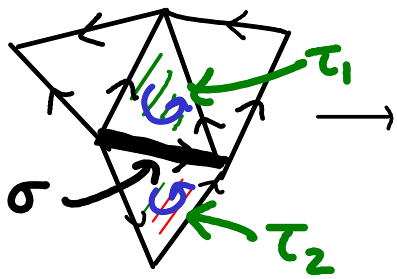
Rmk: $\bigcup \tau$ with $ht(v_\sigma)$ max will give back σ.
Thus $\hat{\sigma}, \sigma$ intersect transversely at v_σ.
One can also describe $\hat{\sigma}$ as

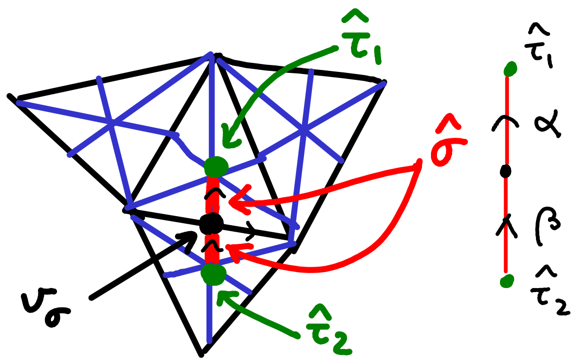
$$\hat{\sigma} = \bigcap_{\substack{\text{vertices} \\ v \in \sigma}} \overline{\mathrm{Star}}_{S(K)}(v)$$

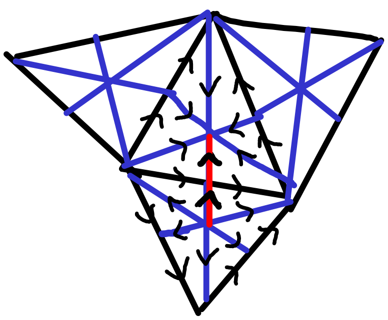
$\leftarrow$ (closed star is the union of simplices of $S(K)$ having v as a face)

FACTS

- $\dim \hat{\sigma} = n - \dim \sigma$
- dual cells $\hat{\sigma}$ give a cell decomposition of M ("polygonal" complex rather than Δ-cx)
- (✪) $\partial \hat{\sigma} = \sum_{\substack{\text{facet } \sigma \subsetneq \tau \\ \tau \in K}} \pm \hat{\tau}$ ← need compare orientations of σ, τ (+ if σ as a facet of τ has boundary orientation)

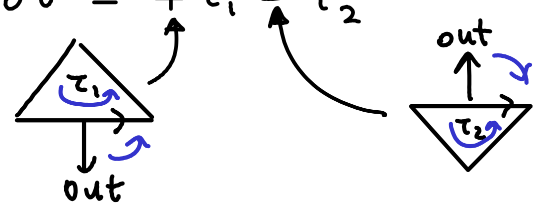






$\hat{\sigma} = \alpha + \beta$

$\partial \hat{\sigma} = +\hat{\tau}_1 - \hat{\tau}_2$



4) **dual chain complex**

D_{n-k} = free abelian group on dual chains $\hat{\sigma}$

$H_*(M) \cong H_*(D_*, \partial_*)$ (since $\hat{\sigma}$ give a cell decomp. of M)

5) $\varphi : D_{n-k} \longrightarrow C^k(M)$

$\hat{\sigma} \longmapsto \sigma^*$ where $\sigma^*(\alpha) = \begin{cases} 0 & \text{for } k\text{-cells } \alpha \neq \sigma \\ 1 & \text{for } \alpha = \sigma \end{cases}$

- φ linear bijection ✓
- chain map:

$\varphi(\partial \hat{\sigma}) = \varphi \sum \pm \hat{\tau} = \sum \pm \tau^*$ (← see ✪)

$\partial^* \varphi(\hat{\sigma}) = \partial^* \sigma^* = \left(\sigma^* \circ \partial : \tau \overset{\partial}{\longmapsto} \sum_{\text{facets } \sigma_i \text{ of } \tau} \pm \sigma_i \overset{\sigma^*}{\longmapsto} \begin{cases} \pm 1 & \text{if one of } \sigma_i = \sigma \\ 0 & \text{else} \end{cases} \right)$

