Homological algebra

André Henriques

$\mathbb{Z}$-modules are the same thing as abelian groups. The direct sum of R-modules M_i is defined by

$$\bigoplus_{i\in\mathcal{I}} M_i := \Big\{ f:\mathcal{I}\to\bigcup_{i\in\mathcal{I}} M_i \,\Big|\, f(i)\in M_i \text{ and } \#\{i\in\mathcal{I}: f(i)\neq 0_{M_i}\}<\infty\Big\}.$$

The product of modules M_i is given by

$$\prod_{i\in\mathcal{I}} M_i := \Big\{ f:\mathcal{I}\to\bigcup_{i\in\mathcal{I}} M_i \,\Big|\, f(i)\in M_i\Big\}.$$

The R-module structure is given by $(f+g)(i)=f(i)+_{M_i} g(i)$ and $(r\cdot f)(i)=r\cdot_{M_i}(f(i))$.

An inclusion of of R-modules $N\subset M$ is called *split* if there exists another submodule $N'\subset M$ such every element of M can be uniquely written as a sum of an element of N and an element of N'. *Example:* the inclusion $\mathbb{Z}/2\hookrightarrow\mathbb{Z}/4$ is not split, but the inclusion $\mathbb{Z}/2\hookrightarrow\mathbb{Z}/6$ is split. A sequence

$$0\to A\stackrel{i}{\to} B\stackrel{\pi}{\to} C\to 0$$

is called a *short exact sequence* if i is injective, π is surjective, and $\ker(\pi)=\mathrm{im}(i)$. Equivalently, if π exhibits C as the cokernel of f and i exhibits A as the kernel of π.

The *kernel* of a map $f:X\to Y$ is a morphism $i:K\to X$ which is universal w.r.t the property that $f\circ i=0$. This means the following: it's an object K along with a morphism $i:K\to X$ satisfying $f\circ i=0$, such that for every object $\tilde K$ and every morphism $\tilde i:\tilde K\to X$ satisfying $f\circ\tilde i=0$, there exists a unique morphism $g:\tilde K\to K$ such that $\tilde i=i\circ g$. Dually, the *cokernel* of a map $f:X\to Y$ is a morhpism $q:Y\to C$ which is universal w.r.t the property that $q\circ f=0$.

A short exact sequence is split if it is isomorphic to one of the form $0\to A\to A\oplus C\to C\to 0$.

Lemma 1. *A short exact sequence $0\to A\stackrel{i}{\to} B\stackrel{\pi}{\to} C\to 0$ is split iff there exists a retraction $r:B\to A$ (a map satisfying $ri=1_A$) iff there exists a section $s:C\to B$ (a map satisfying $\pi s=1_C$).*

Proof. Assuming the existence of a retraction $r:B\to A$, we construct a section $s:C\to B$. Consider the map $1-ir:B\to B$. This map is zero on $i(A)\subset B$, and therefore descends to a map $s:B/i(A)\cong C\to B$. We check that for $c\in C$, we have $\pi s(c)=(c)$:

$$\pi s(c) \overset{\substack{\text{pick } b\in B,\\ \pi(b)=c\\ \downarrow}}{=} \pi\big(b-ir(b)\big)=\pi(b)-\pi ir(b)\overset{\substack{\pi\circ i=0\\ \downarrow}}{=}\pi(b)=c$$

Assuming the existence of a section $s:C\to B$, we construct a retraction $r:B\to A$. Consider the map $1-s\pi:B\to B$. The composite of this map with the projection $\pi:B\to C$ is zero. Its image therefore lands in $\ker(\pi)=i(A)\subset B$. Let $r:=i^{-1}(1-s\pi)$. We check:

$$ri(a)=i^{-1}\big(i(a)-s\pi i(a)\big)\overset{\substack{\pi i=0\\ \downarrow}}{=} i^{-1}i(a)=a.$$

Finally, assuming the existence of a section s and a retraction r, we can identify B with the direct sum $A\oplus C$ via the maps $B\to A\oplus C: b\mapsto(r(b),\pi(b))$ and $A\oplus C\to B:(a,c)\mapsto i(a)+s(c)$. □

Given a ring R, the tensor product over R of a right module M with a left module N is denoted $M \otimes_R N$. It is the abelian group generated by symbols $m_1 \otimes n_1 + \ldots + m_k \otimes n_k$, under the equivalence relation generated by

$$\begin{aligned} (m+m') \otimes n &= m \otimes n + m' \otimes n, \\ m \otimes (n+n') &= m \otimes n + m \otimes n', \\ \text{and} \qquad mr \otimes n &= m \otimes rn. \end{aligned}$$

If R is non-commutative, then $M \otimes_R N$ is just an abelian group. If R is commutative, then it is an R-module, via $r \cdot (\sum m_i \otimes n_i) := \sum rm_i \otimes n_i$.

