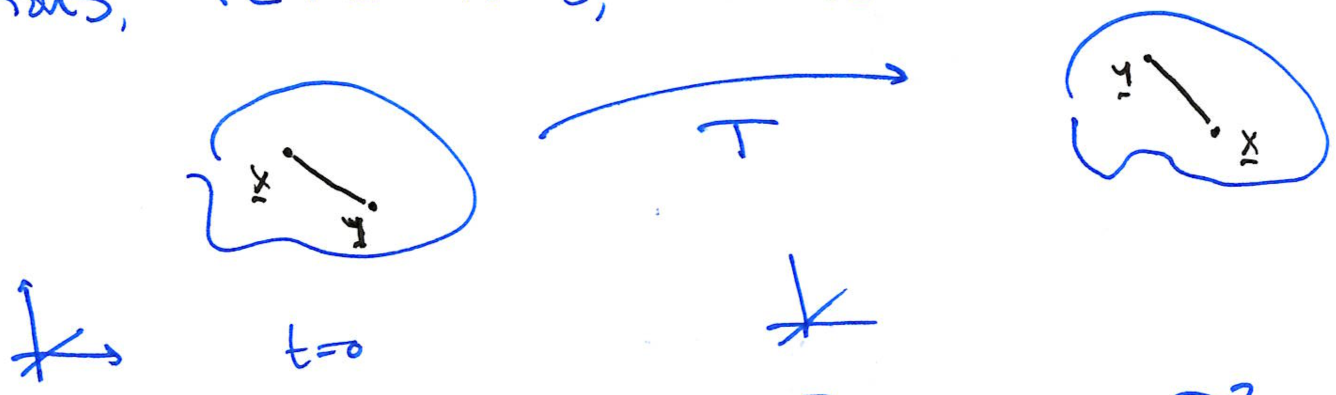
Isometries and Orthogonal Matrices

Def'n An isometry $T : \mathbb{R}^n \to \mathbb{R}^n$ is a distance preserving map:

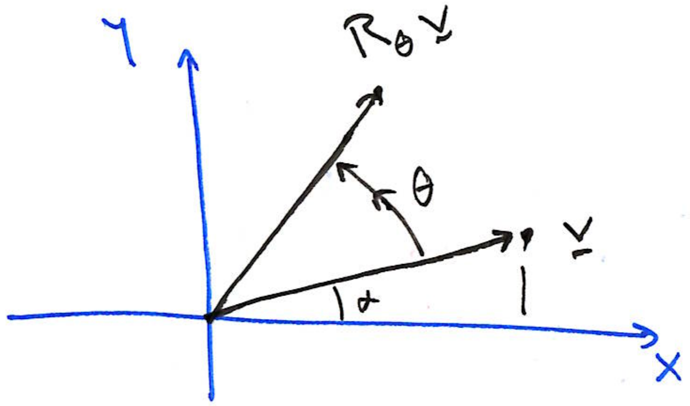
$$|T(\underset{\sim}{x}) - T(\underset{\sim}{y})| = |\underset{\sim}{x} - \underset{\sim}{y}| \quad \forall \underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$$

Ex. Rotations, reflections, translations (consider throwing a rock)



Q. Can we describe w/ matrix? in $\mathbb{R}^2$

- **Rotations** about origin by angle θ



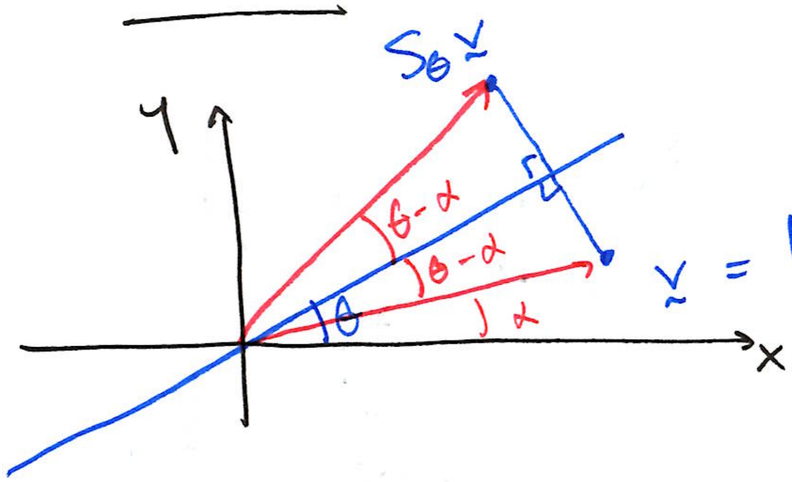
- in polar coords: $\underset{\sim}{v} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}$

- transforms to $R_\theta \underset{\sim}{v} = \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix}$ ← expand

$$= \begin{pmatrix} r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \\ r(\cos\theta\sin\alpha + \sin\theta\cos\alpha) \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{R_\theta} \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} =: R_\theta \underset{\sim}{v}$$

$\therefore$ For any $\underset{\sim}{v}$, multiply by $R_\theta \to$ rotated vec. by θ

Reflection about line $y = \tan\theta \, x$

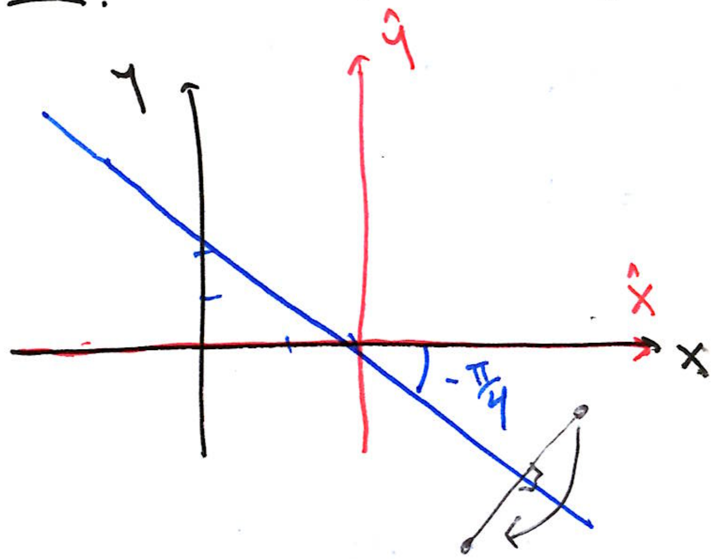


$$\underset{\sim}{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}$$

$$S_\theta \underset{\sim}{v} = \begin{pmatrix} r\cos(2\theta-\alpha) \\ r\sin(2\theta-\alpha) \end{pmatrix}$$

$$\underset{\text{expand}}{=} \underbrace{\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}}_{S_\theta} \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} =: S_\theta \underset{\sim}{v}$$

Ex. 61 Reflection about line $x + y = 2$



1. Shift coords: $\hat{x} = x - 2$, $\hat{y} = y$

$\rightarrow$ in $(\hat{x}, \hat{y})$, line is thru origin: $\hat{y} = -\hat{x} = \tan\left(-\frac{\pi}{4}\right)\hat{x}$

$$\Rightarrow \hat{S}_\theta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \quad \text{Thus } \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$

2. "To unshift" $\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \underbrace{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{= \begin{pmatrix} 0 \\ -2 \end{pmatrix}}$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Are the maps R_θ, S_θ isometries?