$= \sum \pm \tau^* = \varphi(\partial \hat{\sigma})$ ✓

UPSHOT φ is chain iso so get iso:

$$H_*(M) \underset{④}{\cong} H_*(D_*, \partial_*) \xrightarrow[\varphi]{\cong} H^{n-*}(M)$$

Cor $\chi(\text{odd dimensional closed orientable mfd}) = 0$

Pf $b_i = \operatorname{rank} H_i(M)$ (**Betti numbers**)

$$\chi(M) = b_0 - b_1 + \cdots + b_{\dim M - 1} - b_{\dim M}$$

equal by Poincaré duality □

(Poincaré–)Lefschetz duality

Theorem
M compact oriented n-mfd
n-mfd with boundary

$$H^k(M) \cong H_{n-k}(M, \partial M)$$

$1 \in H^0(M) \longmapsfrom [M, \partial M] \in H_n(M, \partial M)$ — **relative fundamental class**

$$H_k(M) \cong H^{n-k}(M, \partial M)$$

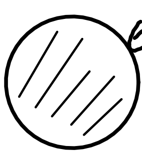
Non-oriented $\Longrightarrow$ same holds with $\mathbb{F}_2$ coefficients.

Pf basically same as Poincaré duality. □

Cor M compact, connected, $\partial M \neq \emptyset \Longrightarrow H^n(M) = H_0(M, \partial M) = 0$

Examples

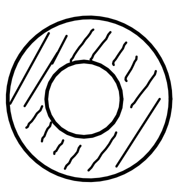
1) $\mathbb{D}^n$



$\partial \mathbb{D}^n = S^{n-1}$

$$\mathbb{Z} \cong H^0 \mathbb{D}^n \cong H_n(\mathbb{D}^n, S^{n-1})$$

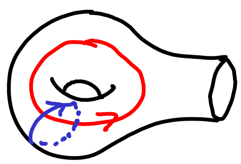
2)



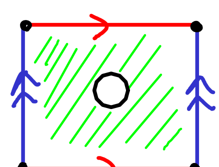
$A = \text{annulus} \subseteq \mathbb{R}^2$
$\simeq S^1$

$$\mathbb{Z} \cong H^0 A \cong H_2(A, \partial A)$$
$$\mathbb{Z} \cong H^1 A \cong H_1(A, \partial A)$$
$$0 \cong H^2 A \cong H_0(A, \partial A)$$

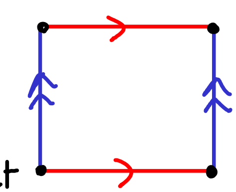
3) $M = T^2 \setminus$ open ball =



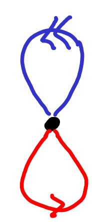
=



$\simeq$ def. retract



$\simeq$



$\simeq S^1 \vee S^1$

$$\Rightarrow H_*(M, \partial M) \cong H^{2-*}(S^1 \vee S^1) = \begin{cases} \mathbb{Z}^2 & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$$

$* = 1$ ← gen. by 2 loops

$* = 2$ ← gen. by $[M, \partial M]$

What happens in the non-compact case?

Locally finite homology (Borel-Moore)

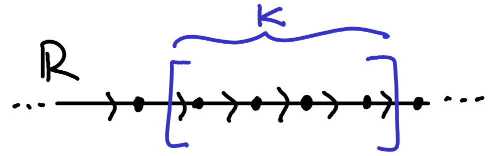
$C_*^{lf}(X)$ allow infinite sums $\sum n_i \sigma_i$ (with $n_i \in \mathbb{Z}$; σ_i ← generators of $C_*(X)$)

s.t. given any compact subset $K \subseteq X$,

$$\#\{n_i \neq 0 : K \cap \text{Im}\, \sigma_i \neq \emptyset\} < \infty.$$

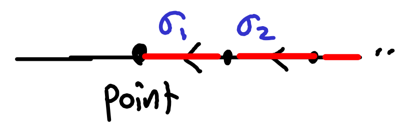
Examples

- $C_1^{lf}(\mathbb{R}) \ni \sum_{m \in \mathbb{Z}} \sigma_m$



$\Rightarrow$ get cycle $[\mathbb{R}] \in H_1^{lf}(\mathbb{R})$ $\sigma_m : I \cong [m, m+1] \subseteq \mathbb{R}$

