Infinite groups: Sheet 4

November 28, 2019

Exercise 1. Let G be a polycyclic group. Suppose that every finite quotient group of G is nilpotent. Prove that G is nilpotent. [*Hint*: use 'Noetherian induction'. If $A \cong \mathbb{Z}^d$ is an abelian normal subgroup of G, think about the lower central series of G/A^p for a prime p.]

Exercise 2. Suppose that $G \leq \mathrm{GL}_n(\mathbb{K})$ is completely reducible and that $g^e = 1,\ \forall g \in G$. Prove that $|G| \leq e^{n^3}$. [*Hint*: first consider the irreducible case.]

Exercise 3. Let G be a linear group. Prove that G is solvable if one of the following holds:

(i) every finitely generated subgroup of G is solvable;

(ii) G is finitely generated and every finite quotient group of G is solvable.

Exercise 4. Consider a semidirect product $\mathbb{Z}^n \rtimes_\varphi \mathbb{Z}$, defined by a homomorphism $\varphi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}^n) = GL(n, \mathbb{Z})$, hence by the matrix $\varphi(1) = M \in GL(n, \mathbb{Z})$. In what follows we use the notation $\mathbb{Z}^n \rtimes_M \mathbb{Z}$ instead of $\mathbb{Z}^n \rtimes_\varphi \mathbb{Z}$.

1. Prove, by induction on n, that if M has all eigenvalues equal to 1 then $\mathbb{Z}^n \rtimes_M \mathbb{Z}$ is nilpotent.

2. Deduce that, if M has all eigenvalues roots of unity, then $\mathbb{Z}^n \rtimes_M \mathbb{Z}$ is virtually nilpotent.

Exercise 5. Let G be a finitely generated nilpotent group and let $\varphi \in Aut(G)$. Prove that the polycyclic group $P = G \rtimes_\varphi \mathbb{Z}$ is

1. either virtually nilpotent;

2. or has exponential growth.

Exercise 6. Let G be a finitely generated group G of sub-exponential growth.

The goal of this exercise is to prove that for all $\beta_1, \ldots, \beta_m, g \in G$, the set of conjugates

$$\{g^k \beta_i g^{-k} \mid k \in \mathbb{Z}, i = 1, \ldots, m\}$$

generates a finitely generated subgroup $N \leqslant G$.

1. Prove that the statement for $m = 1$ implies the statement for every integer $m \geq 1$.

2. In what follows we therefore assume $m = 1$, we set $\alpha := \beta_1$ and let α_k denote $g^k \alpha g^{-k}$ for $k \in \mathbb{Z}$. The goal is to prove that finitely many elements in the set $\{\alpha_k \mid k \in \mathbb{Z}\}$ generate the subgroup N.

   Verify that
   $$g\alpha^{s_0} g\alpha^{s_1} \cdots g\alpha^{s_m} = \alpha_1^{s_0} \alpha_2^{s_1} \cdots \alpha_{m+1}^{s_m} g^{m+1}.$$

3. Prove that if for every integer $m \geq 1$ the map
   $$\mu = \mu_m : \prod_{i=0}^{m} \mathbb{Z}_2 \to G$$
   $$\mu : (s_i) \mapsto g\alpha^{s_0} g\alpha^{s_1} \cdots g\alpha^{s_m}.$$
   is injective then G must have exponential growth.

4. Deduce from the fact that $\mu_m, m \geq 1$, cannot be all injective the fact that N is finitely generated.