Want $|R_\theta \underset{\sim}{u} - R_\theta \underset{\sim}{v}| = |\underset{\sim}{u} - \underset{\sim}{v}|$ $= |R_\theta(\underset{\sim}{u} - \underset{\sim}{v})| = |\underset{\sim}{u} - \underset{\sim}{v}|$ if $|R_\theta \underset{\sim}{x}| = |\underset{\sim}{x}|$ $\forall \underset{\sim}{x}$

Let $\underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, recall $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \rightarrow |R_\theta \underset{\sim}{x}|^2 = (\cos\theta x - \sin\theta y)^2 + (\sin\theta x + \cos\theta y)^2$

$= x^2 + y^2 = |\underset{\sim}{x}|^2$ ✓ (similar for S_θ)

$\underline{Q}$. Let A be $n \times n$ matrix. When is $\underset{\sim}{v} \mapsto A\underset{\sim}{v}$ an isometry?

First, a note about $\underline{\text{angles}}$. Let T be an isometry that fixes the origin ($T\underset{\sim}{0} = \underset{\sim}{0}$)

$$|T\underset{\sim}{x} - T\underset{\sim}{y}|^2 = \cancel{|T\underset{\sim}{x}|^2} - 2\,T\underset{\sim}{x} \cdot T\underset{\sim}{y} + \cancel{|T\underset{\sim}{y}|^2} \overset{\text{by isom.}}{=} |\underset{\sim}{x} - \underset{\sim}{y}|^2 = \cancel{|\underset{\sim}{x}|^2} - 2\underset{\sim}{x} \cdot \underset{\sim}{y} + \cancel{|\underset{\sim}{y}|^2}$$

Also, $|T\underset{\sim}{x}| = |\underset{\sim}{x}|$, $|T\underset{\sim}{y}| = |\underset{\sim}{y}|$ $\Rightarrow$ $T\underset{\sim}{x} \cdot T\underset{\sim}{y} = \underset{\sim}{x} \cdot \underset{\sim}{y}$ Thus T $\underline{\text{preserves angles}}$

- We seek A st $\boxed{A\underset{\sim}{v} \cdot A\underset{\sim}{w} = \underset{\sim}{v} \cdot \underset{\sim}{w}}$ $\forall \underset{\sim}{v}, \underset{\sim}{w} \in \mathbb{R}^n$

column vectors

[$\underset{\sim}{v} = \underset{\sim}{w} \Rightarrow$ length preserving, and $A\underset{\sim}{v} \cdot A\underset{\sim}{w} = |A\underset{\sim}{v}|\,|A\underset{\sim}{w}| \cos\theta$]

$\Rightarrow$ angle preserving

$A\underset{\sim}{v} - A\underset{\sim}{w} = A(\underset{\sim}{v} - \underset{\sim}{w})$]

$$\underset{\sim}{v} \cdot \underset{\sim}{w} = \sum_{i=1}^{n} v_i w_i = (v_1 \cdots v_n) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = v^T w$$

So $A\underset{\sim}{v} \cdot A\underset{\sim}{w} = (A\underset{\sim}{v})^T A\underset{\sim}{w} = (v^T A^T)(Aw) = v^T A^T A w$ $) = $ if $A^T A = \mathbb{1}$

transpose rule

$\underset{\sim}{v} \cdot \underset{\sim}{w} = v^T w$

So $\underset{\sim}{A v} \cdot A \underset{\sim}{w} = \underset{\sim}{v} \cdot \underset{\sim}{w} \quad \forall \underset{\sim}{v}, \underset{\sim}{w}$ iff $A^T A = \mathbb{1}$

[let $\underset{\sim}{v} = (1, 0, \dots 0)$ then $\underset{\sim}{v} \cdot \underset{\sim}{w} = 1$ $\quad A\underset{\sim}{v} \cdot A\underset{\sim}{w} = v^T \underbrace{A^T A}_{M} w = (1 \dots 0) \begin{pmatrix} & & \\ & M & \\ & & \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} = M_{11}$

$\underset{\sim}{w} = (1, 0, \dots 0)$

Def'n A real square matrix A is orthogonal if $A^T A = \mathbb{1}_n$ ie $A^{-1} = A^T$

Note: for $\underset{\sim}{v} \in \mathbb{R}^n$, a linear map $\underset{\sim}{v} \mapsto T(\underset{\sim}{v})$ can be represented by a matrix: $T(\underset{\sim}{v}) = A\underset{\sim}{v}$

Thus all linear isometries of $\mathbb{R}^n$ have form $T(\underset{\sim}{v}) = A\underset{\sim}{v}$ w/ A orthogonal

$T(\alpha \underset{\sim}{a} + \beta \underset{\sim}{b}) = \alpha T(\underset{\sim}{a}) + \beta T(\underset{\sim}{b})$. Let $\underset{\sim}{v} = v_1 \underset{\sim}{e}_1 + \dots + v_n \underset{\sim}{e}_n \rightarrow T(\underset{\sim}{v}) = \sum_{i=1}^{n} v_i T(\underset{\sim}{e}_i) = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ T(e_1) & T(e_2) & \dots & T(e_n) \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

- But $T(\underset{\sim}{v}) = A\underset{\sim}{v} + \underset{\sim}{b}$ for $\underset{\sim}{b} \in \mathbb{R}^n$, $A^T A = \mathbb{1}$ is also an isometry, but not linear $\quad [\, T(\underset{\sim}{x} + \underset{\sim}{y}) = A(\underset{\sim}{x} + \underset{\sim}{y}) + \underset{\sim}{b} = A\underset{\sim}{x} + A\underset{\sim}{y} + \underset{\sim}{b} \neq T\underset{\sim}{x} + T\underset{\sim}{y} \,]$

Check: $|T\underset{\sim}{x} - T\underset{\sim}{y}| = |A\underset{\sim}{x} + \underset{\sim}{b} - (A\underset{\sim}{y} + \underset{\sim}{b})| = |A\underset{\sim}{x} - A\underset{\sim}{y}| = |\underset{\sim}{x} - \underset{\sim}{y}|$ ✓

Isometry

$|T\underline{u} - T\underline{v}| = |\underline{u} - \underline{v}|$

length preserving

Linear ($T\underline{0} = \underline{0}$)

$\downarrow$

Matrix repr.