- $C_0^{lf}(\mathbb{R}) \ni [\text{point}] \ni \partial(\sigma_1 + \sigma_2 + \cdots)$ is a boundary:



exercise $H_*^{lf}(\mathbb{R}) = \begin{cases} \mathbb{Z} & *=1 \\ 0 & \text{else} \end{cases}$ $(\cong H^{1-*}(\mathbb{R}))$

FACT Theorem M orientable n-mfd (possibly not compact) $\Rightarrow$ $\boxed{H^*(M) \cong H_{n-*}^{lf}(M)}$

cohomology with compact supports $H_c^*(X)$

$C_c^*(X)$: only allow cochains $\phi : C_*X \to \mathbb{Z}$ s.t. $\exists$ compact $K \subseteq X$ (depends on ϕ) with

$$\phi(C_*(X \setminus K)) = 0 \quad \text{(vanish on chains in } X\setminus K)$$

Example $c \in C_*(X)$ cycle $\Rightarrow$ $\phi(\alpha)$ = signed # intersections of c with α (geometric intersection #)

$\Rightarrow \phi \in C_c^*(X)$ since $\phi(\alpha) = 0$ if $\alpha \subseteq X \setminus \text{Im}(c)$

Thm M orientable n-mfd (possibly not compact) $\Rightarrow$ $\boxed{H_*(M) \cong H_c^{n-*}(M)}$

Warning H_*^{lf}, H_c^* are not homotopy invariant (indeed non-trivial for $\mathbb{R}^n$)
Caused because they are not functorial. They are however functorial for **proper maps** (preimages of compact sets are compact)
Mayer-Vietoris holds for H_c^* but not for H_*^{lf}.

Fact $H_c^*(X) \cong \varinjlim H^*(X, X\setminus K)$ where compacts $K_1 \subseteq K_2$ give $H^*(M, M\setminus K_1) \to H^*(M, M \setminus K_2)$

Direct limit $\varinjlim G_i$ via maps $G_i \to G_j$ means $\sqcup G_i /$ identify $g \in G_i$ with its images under those maps
(The indices are partially ordered & directed: $\forall i,j, \exists k \geq i,j$ so can compare G_i, G_j inside G_k (via $G_i \to G_k$, $G_j \to G_k$)

Fact $\varinjlim$ is an exact functor.

Cap product and Poincaré duality revisited

X space, $k \geq \ell$

$$\cap : C_k(X) \times C^\ell(X) \longrightarrow C_{k-\ell}(X)$$ **cap product**

$$(\sigma : \Delta^k \to X) \cap (\phi : C_\ell(X) \to \mathbb{Z}) = \underbrace{\phi(\underbrace{\sigma|_{[e_0,\dots,e_\ell]}}_{\text{"bottom face"}})}_{\in \mathbb{Z}} \cdot \underbrace{\underbrace{\sigma|_{[e_\ell,\dots,e_k]}}_{\text{"top face"} \cong \Delta^{k-\ell}}}_{\in C_{k-\ell}(X)}$$

(easy) Properties

- $\cap$ bilinear
- $\partial(\sigma \cap \phi) = (-1)^\ell (\partial\sigma \cap \phi - \sigma \cap \partial^* \phi)$
- cycle $\cap$ cycle is cycle
- boundary $\cap$ cycle / cycle $\cap$ boundary are boundaries

$$\Longrightarrow \boxed{\cap : H_k(X) \times H^\ell(X) \longrightarrow H_{k-\ell}(X)} \quad \text{bilinear}$$

Theorem (Poincaré duality) The map $\phi \longmapsto [M] \cap \phi$ gives following isos

① For M closed oriented n-mfd

$$[M] \cap \cdot \ : H^*(M) \xrightarrow{\cong} H_{n-*}(M)$$

② For M non-compact oriented n-mfd,

$$[M] \cap \cdot \ : H^*_c(M) \xrightarrow{\cong} H_{n-*}(M) \qquad (\star)$$

$$[M] \cap \cdot \ : H^*(M) \xrightarrow{\cong} H^{\ell f}_{n-*}(M)$$

Sketch Pf of ① for smooth mfds (Non-examinable)

If M smooth $\Rightarrow \exists$ "good cover" U_i of M meaning open cover s.t.