Given two left R-modules M and N, we write $\mathrm{Hom}_R(M, N)$ for the set of R-module homomorphism from M to N. If R is non-commutative, then $\mathrm{Hom}_R(M, N)$ is just an abelian group. If R is commutative, then it is an R-module, with $(r \cdot f)(m) := r \cdot (f(m))$.

There are canonical isomorphisms

$$\Big(\bigoplus A_i\Big) \otimes B \cong \bigoplus (A_i \otimes B), \qquad \mathrm{Hom}\Big(\bigoplus A_i, B\Big) \cong \prod \mathrm{Hom}(A_i, B), \qquad \mathrm{Hom}\Big(A, \prod B_i\Big) \cong \prod \mathrm{Hom}(A, B_i).$$

There are also canonical isomorphisms $R \otimes_R N \cong N$, $M \otimes_R R \cong M$, and $\mathrm{Hom}_R(R, N) \cong N$. More generally, if $I < R$ is a left ideal, then there are canonical isomorphisms

$$M \otimes_R R/I \cong M/MI, \qquad \text{and} \qquad \mathrm{Hom}_R(R/I, N) \cong \{n \in N \,|\, rn = 0 \;\forall r \in I\}.$$

We provide a proof for the first isomorphism:

Proof. The isomorphism $M \otimes_R R/I \to M/MI$ is given by

$$\sum m_i \otimes [r_i] \mapsto \Big[\sum m_i \otimes r_i\Big].$$

This map is well defined because (1) $(m+m') \otimes [r]$ and $m \otimes [r] + m' \otimes [r]$ map to the same element $[(m+m')r]$ of M/MI, (2) $m \otimes ([r]+[r'])$ and $m \otimes [r] + m \otimes [r']$ map to the same element $[m(r+r')]$of M/MI, (3) $mr_1 \otimes [r]$ and $m \otimes r_1[r]$ map to the same element $[mr_1 r]$ of M/MI, and (4) for any $a \in I$, the elements $m \otimes [r]$ and $m \otimes [r+a]$ map to the same element $[mr] = [m(r+a)]$ of M/MI. The inverse map is given by

$$M/MI \to M \otimes_R R/I \; : \; [m] \mapsto m \otimes [1].$$

It is well defined because for $m = m'a$ with $a \in I$, the image of $[m]$ under that map is given by $m'a \otimes [1] = m' \otimes a[1] = m' \otimes 0$, which is zero in $M \otimes_R R/I$.

The composite $M/MI \to M \otimes_R R/I \to M/MI$ is obviously the identity. The other composite $M \otimes_R R/I \to M/MI \to M \otimes_R R/I$ sends $\sum m_i \otimes [r_i]$ to $\big(\sum m_i r_i\big) \otimes [1]$. It is the identity since

$$\Big(\sum m_i r_i\Big) \otimes [1] = \sum \big(m_i r_i \otimes [1]\big) = \sum m_i \otimes r_i[1] = \sum m_i \otimes [r_i].$$

□

A chain complex of R-modules $C_\bullet = (C_n, d_n)_{n \in \mathbb{Z}}$ is a collection of R-modules C_n and R-module maps $d_n : C_n \to C_{n-1}$, called 'differentials', subject to the axiom $d_n \circ d_{n+1} = 0$. This axiom is sometimes abusively abbreviated $d^2 = 0$. A chain complex is called *exact* if $\ker(d_n) = \mathrm{im}(d_{n+1})$.

The *homology* of a chain complex of R-modules $C_\bullet = (C_n, d_n : C_n \to C_{n-1})_{n \in \mathbb{Z}}$ is defined by

$$H_n(C_\bullet) := \frac{Z_n}{B_n} := \frac{\ker(d_n : C_n \to C_{n-1})}{\mathrm{im}(d_{n+1} : C_{n+1} \to C_n)}$$

Here Z_n are called the *cycles*, and B_n are called the *boundaries*. If $C_\bullet$ is a chain complex in an arbitrary abelian category (to be defined later), the object $H_n(C_\bullet)$ can be defined in purely categorical terms, as the cokernel of the canonical map $C_{n+1} \to \ker(d_n : C_n \to C_{n-1})$.

A morphism of chain complexes $f_\bullet : C_\bullet \to D_\bullet$ is called a *quasi-isomorphism* if it induces an isomorphism at the level of homology: $H_n(f_\bullet) : H_n(C_\bullet) \overset{\cong}{\to} H_n(D_\bullet), \forall n \in \mathbb{Z}$.