$T(\underline{v}) = A\underline{v}$

w/ $A^TA = \mathbb{1}$

$\downarrow$

Angle preserving

$T\underline{v}\cdot T\underline{w} = \underline{v}\cdot\underline{w}$

Nonlinear $\searrow$? (No)

$\downarrow$

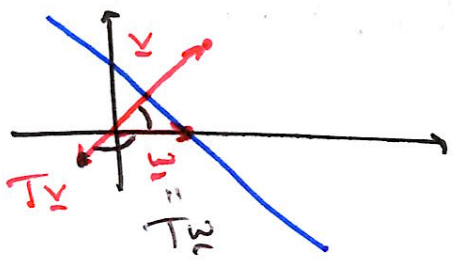
$T(\underline{v}) = A\underline{v} + \underline{b}$, $\quad (T(\underline{0}) \neq \underline{0})$

$A^TA = \mathbb{1}$, $\underline{b}$ const

$\downarrow$

Not angle preserving

Ex.



$$\therefore\ A = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\substack{\parallel \\ R_\theta \text{ rotation}}} \quad \text{OR} \quad A = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}}_{S_{\frac{\theta}{2}}:\ \text{reflection about } y = \tan\frac{\theta}{2}x}$$

$$\boxed{A^TA = \mathbb{1}}$$

Note: $\det(AB) = \det(A)\det(B)$

$\det(A^T) = \det(A)$

$\Rightarrow \det(A^TA) = \left(\det(A)\right)^2 = 1$

$\Rightarrow \underline{\det A = \pm 1}$ "signature" for orthog matrix

- $A^TA = \mathbb{1}$, $B^TB = \mathbb{1} \Rightarrow \cancel{AB}\ (AB)^T AB = \mathbb{1}$

LHS $= B^TA^TAB = B^TB = \mathbb{1}$

$\therefore$ Product of 2 orthog matrices is orthog.

in $\mathbb{R}^2$ | Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow A^TA = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix}$

$A^TA = \mathbb{1}_2 \Rightarrow a^2 + c^2 = 1$; $b^2 + d^2 = 1$, $ab + cd = 0$

$a = \cos\theta$, $c = \sin\theta$, for some $\theta \in [0, 2\pi)$

$b = \cos\phi$, $d = \sin\phi$, $\phi \in [0, 2\pi)$

$\cos(\theta - \phi) = 0 \Rightarrow \phi = \theta \pm \frac{\pi}{2}$

$\Rightarrow b = \mp\sin\theta$, $d = \pm\cos\theta$

Observe $\det R_\theta = 1$, $\det S_{\frac{\theta}{2}} = -1$

On Coords. and Measurement

- We make calcs. using variables defined in a coord. sys.
- But a coord. sys. is effectively <u>arbitrary</u>

→ <u>Q</u> How much are our calcs. tied to our choice of coord. sys?

- Distinguish between:

<u>Geometric truths</u>	<u>System dependent statements/measurements</u>
 "these lines are perpendicular"	 "this line has length 5"
 "this is a circle"	"$x^2+y^2=1$ is a circle"
 "$\underset{\sim}{a}$ is twice the length of $\underset{\sim}{b}$"	"the length of $\underset{\sim}{a}$ is $\sqrt{a_1^2+a_2^2}$"

- Measurements require coordinates, coords. require axes - ie a <u>basis</u>

<u>Def'n</u> $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots \underset{\sim}{v}_n \in \mathbb{R}^n$ form a basis for $\mathbb{R}^n$ if we can write every $\underset{\sim}{x} \in \mathbb{R}^n$ as $\underset{\sim}{x} = \alpha_1 \underset{\sim}{v}_1 + \ldots + \alpha_n \underset{\sim}{v}_n$ for some unique numbers $\alpha_1, \ldots \alpha_n$

- the α_i are the <u>coords</u> of $\underset{\sim}{x}$ in the basis.

- We're been using standard basis

$$\underset{\sim}{e}_i = (0, \ldots, \underset{\uparrow_{i\text{th}}}{1}, 0 \ldots 0)$$

- Only require $\{\underset{\sim}{v}_i\}$ is
  - <u>lin. indep</u> ← "not too many"
  - <u>spanning</u> ← "enough"

<u>Q</u>. What makes a "good" basis?

"Good" means: our formulas hold;

$$|\underset{\sim}{x}|^2 = x_1^2 + x_2^2 + \ldots + x_n^2$$

$$\underset{\sim}{x} \cdot \underset{\sim}{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

The point: $\underset{\sim}{x} \cdot \underset{\sim}{y}$ is a geometric truth

__Intuition__ : formulas rely on 1. Pythagoras, ie right triangles

2. x_i is __distance__ in direction of i^{th} axis.

⇝ suggests: a "good" basis has __unit vectors__ that are __mutually orthogonal__

Supp. $B_1 : \{v_1, v_2, \dots, v_n\}$, $B_2 : \{u_1, \dots, u_n\}$ form bases for $\mathbb{R}^n$

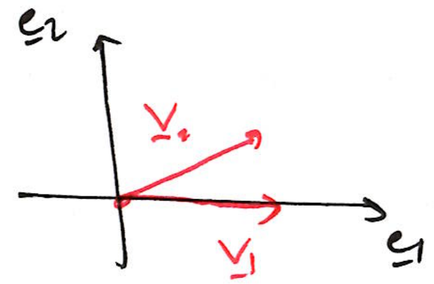
x : "an arrow" : $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in B_1 __means__ $x = \sum x_i v_i$

$x = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ in B_2 means $x = \sum X_i u_i$

__Observe__: by construction,

$$v_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow i^{th} \quad \text{in } B_1$$

__Claim__ $x \cdot y = x_1 y_1 + \dots + x_n y_n$, where $\{x_i\}$, $\{y_i\}$ are coords in basis $\{v_i\}$ holds iff $v_i \cdot v_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$



__Pf__ ($\Rightarrow$) $v_i \cdot v_j = \delta_{ij}$ b/c $v_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$

($\Leftarrow$) $x \cdot y = \left(\sum_{i=1}^{n} x_i v_i\right) \cdot \left(\sum_{j=1}^{n} y_j v_j\right)$

$$= \sum_i \sum_j x_i y_j \underbrace{v_i \cdot v_j}_{\delta_{ij}} = \sum_i x_i y_i \quad ✓$$