FACT from Riemannian geometry ("convex neighbourhoods")

$U_i \cong \mathbb{R}^n$

$U_{i_1} \cap \dots \cap U_{i_k} \cong \mathbb{R}^n$ or $\emptyset$

Then compute $H^*_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$ and $(\star)$ holds for $\mathbb{R}^n$.

$\Rightarrow (\star)$ holds $\forall\, U_i$

$\Rightarrow$ by naturality of $(\star)$ and of Mayer-Vietoris get $(\star)$ for $\bigcup_{\text{finite}} U_i$ (use 5-lemma)

$\Rightarrow (\star)$ for M, which is ①. □

General Pf of Poincaré duality ← Non-examinable

Step 1 : holds for $\mathbb{R}^n$

Pf $H^k_c(\mathbb{R}^n) \cong H^k(D^n, S^{n-1}) \cong \begin{cases} 0 & k\neq n \\ \mathbb{Z} & k=n \end{cases} \cong H_{n-k}(\mathbb{R}^n)$

(↑ first $\cong$:) can make K larger by picking K = large ball
then $H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong H^k(K, \partial K)$

Pick Δ-cx structure for $\mathbb{R}^n$. So $[\mathbb{R}^n] = \sum \pm \sigma_i$ ← sum over n-simplices.
Say $\exists$ simplex $\sigma_0 : \Delta^n \to \mathbb{R}^n$. Define $\phi : C^{CW}_c(\mathbb{R}^n) \to \mathbb{Z}$, $\phi(\sigma_0) = \pm 1$ (✪), $\phi(\text{other simplices}) = 0$
$\Rightarrow \delta\phi = 0$ for dim reasons

$[\mathbb{R}^n] \cap \phi = \sum \phi(\pm\sigma_i) = \phi(\pm\sigma_0) = 1$ (Pick sign in ✪)

Step 2 holds for $A, B, A\cap C \Rightarrow$ holds for $A \cup B$

Pf Mayer-Vietoris for H^*_c, naturality, 5-Lemma ✓

Step 3 holds for A_i, and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \Rightarrow$ holds for $\cup A_i$

Pf By applying $\varinjlim$: both sides of P.D. iso commute with limits ✓

Step 4 holds for open subsets in $\mathbb{R}^n$

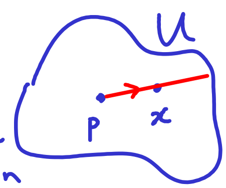
Pf Every such set is a union of convex open sets (e.g. balls)

By Step 3 enough to consider case of finite union.

By induction on # convex open sets:

1 convex set $U \cong \mathbb{R}^n$ via a proper homeomorphism,
now use Step 1 ✓

fix $p \in U$
$U \longrightarrow \mathbb{R}^n$
$x \longmapsto \dfrac{x-p}{\text{dist } p \text{ to } \partial U \text{ along ray to } x}$



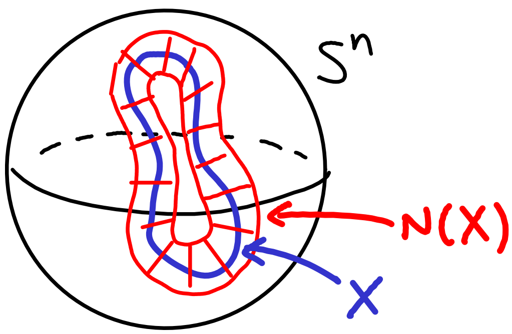
2 convex sets : KEY TRICK convex set $\cap$ convex set is convex in $\mathbb{R}^n$!
$\Rightarrow$ use Step 2 & previous case

k+1 convex sets: $A = \bigcup\{\text{first } k \text{ convex sets}\}$, B = last convex set
$\Rightarrow A \cap B \subseteq B$ is a union of k convex sets
$\Bigg\} \Rightarrow$ use Step 2 & Inductive hypothesis ✓

Step 5 holds for mfd M

Consider open sets in M for which it holds.
By a Zorn's Lemma argument we get a maximal open subset U where holds.
If $U \neq M$ pick $p \in M \setminus U$ and nbhd $V \cong \mathbb{R}^n$ of p. Then holds for $U, V, U\cap V$
(note $U \cap V \subseteq V \cong \mathbb{R}^n$ open, so Step 4 applies) so by Step 2 holds for $U \cup V$
Contradicts maximality. ✓ □

Alexander duality



(in fact, enough to assume X is locally contractible)

$\emptyset \neq X \subsetneq S^n$ compact subset s.t.