An additive functor between abelian categories (to be defined later) is called *exact* if it sends exact sequences to exact sequences, equivalently, if it sends short exact sequences to short exact sequences. Note that the functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$ is not exact: it sends the exact sequence

$$0 \to \mathbb{Z} \overset{\cdot 2}{\to} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0$$

to the sequence $0 \to \mathbb{Z}/2 \overset{0}{\to} \mathbb{Z}/2 \overset{\simeq}{\to} \mathbb{Z}/2 \to 0$ which is not exact. Similarly, the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$ sends the exact sequence

$$0 \to \mathbb{Z} \overset{\cdot 2}{\to} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0$$

to the sequence $0 \to 0 \to 0 \to \mathbb{Z}/2 \to 0$ which is not exact. Finally, the contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2)$ sends the exact sequence

$$0 \to \mathbb{Z} \overset{\cdot 2}{\to} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0$$

to the sequence $0 \leftarrow \mathbb{Z}/2 \overset{0}{\leftarrow} \mathbb{Z}/2 \overset{\simeq}{\leftarrow} \mathbb{Z}/2 \leftarrow 0$ which is not exact. The functors $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$ and $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2)$ are therefore not exact.

A functor F is *right exact* if for every short exact sequence $0 \to A \to B \to C \to 0$, the sequence $F(A) \to F(B) \to F(C) \to 0$ is exact. Similarly, a functor F is *left exact* if whenever $0 \to A \to B \to C \to 0$ is exact, then $0 \to F(A) \to F(B) \to F(C)$ is exact.

Lemma 2. *Let $\mathcal{A}$ be an abelian category, and let $M \in \mathcal{A}$ be an object. Then the functor* $\mathrm{Hom}_{\mathcal{A}}(M, -) : \mathcal{A} \to \mathrm{AbGrp}$ *is left exact.*

Proof. Let $0 \to A \overset{\iota}{\to} B \overset{\pi}{\to} C \to 0$ be a short exact sequence in $\mathcal{A}$. We need to show that $0 \to \mathrm{Hom}(M, A) \xrightarrow{\iota_*} \mathrm{Hom}(M, B) \xrightarrow{\pi_*} \mathrm{Hom}(M, C)$ is exact.

- $\underline{\iota_* \text{ is injective}}$. Let $\alpha \in \mathrm{Hom}(M, A)$ be an element that maps to zero in $\mathrm{Hom}(M, B)$. Since $\iota \circ \alpha = 0$, and ι is a monomorphism (see Lemma 4 below), $\alpha = 0$. So ι_* is injective.
- $\underline{\mathrm{im}}(\iota_*)\underline{\subset \ker}(\pi_*)$. Follows trivially from the fact that $\pi \circ \iota = 0$.
- $\underline{\ker}(\pi_*)\underline{\subset \mathrm{im}}(\iota_*)$. Let $\beta \in \mathrm{Hom}(M, B)$ be an element that maps to zero in $\mathrm{Hom}(M, C)$. Since $\pi \circ \beta = 0$, the map $\beta : M \to B$ factors through $\ker(\pi) = A$. So we can write β as $\iota \circ \alpha$ for some $\alpha \in \mathrm{Hom}(M, A)$. We have $\beta = \iota_*(\alpha)$, and hence $\beta \in \mathrm{im}(\iota_*)$. $\square$

Corollary. Let R be a ring and let M be an R-module. Then the functors

$$\mathrm{Hom}_R(M, -) : R\text{-Mod} \to \mathrm{AbGrp} \qquad \text{and} \qquad \mathrm{Hom}_R(-, M) : R\text{-Mod}^{op} \to \mathrm{AbGrp}$$

are left exact.

Lemma 3. *The functor $- \otimes_R N$ is right exact.*

Proof. Given a short exact sequence of right R-modules $0 \to A \overset{\iota}{\to} B \overset{\pi}{\to} C \to 0$, we need to show that $A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$ is exact. The surjectivity of $B \otimes_R N \to C \otimes_R N$ is easy, so let us focus on the harder argument: given an element $\sum b_i \otimes n_i \in B \otimes_R N$ that goes to zero in $C \otimes_R N$, we need to show that it comes from $A \otimes_R N$.

Since $\sum \pi(b_i) \otimes n_i = 0$ in $A \otimes_R N$, there exist elements c'_α, c''_α, n_α, c_β, n'_β, n''_β, c_γ, r_γ, n_γ such that

$$\begin{aligned}\sum_i \pi(b_i) \otimes n_i + \sum_\alpha (c'_\alpha + c''_\alpha) \otimes n_\alpha - c'_\alpha \otimes n_\alpha - c''_\alpha \otimes n_\alpha \\ + \sum_\beta c_\beta \otimes (n'_\beta + n''_\beta) - c_\beta \otimes n'_\beta - c_\beta \otimes n''_\beta \\ + \sum_\gamma c_\gamma r_\gamma \otimes n_\gamma - c_\gamma \otimes r_\gamma n_\gamma\end{aligned}$$

is zero in the free abelian group on the set of symbols "$c \otimes n$". If we mod out that free abelian group by the first set of relations $(c' + c'') \otimes n = c' \otimes n + c'' \otimes n$, then we get the abelian group $\bigoplus_{n \in N} C$. So, another way of saying that $\sum \pi(b_i) \otimes n_i$ is zero in $A \otimes_R N$ is to say that there exist elements c_β, n'_β, n''_β, c_γ, r_γ, n_γ such that

$$\sum_i \pi(b_i) \otimes n_i + \sum_\beta c_\beta \otimes (n'_\beta + n''_\beta) - c_\beta \otimes n'_\beta - c_\beta \otimes n''_\beta + \sum_\gamma c_\gamma r_\gamma \otimes n_\gamma - c_\gamma \otimes r_\gamma n_\gamma = 0 \text{ in } \bigoplus_{n \in N} C,$$

where "$c \otimes n$" now stands for the element c put in the n-th copy of C.