Defn An <u>orthonormal</u> basis satisfies $\underset{\sim}{v}_i \cdot \underset{\sim}{v}_j = \delta_{ij}$

<u>Notes</u>: • n orthonormal vectors forms a basis for $\mathbb{R}^n$

• if $\underset{\sim}{x} = x_1 \underset{\sim}{v}_1 + \dots + x_n \underset{\sim}{v}_n$ and $\{\underset{\sim}{v}_i\}$ is orthonormal, then

$x_i = \underset{\sim}{x} \cdot \underset{\sim}{v}_i$, and this is the distance along $\underset{\sim}{v}_i$

• Connection to orthog. matrices : Let A be matrix w/ columns $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n$

then
$$A^T A = \begin{pmatrix} \leftarrow \underset{\sim}{v}_1 \rightarrow \\ \vdots \\ \leftarrow \underset{\sim}{v}_n \rightarrow \end{pmatrix} \begin{pmatrix} \uparrow & & \uparrow \\ \underset{\sim}{v}_1 & \cdots & \underset{\sim}{v}_n \\ \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_1 & \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 & & \\ \underset{\sim}{v}_2 \cdot \underset{\sim}{v}_1 & \underset{\sim}{v}_2 \cdot \underset{\sim}{v}_2 & & \\ & & \ddots & \\ & & & \underset{\sim}{v}_n \cdot \underset{\sim}{v}_n \end{pmatrix}$$

$\therefore A^T A = \mathbb{1}$ iff $\underset{\sim}{v}_i \cdot \underset{\sim}{v}_j = \delta_{ij}$

So A is an orthog. matrix iff columns of A form orthon. basis

[rows too, since $\underline{A^T A = \mathbb{1}} \Leftrightarrow \underline{A A^T = \mathbb{1}}$]

$\cancel{AA^TAA^T} \Rightarrow AA^T$

$AA^T = (A^T A)^T = \mathbb{1}$

Supp. 2 bases $B_1 : \underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n$ st $\underset{\sim}{v}_i \cdot \underset{\sim}{v}_j = \delta_{ij}$.

$B_2 : \underset{\sim}{u}_1, \dots, \underset{\sim}{u}_n$

Supp. pt P has coords $x_1, x_2, \dots, x_n$ in B_1, and $X_1, \dots, X_n$ in B_2

ie can express P via $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in B_1 OR $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ in B_2

• We know: □ $\underset{\sim}{x} \cdot \underset{\sim}{y} = x_1 y_1 + \dots + x_n y_n = x^T y$ since $\mathcal{B}_1$ orthon.

□ $\underset{\sim}{x} \cdot \underset{\sim}{y} = \underset{\sim}{X} \cdot \underset{\sim}{Y}$ "since geometric truth"

<u>Q.</u> $\underset{\sim}{X} \cdot \underset{\sim}{Y} = X^T Y$?

· We'll answer by considering <u>change</u> of <u>coords</u> (change of basis)

- Let's stick to $n=2$: can write $\underset{\sim}{u}_1 = \alpha_1 \underset{\sim}{v}_1 + \beta_1 \underset{\sim}{v}_2$

$\underset{\sim}{u}_2 = \alpha_2 \underset{\sim}{v}_2 + \beta_2 \underset{\sim}{v}_2$

$$\text{Then } \underset{\sim}{X} = X_1 \underset{\sim}{u}_1 + X_2 \underset{\sim}{u}_2 = \Big(\underbrace{X_1 \alpha_1 + X_2 \alpha_2}_{x_1}\Big) \underset{\sim}{v}_1 + \Big(\underbrace{X_1 \beta_1 + X_2 \beta_2}_{x_2}\Big) \underset{\sim}{v}_2 = \underset{\sim}{x}$$

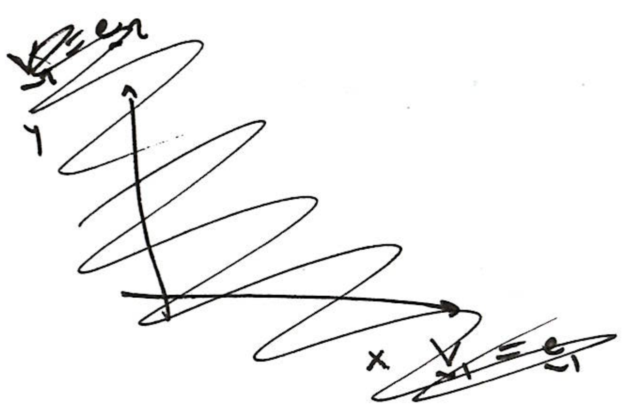
$$\text{So } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}}_{\text{call } P} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

Thus $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = P \underset{\sim}{X}$

Now consider

$$\underset{\sim}{x} \cdot \underset{\sim}{y} = x^T y = (PX)^T (PY)$$

$$= X^T P^T P Y = \underset{\sim}{X} \cdot \underset{\sim}{Y}$$



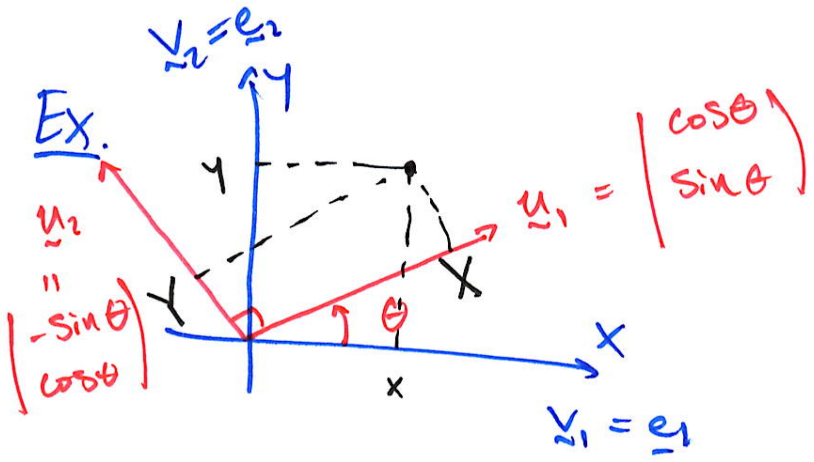
$\therefore \underset{\sim}{X} \cdot \underset{\sim}{Y} = X^T Y = X_1 Y_1 + \dots + X_n Y_n$ iff $P^T P = \mathbb{1}$

ie if change of basis matrix is orthog.

Note: if $\underset{\sim}{v}_i = \underset{\sim}{e}_i$ - "standard basis",

then $P = \begin{pmatrix} \uparrow & & \uparrow \\ \underset{\sim}{u}_1 & \cdots & \underset{\sim}{u}_n \\ \downarrow & & \downarrow \end{pmatrix}$

Ex.



