

C3.1 Algebraic Topology

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Sheet 3

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Convention: all spaces are topological spaces,
maps of spaces are always continuous.

- 1) Construct a degree d map $S^n \rightarrow S^n$ for any $n \geq 1$.
- 2) Given finitely generated abelian groups A_1, \dots, A_n , construct a space with $H_*(X) \cong \begin{cases} \mathbb{Z} & * = 0 \\ A_k & * = k \in \{1, \dots, n\} \\ 0 & \text{else} \end{cases}$ (Hint. CW-complex)

- 3) Let $f, g: S^n \rightarrow S^n$ satisfy $f(x) \neq g(x), \forall x \in S^n$.
Prove that $f \simeq -\text{id} \circ g$. (Hint. consider $\frac{\varphi_t}{\|\varphi_t\|}$ where $\varphi_t = tf - (1-t)g$)
Deduce that

- if $f: S^n \rightarrow S^n$ has no fixed point then $f \simeq -\text{id}$.
- if G is a group acting continuously and freely on S^{2n} then $G = 1$ or $\mathbb{Z}/2$. (Hint. degree)
 \uparrow
 $g \neq e \in G$ has no fixed points

- 4) a) In the CW complex for $\mathbb{C}P^n$ from the course notes, show that the attaching maps commute with the obvious inclusions $S^{k-1} \subseteq S^k$ via $\mathbb{R}^k \cong \mathbb{R}^k \times 0 \subseteq \mathbb{R}^{k+1}$, and $\mathbb{C}P^k \subseteq \mathbb{C}P^{k+1}$ via $\mathbb{C}^{k+1} \cong \mathbb{C}^{k+1} \times 0 \subseteq \mathbb{C}^{k+2}$.

(You have to decide in which dimensions to consider these inclusions, and also recall $\mathbb{R}^2 \cong \mathbb{C}, (x,y) \mapsto x+iy$)

- b) Explain why $\mathbb{R}P^n \cong \mathbb{D}^n / (\pm \text{id action on } \partial \mathbb{D}^n)$.

Under this identification, show that the i -th hyperplane $x_i = 0$ intersects $\mathbb{R}P^n$ in a copy of $\mathbb{R}P^{n-1}$. Show that the corresponding inclusion $\text{incl}_i: \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^n$ induces isomorphisms $H_*(\mathbb{R}P^{n-1}; \mathbb{Z}_2) \rightarrow H_*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ for $* \neq n$. (Hint. "homotope it")

State and prove an analogous result for $\mathbb{C}P^n$ (using \mathbb{Z}).

- c) Use the Cultural Remark on page 57 of the notes for this exercise:
Compute the cup product to deduce

$$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x] / x^{n+1} \quad |x| = 2$$

$$H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[y] / y^{n+1} \quad |y| = 1$$

means: replace $\mathbb{Z}, \text{Hom}(\cdot, \mathbb{Z})$ by $\mathbb{Z}_2, \text{Hom}(\cdot, \mathbb{Z}_2)$ in definitions of C_x, C^*

You may assume as known that $\mathbb{C}P^n, \mathbb{R}P^n$ are compact connected smooth manifolds, and that $\mathbb{C}P^n$ is orientable.

5) Let $\mathbb{C}P^\infty = \bigcup_{n \geq 0} \mathbb{C}P^n$, $S^\infty = \bigcup_{n \geq 0} S^n$, $\mathbb{R}P^\infty = \bigcup_{n \geq 0} \mathbb{R}P^n$ (using the natural inclusions from 4(a))

a) Describe a CW-complex structure on these spaces and compute H_* .

b) Compute $H_*(\mathbb{R}P^\infty; \mathbb{Z}/2)$

c) Describe the ring structure on their cohomologies and on $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$.

6) $Y = (X \cup \mathbb{D}^m) / (\text{attaching map } \varphi: \partial \mathbb{D}^m \rightarrow X)$ ← so identify $x \sim \varphi(x)$
 $\partial \mathbb{D}^m \xrightarrow{m \geq 1} X$

Prove: $H_*(Y) \cong \begin{cases} H_*(X) & * \neq m-1, m \\ H_{m-1}(X) / \text{Im } \varphi_* & * = m-1 \\ H_m(X) \oplus \text{Ker } \varphi_* & * = m \end{cases}$

← (Hint: Consider $(Y, Y \setminus D)$ where $D \subseteq \mathbb{D}^m$ is a closed disc in the interior of \mathbb{D}^m)

7) a) Prove that if each $x_i \in X_i$ has a contractible neighbourhood, then:

$$H^*(\bigvee_i X_i) \cong \prod_i H^*(X_i) \text{ for } * \geq 1 \text{ is an iso of rings.}$$

b) Show that $S^1 \vee S^1 \vee S^2$ and T^2 have the same homology, but different cohomology rings.

8) a) Let $X = \text{Moore space } M(\mathbb{Z}/m, n) = S^n \cup \frac{\mathbb{D}^{n+1}}{\varphi: \partial \mathbb{D}^{n+1} = S^n \rightarrow S^n \text{ of degree } m}$
 Compute $H_*^{CW}(X)$ and $H_*^{CW}(X)$.

b) Let $Y = (\mathbb{C}P^2 \cup \mathbb{D}^3) / (\text{attaching map } \varphi: \partial \mathbb{D}^3 = S^2 \xrightarrow{\text{degree } p \text{ map}} S^2 \cong \mathbb{C}P^1 \subseteq \mathbb{C}P^2)$
 Compute $H_*^{CW}(Y)$.

c) For $X = M(\mathbb{Z}/p, 2)$:

Show that $H^*(Y) \cong H^*(X \vee S^4)$ as rings

but $H^*(Y; \mathbb{Z}/p) \not\cong H^*(X \vee S^4; \mathbb{Z}/p)$

← (means: replace $\mathbb{Z}, \text{Hom}(\cdot, \mathbb{Z})$ by $\mathbb{Z}/p, \text{Hom}(\cdot, \mathbb{Z}/p)$ in constructions of C_* and C^*)

9) Compute directly the cup product structure

on $H^*(K)$ and $H^*(K; \mathbb{Z}/2)$, where $K = \text{Klein bottle}$.

(i.e. do not use intersection theory, only use CW-complexes and the definition of \cup)

10) Let $I = [0, 1]$. Build orientation-preserving homeomorphisms of pairs

$$\begin{aligned} (\mathbb{D}^n, S^{n-1}) &\cong (I^n, \partial I^n) \cong (I^l \times I^k, \partial I^l \times I^k \cup I^l \times \partial I^k) \\ &\cong (\mathbb{D}^l \times \mathbb{D}^k, S^{l-1} \times \mathbb{D}^k \cup \mathbb{D}^l \times S^{k-1}) \end{aligned}$$

where $l+k=n$.