Pick preimages b_β, $b_\gamma \in B$ of $c_\beta, c_\gamma \in C$, and consider the element

$$y := \sum_i b_i \otimes n_i + \sum_\beta b_\beta \otimes (n'_\beta + n''_\beta) - b_\beta \otimes n'_\beta - b_\beta \otimes n''_\beta + \sum_\gamma b_\gamma r_\gamma \otimes n_\gamma - b_\gamma \otimes r_\gamma n_\gamma \quad \in \bigoplus_{n \in N} B.$$

This element goes to 0 in $\bigoplus_{n \in N} C$ and therefore comes from some $x \in \bigoplus_{n \in N} A$.

Let $[x]$ denote the image of x in $A \otimes_R N$ and let $[y]$ denote the image of y in $B \otimes_R N$. Since $x \mapsto y$, it follows that $[x] \mapsto [y]$. We are done since $[y] = \sum_i b_i \otimes n_i$ in $B \otimes_R N$. □

A *terminal object* is an object that admits exactly one morphism to it from any other object. An *initial object* is an object that admits exactly one morphism from it to any other object. A *zero object* is an object that admits exactly one morphism to it from any other object and exactly one morphism from it to any other object, i.e., is both initial and terminal.

A *monomorphism* is a morphism f that satisfies $(f \circ g_1 = f \circ g_2) \Rightarrow (g_1 = g_2)$. Equivalently, it is a morphism $f : X \to Y$ with the property that whenever two morphisms $g_1, g_2 : Z \to X$ are distinct, they remain distinct after composing them with f. Dually, an *epimorphism* is a map f that satisfies $(g_1 \circ f = g_2 \circ f) \Rightarrow (g_1 = g_2)$.

The *direct sum* of two objects X_1 and X_2 is an object Z equipped with maps $i_1 : X_1 \to Z$, $i_2 : X_2 \to Z$, $p_1 : Z \to X_1$, $p_2 : Z \to X_2$ satisfying $p_1 \circ i_1 = \text{id}$, $p_2 \circ i_2 = \text{id}$, $p_1 \circ i_2 = 0$, $p_2 \circ i_1 = 0$, and $i_1 \circ p_1 + i_2 \circ p_2 = \text{id}$.

An pre-additive category is a category such that all the hom-sets are equipped with the structure of abelian groups and such that composition $\text{Hom}(x, y) \times \text{Hom}(y, z) \to \text{Hom}(x, z)$ is bilinear. An *additive category* is a category which is preadditive, admits a zero object, and admits all direct sums.

An additive category is called *abelian* if for every monomorphism $f : A \rightarrowtail B$, the pair (A, f) is a kernel of the morphism $B \to \text{coker}(f)$, and for every epimorphism $f : A \twoheadrightarrow B$ the pair (B, f) is a cokernel of the morphism $\ker(f) \to A$.

A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact iff (A, f) is a kernel of g and (C, g) is a cokernel of f.

The homology of a chain complex $\ldots C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \ldots$ is the cokernel of the map $C_{n+1} \to \ker(d_n)$.

Lemma 4. *Kernels are monomorphisms; cokernels are epimorphisms.*

Proof. Let $f : A \to B$ be a morphism. Consider two morphisms $a, b : X \to \ker(f)$ with the property that $\iota a = \iota b$:

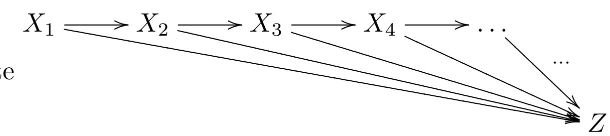
$$X \underset{b}{\overset{a}{\rightrightarrows}} \ker(f) \xrightarrow{\iota} A \xrightarrow{f} B$$

Since $f\iota a = 0$, by the universal property of $\ker(f)$, there exists a unique morphism $X \to \ker(f)$ whose composition with ι yields ιa. Both a and b satisfy that property. So they're equal. $\square$

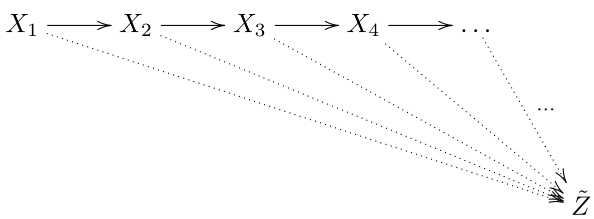
Lemma 5 (exercise)**.** *A morphism f is an epimorphism if and only if* $\operatorname{coker}(f) = 0$.