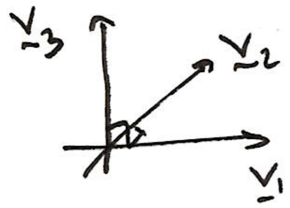
$\rightarrow P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$= R_\theta$

$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} X \\ Y \end{pmatrix}$

Orthogonal Change of Coords

Summary



- orthonormal basis - "good" basis

$$\underset{\sim}{x} \cdot \underset{\sim}{y} = \sum_{i=1}^{n} x_i y_i$$

· orthogonal matrix P ($P^T P = \mathbb{1}$)

transforms coords b/t orthon. bases:

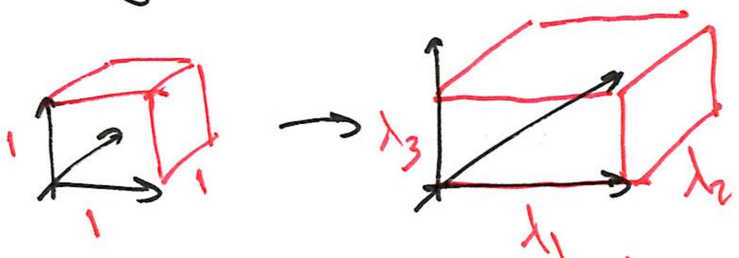
$$\underset{\sim}{x} = P \underset{\sim}{X}$$

→ Q. Can we solve a problem by converting coordinates? ("ideal" basis?)

- if view matrix as map $\underset{\sim}{x} \mapsto M\underset{\sim}{x}$

"simplest" map is diagonal

$$M = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$



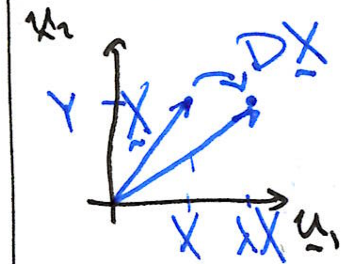
$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad M\underset{\sim}{x} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \lambda_3 x_3 \end{pmatrix}$$

Spectral Theorem

Let A be a square symmetric matrix. Then $\exists$ orthog. P and diagonal D st $P^T A P = D$

- equiv., $\boxed{A = PDP^T}$

Ex. Consider 2D transformation: stretch by λ in direction $\underset{\sim}{u}_1$

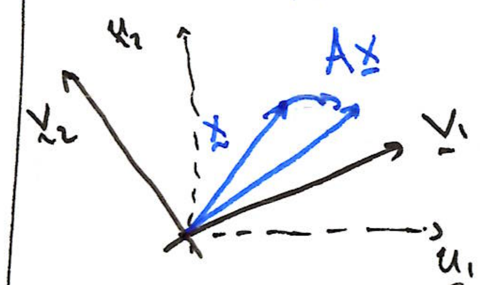


in B_1: $D = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$

B_1: $\underset{\sim}{u}_1, \underset{\sim}{u}_2$

$$\underset{\sim}{X} \mapsto D\underset{\sim}{X} = \begin{pmatrix} \lambda X \\ Y \end{pmatrix}$$

· in another basis B_2: $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$



A not diagonal!

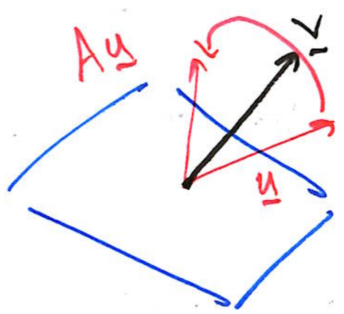
$\exists P$ orthog: $B_1 \to B_2$

Spec. Thm: $A\underset{\sim}{x} = \underbrace{P \underbrace{D \underbrace{P^T \underset{\sim}{x}}_{\text{rotate to } B_1}}_{\text{apply stretch}}}_{\text{rotate back}}$

3x3 orthogonal matrices

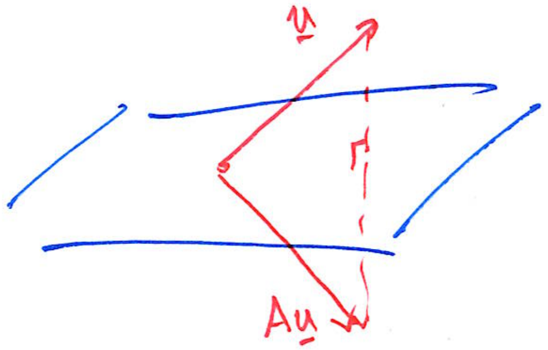
• A is orthog $\Rightarrow A^TA = \mathbb{1}$, $\det A = \pm 1$

Key:



rotation has axis of rot.
- axis forms a 1D invariant space of map : $A\underset{\sim}{v} = \underset{\sim}{v}$

$\Rightarrow A - \mathbb{1}$ has 1D null space



reflection requires plane of refl. - forms a 2D invariant space.

$\Rightarrow A\underset{\sim}{v} = \underset{\sim}{v}$ has 2 indep. sol'ns directions

ie $A - \mathbb{1}$ has 2D null space

Ex 72

$$A = \frac{1}{25}\begin{pmatrix} 20 & 15 & 0 \\ -12 & 16 & 15 \\ 9 & -12 & 20 \end{pmatrix}, \quad B = \frac{1}{25}\begin{pmatrix} -7 & 0 & -24 \\ 0 & 25 & 0 \\ -24 & 0 & 7 \end{pmatrix}$$

- We're told : $A^TA = \mathbb{1} = B^TB$, one is a reflection, one a rotation.