$\exists$ open neighbourhood $N(X)$ which deformation retracts to X such that $\overline{N(X)} \subseteq S^n$ is an n-mfd with boundary.

Theorem $\boxed{\tilde{H}_*(X) \cong \tilde{H}^{n-*-1}(S^n \setminus X)}$

Pf later

Example $X \subseteq S^3$ **knot** (i.e. $X = \text{image}(S^1 \xrightarrow{} S^3)$)

homeomorphism onto the image ← **embedding**

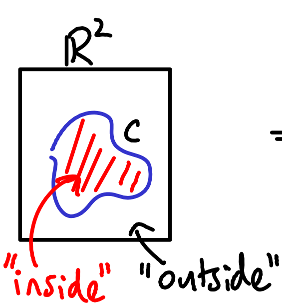
$\Longrightarrow$ $N(X) \cong$ solid torus $\simeq S^1$

$\Longrightarrow$

$$\tilde{H}_0(X) = 0 = \tilde{H}^2(S^3 \setminus X)$$
$$\tilde{H}_1(X) = \mathbb{Z} = \tilde{H}^1 \quad ''$$
$$\tilde{H}_2(X) = 0 = \tilde{H}^0 \quad ''$$

so the homology of a **knot complement** does not tell knots apart (always same)

Theorem (**Jordan curve Theorem**)



$C \simeq S^1$ closed curve in $\mathbb{R}^2 \subseteq S^2$ ← e.g. by stereographic projection $S^2 \cong \mathbb{C} \cup \infty$

$\Longrightarrow$ $\mathbb{R}^2 \setminus C$ has 2 path-components (= connected components)

Similarly for $C \simeq S^n \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}$.

Pf $C \simeq S^n \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}$ $\Rightarrow$ $\tilde{H}_n(S^n) \cong \mathbb{Z} \cong \tilde{H}^0(S^{n+1} \setminus C)$

$\Rightarrow$ $H^0(S^{n+1} \setminus C) \cong \mathbb{Z}^2$

$\Rightarrow$ $S^{n+1} \setminus C$ has 2 path components. $\square$

Proof Alexander duality

$$Y := S^n \setminus N(X) \qquad (\simeq S^n \setminus X)$$

for $* < n-1$

$$\tilde H^{n-*-1}(Y) = H^{n-*-1}(Y)$$
$$\underset{\text{Lefschetz}}{\cong} H_{*+1}(Y, \partial Y)$$
$$\underset{\text{exc.}}{\cong} H_{*+1}(S^n, \overline{N(X)})$$
$$\underset{\substack{\text{LES} \\ \text{using } * < n-1}}{\cong} \tilde H_{*+1}(\underbrace{\overline{N(X)}}_{\simeq X})$$

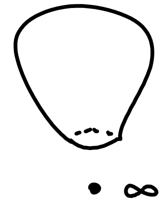
for $* = n-1$

$$\tilde H^0(Y) \oplus \mathbb{Z} \cong H^0(Y)$$
$$\underset{\text{Lef.}}{\cong} H_n(Y, \partial Y)$$
$$\underset{\text{exc.}}{\cong} H_n(S^n, \overline{N(X)})$$
$$\cong \tilde H_{n-1}(\underbrace{\overline{N(X)}}_{\simeq X}) \oplus \mathbb{Z}$$

$$0 \to \tilde H_n(S^n) \to H_n(S^n, \overline{N(X)}) \to \tilde H_{n-1}(\overline{N(X)}) \to 0$$

$$H_n(S^n, \overline{N(X)}) \downarrow \quad \tilde H_n(S^n) \xrightarrow{\cong} H_n(S^n, S^n \setminus \infty) \cong \mathbb{Z}$$

$$\uparrow \; S^n = \mathbb{R}^n \cup \infty$$



for $* = n$

$$H^{n-*-1}(Y) = H^{-1}(Y) = 0$$

$$H_n(X) \cong H_n(N(X)) \underset{\text{Lef.}}{\cong} 0 \quad \text{n-mfd with bdry} \neq \emptyset.$$

□