A *colimit* (also called *direct limit*) of a sequence of morphisms $X_1 \to X_2 \to X_3 \to \ldots$ is an object Z along with morphisms $X_i \to Z$ such that

all little triangles commute

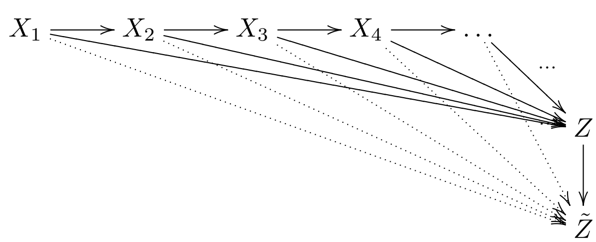


and such that for every other diagram



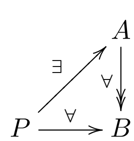
all of whose little triangles commute

there exists a unique morphism $Z \to \tilde{Z}$ such that all the triangles in this big diagram commute:



The colimit can be denoted $\operatorname{colim} X_i$ or $\varinjlim X_i$. Quite often 'colimit' means the same thing as 'union'. The dual notion is called a *limit*. It is denoted $\lim X_i$ or $\varprojlim X_i$.

An object P is projective if the functor $\operatorname{Hom}(P, -)$ sends epimorphisms to epimorphisms. Equivalently, if for every epimorphism $f : A \twoheadrightarrow B$, the map $f \circ - : \operatorname{Hom}(P, A) \to \operatorname{Hom}(P, B)$ is surjective. Equivalently, an object P of an abelian category is called *projective* if for every epimorphism $A \to B$ and every morphism $P \to B$, there exists a morphism $P \to A$ such that the triangle commutes:



Lemma 6. *An R-module is projective if it is a direct summand of a free module.*

The next exercise is a long and painful one which I don't expect you (or want you) to finish. But I do want you to start it. Write down what you think is approximately 50% of the proof, and then write "I give up" (or, if you don't want to give up, you may hand in a complete answer):

A projective resolution $P_\bullet \to M$ of an R-module M is an exact sequence of R-modules $\ldots \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to 0$, where all the P_n are projective R-modules.

Let M be a right R-module and N a left R-module. Then:

$$\mathrm{Tor}_i^R(M,N) := H_i(P_\bullet \otimes_R N) = H_i(M \otimes_R Q_\bullet)$$

where $P_\bullet$ is a projective resolution of M, or $Q_\bullet$ is a projective resolution of N. Implicit in the above definition is the fact that $\mathrm{Tor}_i^R(M,N)$ doesn't depend on the choice of projective resolution, and doesn't depend on whether one resolves M or N.

Let M and N be R-modules (either both right modules or both left modules). Then:

$$\mathrm{Ext}^i_R(M,N) := H^i(\mathrm{Hom}_R(P_\bullet, N)) = H^i(\mathrm{Hom}_R(M, I^\bullet)).$$

Here, $P_\bullet$ is a projective resolution of M and $I^\bullet$ is an injective resolution of N (injective objects are defined below). Once again, the choice of resolution doesn't matter, neither does the choice of which of the two modules one decides to resolve.

Here, a module E is called injective if for every monomorphism $i : A \to B$ and for every map $f : A \to E$, one can factorise f as $f = gi$ for some $g : B \to E$.

If $R = \mathbb{Z}$, then every module admits a resolution of length 1. This implies that $\mathrm{Tor}_i^{\mathbb{Z}}$ and $\mathrm{Ext}^i_{\mathbb{Z}}$ vanishes as soon as $i > 1$. This property is called '$\mathbb{Z}$ has cohomological dimension one'.

An abelian category A has enough projectives is for every object $M \in A$, there exists a projective object $P \in A$ and an epimorphism $P \to M$. The category of left R-modules has enough projectives: Given a module M, pick a set $\{m_i\}_{i \in I} \subset M$ of generators. The free module $F := \bigoplus_I R$ surjects onto M by sending the i-th basis element $e_i \in F$ to the generator $m_i \in M$. Finally, we note that free modules are projective.

Let A and B be abelian categories. Assume that A has enough projectives. Let $F : A \to B$ be an additive functor (often assumed to be right exact). The nth *left derived functor* of F, denoted $L_nF : A \to B$ is defined by $X \mapsto H_n(F(P_\bullet))$, where $P_\bullet \to X$ is a projective resolution.

Assume now that A has enough injectives and that $F : A \to B$ is an additive functor (often assumed to be left exact). The nth *right derived functor* of F, denoted $R^nF : A \to B$ is defined by $X \mapsto H^n(F(I^\bullet))$, where $X \to I^\bullet$ is an injective resolution.