- row reduction of $A - \mathbb{1}$, $B - \mathbb{1}$:

$$A - \mathbb{1} \longleftrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow (A - \mathbb{1})\underset{\sim}{x}$ has sol'n space

$\lambda(3, 1, 3)$ ← axis of rotation

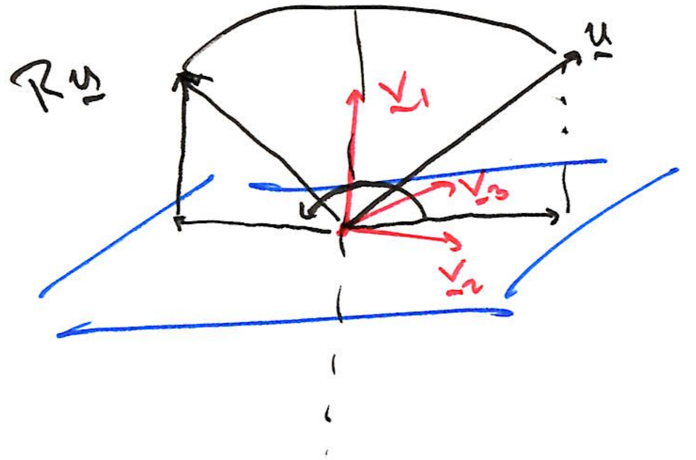
$$B - \mathbb{1} \longleftrightarrow \begin{pmatrix} 4 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so $(B - \mathbb{1})\underset{\sim}{x}$ has sol'n space $4x + 3z = 0$

or spanned by $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3/4 \\ 0 \\ 1 \end{pmatrix}$

- plane of reflection

"Ideal" basis? For a **rotation**, R, let $\underset{\sim}{v}_1$ be a unit vec. along axis of rotation, and $\underset{\sim}{v}_2, \underset{\sim}{v}_3$ be unit vectors in orthog. plane st $\{\underset{\sim}{v}_i\}$ orthonormal



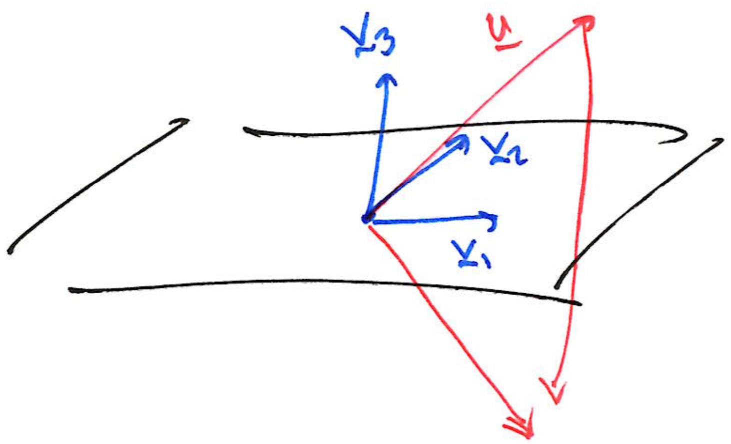
. since $R\underset{\sim}{v}_1 = \underset{\sim}{v}_1$ - "looks like" a 2D rotation in $\underset{\sim}{v}_2$-$\underset{\sim}{v}_3$ plane $\Rightarrow \exists\, \theta$ st

$$R\underset{\sim}{v}_2 = \cos\theta\, \underset{\sim}{v}_2 + \sin\theta\, \underset{\sim}{v}_3$$

$$R\underset{\sim}{v}_3 = -\sin\theta\, \underset{\sim}{v}_2 + \cos\theta\, \underset{\sim}{v}_3$$

Thus, if X_i are coords. in $\{\underset{\sim}{v}_i\}$, $R\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

For **reflection** S let $\underset{\sim}{v}_1, \underset{\sim}{v}_2$ be orthon basis for the plane of reflection, (2D null space of $S - \mathbb{1}$)



$S\underset{\sim}{v}_1 = \underset{\sim}{v}_1$

$S\underset{\sim}{v}_2 = \underset{\sim}{v}_2 \quad \Rightarrow \quad S\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ -X_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

$S\underset{\sim}{v}_3 = -\underset{\sim}{v}_3$

Classifying ("the signature" of reflections, rotations)

Let A be a 3×3 orthog. matrix

(a) if $\det A = 1$, then A is a rotation by angle θ where

$$\operatorname{tr} A = 1 + 2\cos\theta$$

(b) $\det A = -1$ and $\operatorname{tr} A = 1$ iff A is a reflection

[observe: these hold in "ideal" basis, and trace and determinant are invariant under coord. change]

Proof (Outline) (a) • $\det A = 1 \Rightarrow \det(A - \mathbb{1}) = 0 \Rightarrow \exists\, \underset{\sim}{v}_1$ st $A\underset{\sim}{v}_1 = \underset{\sim}{v}_1$
& $A^T A = \mathbb{1}$

$\det(A-\mathbb{1}) = \det(A - A^T A) = \det(\underbrace{\mathbb{1} - A^T}_{(\mathbb{1}-A)^T})$
$= \det(\mathbb{1} - A) = -\det(A - \mathbb{1})$

• Create orthon. basis w/ $\underset{\sim}{v}_2, \underset{\sim}{v}_3$, and show $A\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_1 = 0$
$A\underset{\sim}{v}_3 \cdot \underset{\sim}{v}_1 = 0$

$$[A\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_1 = A\underset{\sim}{v}_2 \cdot A\underset{\sim}{v}_1 = \underset{\sim}{v}_2 \cdot \underset{\sim}{v}_1 = 0]$$

$\Rightarrow$ the effect of A on $\underset{\sim}{v}_2 - \underset{\sim}{v}_3$ plane is 2D

$\Rightarrow A$ has form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$(\tilde{A})$}} \\ 0 & & \end{pmatrix}$. $\det A = 1 \Rightarrow \det \tilde{A} = 1$ so can use 2D results

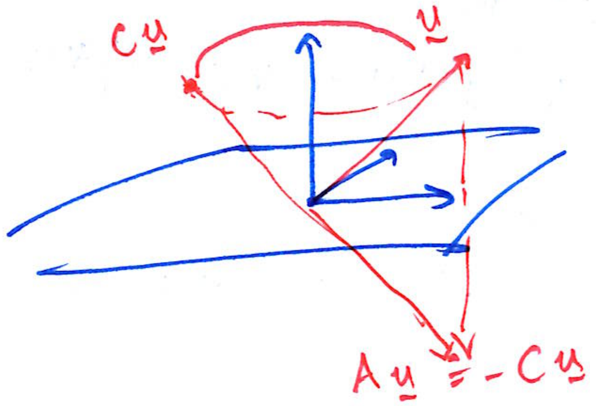
$\rightarrow \exists\,\theta$ st $\tilde{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

• Note $\operatorname{tr} A = 1 + 2\cos\theta$

Proof of (b) $(\Rightarrow)$ • $C = -A$ satisfies $\det C = 1$, so by (a), C is a rotation by θ: $1 + 2\cos\theta = \operatorname{tr} C$

But $\operatorname{tr} C = -\operatorname{tr} A = -1 \Rightarrow 2(1+\cos\theta) = 0$