Lemma 7. *If F is right exact, then $L_0F = F$. (If F is left exact, then $R^0F = F$.)*

Proof. Let $P_\bullet \to M$ be a projective resolution, so that $P_1 \xrightarrow{d} P_0 \xrightarrow{\varepsilon} M \to 0$ is exact. By definition, $L_0F(M) = \mathrm{coker}(F(d))$. Consider the short exact sequence $0 \to K \to P_0 \to M \to 0$, where $K := \ker(\varepsilon)$. The comparison map $P_1 \to K$ is an epimorphism by the exactness of $P_\bullet \to M$. Since right exact functors send epimorphisms to epimorphisms, the map $F(P_1) \to F(K)$ is then also an epimorphism.

By the right exactness of F, the sequence $F(K) \to F(P_0) \to F(M) \to 0$ is exact. So $F(M) = \ker(F(K) \to F(P_0)) = \ker(F(P_1) \to F(P_0)) = L_0F(M)$. The middle equality holds true because composing with an epimorphism (namely with the map $F(P_1) \to F(K)$) doesn't change cokernels; see the next lemma. □

Lemma 8 (exercise)**.** *Given composable morphisms $A \overset{f}{\twoheadrightarrow} B \xrightarrow{g} C \xrightarrow{h} D$, the morphism h is a cokernel of g if and only if it is a cokernel of $g \circ f$.*

A morphism of chain complexes $f_\bullet : C_\bullet \to D_\bullet$ induces a corresponding morphism at the level of cohomology groups $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)$. Two chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are called *chain homotopic* if there exists a degree -1 map $h : C_\bullet \to D_\bullet$ satisfying $hd + dh = f - g$.

There are two ways of making the operation "take a projective resolution" into a functor:

(1) Take P_0 to be the free R-module on the underlying set of M. Take P_1 to be the free R-module

on the underlying set of $\ker(P_0 \to M)$. Take P_2 to be the free R-module on the underlying set of $\ker(P_1 \to P_0)$. Etc.

(2) View the operation "take a projective resolution" as a functor from our abelian category $\mathcal{A}$ to its derived category $D(\mathcal{A})$.

Definition: Let $\mathcal{A}$ be an abelian category. Its *derived category* $D(\mathcal{A})$ has:
• Object = positively graded chain complexes of projectives of $\mathcal{A}$
• Morphisms = chain maps modulo chain homotopy.

The notion of chain homotopy is made so that whenever $f_\bullet : C_\bullet \to D_\bullet$ and $g_\bullet : C_\bullet \to D_\bullet$ are chain homotopic maps, then $H_*(f_\bullet) = H_*(g_\bullet) : H_*(C_\bullet) \to H_*(D_\bullet)$.

Here's a way of defining the nth derived functor of an additive functor $F : \mathcal{A} \to \mathcal{B}$:

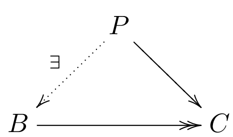
$$L_nF \;:\; \mathcal{A} \xrightarrow[\text{resolution}]{\text{take projective}} D(\mathcal{A}) \xrightarrow{\text{apply } F} \left(\mathrm{Ch}(\mathcal{B}); \begin{smallmatrix}\text{chain maps modulo}\\ \text{chain homotopy}\end{smallmatrix}\right) \xrightarrow{H_n} \mathcal{B}$$

The *total derived functor* of F, or simply "the derived functor of F" is the functor

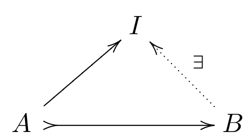
$$LF \;:\; \mathcal{A} \xrightarrow[\text{resolution}]{\text{take projective}} D(\mathcal{A}) \xrightarrow{\text{apply } F} \left(\mathrm{Ch}(\mathcal{B}); \begin{smallmatrix}\text{chain maps modulo}\\ \text{chain homotopy}\end{smallmatrix}\right) \xrightarrow[\text{resolution}]{\text{take projective}} D(\mathcal{B}).$$

Here, a projective resolution of a chain complex $C_\bullet$ is the data of a chain complex of projectives $P_\bullet$ together with a map of chain complexes $P_\bullet \to C_\bullet$ which is a quasi-isomorphism.

Recall that a module (or object of some arbitrary abelian category) P is projective if for every solid arrow diagram there exists a dotted arrow making the diagram commute:



In the same vein, a module (or object of some arbitrary abelian category) I is called injective if for every solid arrow diagram there exists a dotted arrow making the diagram commute:



A module P is *projective* iff $\mathrm{Hom}_R(P, -)$ is exact. A module I is *injective* iff $\mathrm{Hom}_R(-, I)$ is exact. A module F is *flat* if $- \otimes_R F$ is exact. Every projective module is flat. Indeed, if $M = M' \oplus M''$, then we have (M is flat) $\Leftrightarrow$ (M' is flat and M'' is flat). Starting from the obvious fact that free modules are flat, we conclude that every projective module is flat.