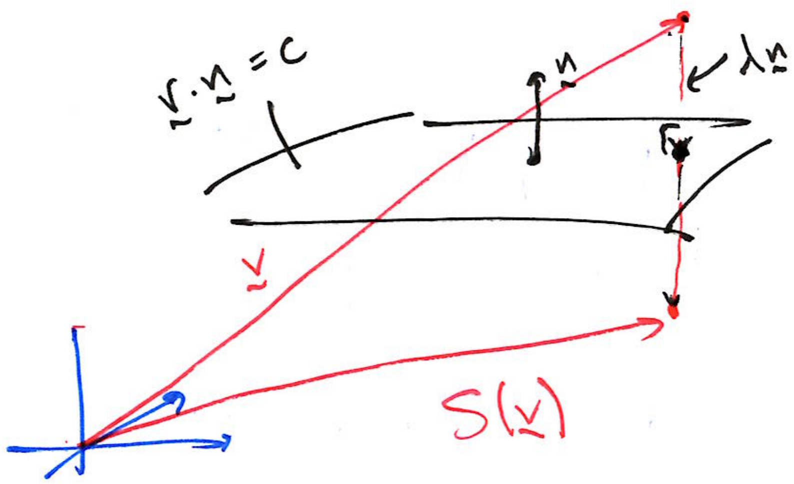
$\Rightarrow \theta = \pi \Rightarrow A = -C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which is reflection in $\underset{\sim}{v}_2 - \underset{\sim}{v}_3$ plane



$(\det A = -1,\ \operatorname{tr} A \neq 1,\ \text{eg } A = -\mathbb{1})$

Ex 75. Let $|\underset{\sim}{n}| = 1$ be given. Show that reflection in plane $\underset{\sim}{r} \cdot \underset{\sim}{n} = c$ is given by $S(\underset{\sim}{v}) = \underset{\sim}{v} - 2(\underset{\sim}{v} \cdot \underset{\sim}{n})\underset{\sim}{n} + 2c\underset{\sim}{n}$

[$c \neq 0 \Rightarrow$ not a plane containing origin, so won't have form just outlined!]



$\exists \lambda$ st $(\underset{\sim}{v} - \lambda\underset{\sim}{n}) \cdot \underset{\sim}{n} = c \Rightarrow \lambda = \underset{\sim}{v} \cdot \underset{\sim}{n} - c$

Thus $S(\underset{\sim}{v}) = \underset{\sim}{v} - 2\lambda\underset{\sim}{n} = v - 2(\underset{\sim}{v} \cdot \underset{\sim}{n})\underset{\sim}{n} + 2c\underset{\sim}{n}$

Products of Rotations

$\left(\det A = 1, \ \det B = 1 \Rightarrow \det(AB) = 1 \right)$

[an even # of reflections equiv. to a rot.!]

$$R_i(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_i(\theta)\,\underset{\sim}{i} = \underset{\sim}{i}, \quad R_j(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \quad R_j(\theta)\,\underset{\sim}{j} = \underset{\sim}{j}$$

<u>Ex</u> 77. Claim: can write any rotation R as $R = R_i(\alpha) R_j(\beta) R_i(\gamma)$ ★
for some $\alpha, \gamma \in (-\pi, \pi]$, $\beta \in [0, \pi]$

$[\, R_j^{-1}(\beta) = R_j(-\beta) \,]$

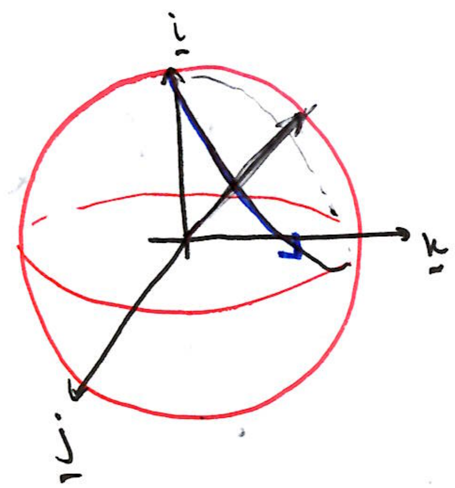
- ★ true iff $R_j^{-1}(\beta) R_i^{-1}(\alpha) R = R_i(\gamma)$

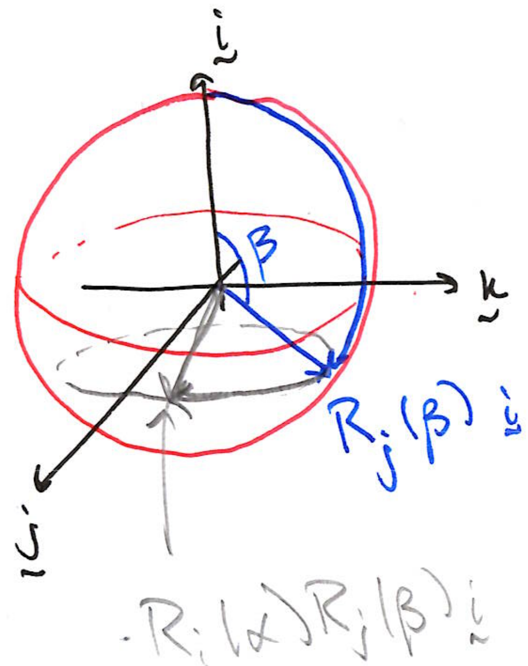
iff $R_j^{-1}(\beta) R_i^{-1}(\alpha) R \underset{\sim}{i} = \underset{\sim}{i}$ iff $R\underset{\sim}{i} = \underbrace{R_i(\alpha) \underbrace{R_j(\beta)\, \underset{\sim}{i}}_{\text{fix latitude}}}_{\text{fix longitude}}$

$$R\underset{\sim}{i} = \underset{\sim}{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \text{ w/ } c_1^2 + c_2^2 + c_3^2 = 1$$

fix latitude

fix ~~lat~~ longitude





Isometries of $\mathbb{R}^n$

: Defn: $T : \mathbb{R}^n \to \mathbb{R}^n$ is isometry if $|T(u) - T(v)| = |u - v| \quad \forall u, v \in \mathbb{R}^n$

- Main result: if T is an isometry, then $\exists$ an orthogonal A and constant vector b st $T(v) = Av + b$

Proof Start w/ case : $S(0) = 0$

1. $|S(v)| = v$ ← by defn w/ $u = 0$

2. $S(u) \cdot S(v) = u \cdot v$ ← expand defn and use 1

3. (Linearity) $S(a + \alpha b) = S(a) + \alpha S(b)$

Pf: Show $|S(a + \alpha b) - (S(a) + \alpha S(b))|^2 = 0$ ← expand plus repeated use of 1 & 2

Let $\{e_i\}$ be an orthonormal basis, Define $A = \begin{pmatrix} \uparrow & & \uparrow \\ Se_1 & \dots & Se_n \\ \downarrow & & \downarrow \end{pmatrix}$

- let $v \in \mathbb{R}^n \Rightarrow v = \sum_{i=1}^{n} \lambda_i e_i$ - in coords: $v = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$

By 3. $S(v) = \sum_{i=1}^{n} \lambda_i S(e_i)$

$$= \begin{pmatrix} \uparrow & & \uparrow \\ Se_1 & \dots & Se_n \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

$$= Av$$

And $A^T A = \begin{pmatrix} \leftarrow Se_1 \rightarrow \\ \vdots \\ \leftarrow Se_n \rightarrow \end{pmatrix} \begin{pmatrix} \uparrow & & \uparrow \\ Se_1 & \dots & Se_n \\ \downarrow & & \downarrow \end{pmatrix}$

So $(A^T A)_{ij} = Se_i \cdot Se_j$

$\underset{(2)}{=} e_i \cdot e_j = \delta_{ij} \quad \therefore A^T A = \mathbb{1}$

- If $T(0) = b \neq 0$

Then $S(v) = T(v) - b$ is an isometry w/ $S(0) = 0$

$\Rightarrow \exists$ orthog. A st $S(v) = Av$

$\Rightarrow T(v) = Av + b$ ✓

Unique? if $T(v) = A_1 v + b_1 = A_2 v + b_2$

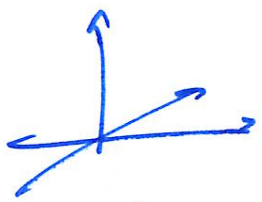
- $v = 0 \Rightarrow b_1 = b_2 \Rightarrow A_1 v = A_2 v \quad \forall v$

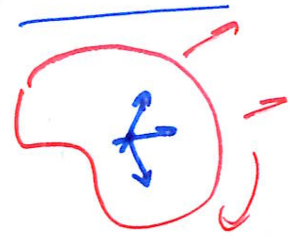
$\Rightarrow A_1 = A_2$

Rotating Frames

Motivation: Rigid body motion

- 3 degrees of freedom - translation
- 3 " " " - rotation





"fixed-in-space" basis B_1 (lab frame)

"fixed-in-body" basis B_2

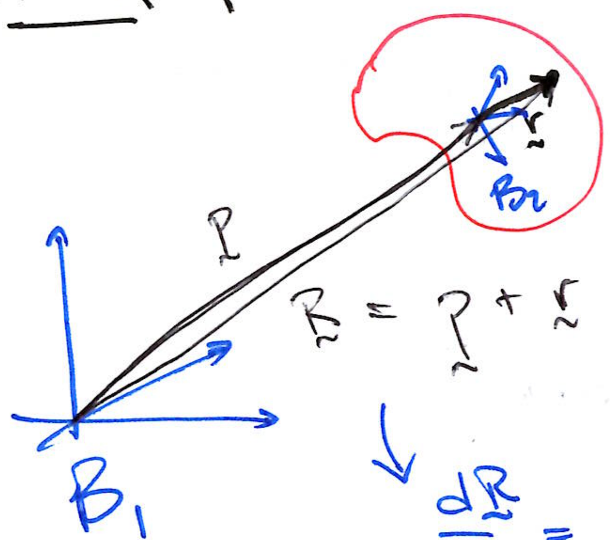
- We've seen: $\exists$ orthogonal matrix $A = A(t)$ relating B_1 to B_2

$$A^T A = \mathbb{1} \quad \forall t \implies \underline{A'(t)A^T(t)} + \underbrace{A(t)(A'(t))^T}_{(A'A^T)^T} = \frac{d}{dt}\mathbb{1} = 0 \implies A'A^T = \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} \overset{\text{call}}{=} M(t)$$

$$A^T A = \mathbb{1} \Leftrightarrow A A^T = \mathbb{1}$$

skew symmetric

Recall (PS 2): if define $\underset{\sim}{\omega}(t) = (\alpha, \beta, \gamma)$, then $M\underset{\sim}{x} = \underset{\sim}{\omega} \wedge \underset{\sim}{x} \quad \forall \underset{\sim}{x}$



$\underset{\sim}{R} = \underset{\sim}{p} + \underset{\sim}{r}$

in B_2: $\underset{\sim}{r}$ constant, ie $\frac{d\underset{\sim}{r}}{dt} = \underset{\sim}{0}$

in B_1: $\underset{\sim}{r} = \underset{\sim}{r}(t) = A(t)\underset{\sim}{r}(0) \implies \frac{d}{dt}\underset{\sim}{r}(t) = A'(t)\underset{\sim}{r}_0 = MA\underset{\sim}{r}_0 = M\underset{\sim}{r}$

$\underset{\sim}{r}(0) \overset{\text{call}}{=} \underset{\sim}{r}_0$

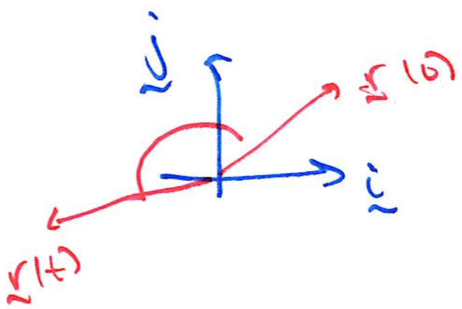
$$\implies \boxed{\frac{d}{dt}\underset{\sim}{r}(t) = \underset{\sim}{\omega}(t) \wedge \underset{\sim}{r}(t)}$$

$$\frac{d\underset{\sim}{R}}{dt} = \frac{d\underset{\sim}{p}}{dt} + \underset{\sim}{\omega} \wedge \underset{\sim}{r}$$

Note
- same $\underset{\sim}{\omega} = \underset{\sim}{\omega}(t)$ for any $\underset{\sim}{r}$ in body
- $\underset{\sim}{\omega}$ is called angular velocity

[More to come in Dynamics HT]

Ex. 2D motion



rotation in plane:

$$A(t) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix}$$

[could write $A = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for 3D]

Then $A'(t) = \begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \cdot \theta'(t) \Rightarrow M = A'A^T = \begin{pmatrix} 0 & -\theta'(t) \\ \theta'(t) & 0 \end{pmatrix}$

so if $\underset{\sim}{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \rightarrow \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\theta'(t)\, y(t) \\ \theta'(t)\, x(t) \end{pmatrix}$

- in 3D: $M = \begin{pmatrix} 0 & -\theta' & 0 \\ \theta' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow (\gamma = \theta', \alpha = \beta = 0)$ $\boxed{\underset{\sim}{\omega}(t) = \theta'(t)\, \underset{\sim}{k}}$

Interpretation: $\underset{\sim}{\omega}$ points along/defines (instantaneous) axis of rotation

$|\underset{\sim}{\omega}|$ is rotation rate