Example: $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. That's because $\mathbb{Q} = \mathrm{colim}(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \xrightarrow{\cdot 5} \ldots)$ and for every abelian group A we have

$$\mathbb{Q} \otimes_{\mathbb{Z}} A = \mathrm{colim}(A \xrightarrow{\cdot 2} A \xrightarrow{\cdot 3} A \xrightarrow{\cdot 4} A \xrightarrow{\cdot 5} \ldots).$$

In order to check that $\mathbb{Q}$ is flat, one needs to check that an injective map $f : A \to B$ remains injective after applying the functor $\mathbb{Q} \otimes_{\mathbb{Z}} -$. This is a diagram chase in the diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{\cdot 2} & A & \xrightarrow{\cdot 3} & A & \xrightarrow{\cdot 4} & \ldots \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\ B & \xrightarrow{\cdot 2} & B & \xrightarrow{\cdot 3} & B & \xrightarrow{\cdot 4} & \ldots \end{array}$$

Lemma 9. *A short exact sequence of chain complexes $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ (which, by definition, means that for each n the sequence $0 \to A_n \to B_n \to C_n \to 0$ is exact) induces a long exact sequence in homology.* See p. 117 of Hatcher's book for a proof.

A bigraded chain complex $C_{\bullet\bullet}$ is a sequence of abelian groups $C_{p,q}$ (or objects of some abelian category) together with maps $d_h : C_{p,q} \to C_{p-1,q}$ and $d_v : C_{p,q} \to C_{p,q-1}$ satisfying $d_h d_h = 0$, $d_v d_v = 0$, and $d_h d_v = d_v d_h$. The total chain complex $\mathrm{Tot}(C_{\bullet\bullet})$ is defined by

$$\big[\mathrm{Tot}(C_{\bullet\bullet})\big]_n = \bigoplus_{p+q=n} C_{p,q}$$

The differential $d^{\mathrm{Tot}} : \big[\mathrm{Tot}(C_{\bullet\bullet})\big]_n \to \big[\mathrm{Tot}(C_{\bullet\bullet})\big]_{n-1}$ is the sum of the maps $d_h : C_{p,q} \to C_{p-1,q}$ and $(-1)^p \cdot d_v : C_{p,q} \to C_{p,q-1}$ over all p,q such that $p+q=n$. There's also a variant of Tot where one uses direct products instead of direct sums

$$\big[\mathrm{Tot}^{\prod}(C_{\bullet\bullet})\big]_n = \prod_{p+q=n} C_{p,q}$$

Lemma 10. *Let $C_{\bullet\bullet}$ be a double complex such that for every n there exists only finitely many pairs (p,q), $p+q=n$, such that $C_{p,q} \neq 0$. Then we have*

$$\Big(C_{\bullet\bullet} \text{ has exact rows}\Big) \Rightarrow \Big(\mathrm{Tot}(C_{\bullet\bullet}) \text{ is exact}\Big)$$

More generally, if $C_{\bullet\bullet}$ is a double complex such that for every n the set $\{p \in \mathbb{Z} \,|\, C_{p,n-p} \neq 0\}$ is bounded below, then

$$\Big(C_{\bullet\bullet} \text{ has exact rows}\Big) \Rightarrow \Big(\mathrm{Tot}^{\prod}(C_{\bullet\bullet}) \text{ is exact}\Big)$$

$\mathrm{Tor}_i^R(M,N)$ and $\mathrm{Ext}^i_R(M,N)$ are independent of the choice of resolution. They can be computed by resolving either M or N.

Let M be a right R-module and N a left R-module, let $P_\bullet$ be a projective resolution of M and $Q_\bullet$ a projective resolution of N. Then we have quasi-isomorphisms

$$P_\bullet \otimes_R N \leftarrow \mathrm{Tot}(P_\bullet \otimes_R Q_\bullet) \to M \otimes_R Q_\bullet$$

inducing isomorphisms

$$H_i(P_\bullet \otimes_R N) \cong H_i(\mathrm{Tot}(P_\bullet \otimes_R Q_\bullet)) \cong H_i(M \otimes_R Q_\bullet).$$

The isomorphism $H_i(P_\bullet \otimes_R N) \stackrel{\cong}{\leftarrow} H_i(\mathrm{Tot}(P_\bullet \otimes_R Q_\bullet))$ is the connecting homomorphism in the LES associated to the short exact sequence

$$0 \to P_\bullet \otimes_R N \to \mathrm{Tot}\big(P_\bullet \otimes_R Q_\bullet \to P_\bullet \otimes_R N\big) \to \mathrm{Tot}\big(P_\bullet \otimes_R Q_\bullet\big) \to 0.$$

The fact that the middle term is acyclic (the words 'acyclic' and 'exact' are synonyms) follows from Lemma 10 below.

Let now M and N be R-modules (either both right modules or both left modules). Let $P_\bullet$ be a projective resolution of M and $I^\bullet$ an injective resolution of N. Then we have quasi-isomorphisms

$$\mathrm{Hom}_R(P_\bullet, N) \to (\mathrm{Tot}(\mathrm{Hom}_R(P_\bullet, I^\bullet)) \leftarrow \mathrm{Hom}_R(M, I^\bullet)$$

and $\mathrm{Ext}^i_R(M,N)$ can be computed in any one of the following ways:

$$H^i(\mathrm{Hom}_R(P_\bullet, N)) \cong H^i(\mathrm{Tot}(\mathrm{Hom}_R(P_\bullet, I^\bullet))) \cong H^i(\mathrm{Hom}_R(M, I^\bullet)).$$

If instead one takes a projective resolution $Q_\bullet$ of N, then one has yet another chain complex that computes $\mathrm{Ext}^*_R(M,N)$, namely $\mathrm{Tot}^{\prod}(\mathrm{Hom}_R(P_\bullet, Q_\bullet))$.

The *pullback* of a diagram of modules $A \xrightarrow{f} C \xleftarrow{g} B$ is the set $\{(a,b) \in A \oplus B : f(a) = g(b)\}$. It is also the limit of the diagram $A \to C \leftarrow B$. The *pushout* of a diagram of modules $A \xleftarrow{f} C \xrightarrow{g} B$ is the quotient $A \oplus B/\{(f(c), -g(c)) : c \in C\}$. It is also the colimit of the diagram $A \leftarrow C \to B$.

A *diagram* of R-modules indexed by a poset P is just a functor $P \to R$-Mod. Concretely, this is the data of R-modules M_α indexed by P, and maps $f_{\alpha\beta} : M_\alpha \to M_\beta$ for all $\alpha < \beta \in P$, satisfying $f_{\beta\gamma} f_{\alpha\beta} = f_{\alpha\gamma}$.

The *limit* of a a diagram $P \to R$-Mod (where P is a poset) can be described concretely as $\big\{(m_\alpha) \in \prod_{\alpha \in P} M_\alpha : f_{\alpha\beta}(m_\alpha) = m_\beta, \forall \alpha < \beta \in P\big\}$. The *colimit* of a a diagram $P \to R$-Mod is given by $\bigoplus_{\alpha \in P} M_\alpha \big/ \mathrm{Span}\{m - f_{\alpha\beta}(m) : m \in M_\alpha\}$. Limits and colimits can alternatively be defined by means of a universal property.

A poset is called *directed* if for every $x, y \in P$, there exists $z \in P$ such that $z \geq x$ and $z \geq y$. If P is a directed poset, then every element of $\mathrm{colim}_{\alpha \in P} M_\alpha$ is represented by some element m of some M_α. Moreover, if P is a direct poset, then an element $m \in M_\alpha$ represents the zero element in $\mathrm{colim}_{\alpha \in P} M_\alpha$ iff there exists some $\beta \geq \alpha$ in P such that m becomes zero in M_β.

The latter fails miserably for e.g. pushout($\mathbb{Z}/2 \leftarrow \mathbb{Z} \to \mathbb{Z}/3$).

Theorem (Baer's criterion)

An R-module E is injective if and only if every left ideal $I < R$ and any map $I \to E$, the extension problem

$$\begin{array}{ccc} E & & \\ \uparrow & \nwarrow & \\ I & \longrightarrow & R \end{array}$$

admits a solution.

See e.g. https://ncatlab.org/nlab/show/Baer's+criterion for a proof.

Corollary of Baer's criterion: if R is a PID, then a module M is injective iff it is *divisible*, i.e. iff for every $x \in M$ and every non-zero $r \in R$ there exists $y \in M$ such that $ry = x$.

Proof: Let M be an injective module. Given $\forall r \in R \backslash \{0\}$, since R is a PID, the map $r\cdot : R \to R$ is injective. Given an element $m \in M$, consider the map $R \to M : 1 \mapsto m$. Since M is injective, we may factor it as a composite $R \xrightarrow{r\cdot} R \xrightarrow{\phi} M$. Let $m' := \phi(1)$. One checks that $m = \phi(r \cdot 1) = r\phi(1) = rm'$, as desired.

Let M be a divisible module. Since R is a PID, the inclusion of a non-zero ideal $I \hookrightarrow R$ is isomorphic to the map $R \xrightarrow{r\cdot} R$, for some $r \in R \setminus \{0\}$. Therefore, an R-module E is injective iff for every $r \in R \setminus \{0\}$ and every morphism $f : R \to E$ there exists a morphism $g : R \to E$ such that $f(x) = g(rx)$, $\forall x \in R$.

Given $r \in R \setminus \{0\}$ and a map $f : R \to M$ as above, we need to find $g : R \to M$ such that $f(x) = g(rx)$. Let $m := f(1)$. Since M is divisible, $\exists m'$ such that $m = rm'$. Then $g : r \mapsto rm'$ is the desired map. *qed.*

An abelian category is said to *have enough projectives* if for every object X, there exists a projective object P and an epimorphism $P \to X$. Dually, an abelian category is said to *have enough injectives* if for every object X, there exists an injective object I and a monomorphism $X \to I$.

It is easy to see that for any ring R, the category of R-modules has enough projectives: take P to be free R module on the underlying set of X (any generating set would also do).

Showing the R-mod has enough injectives is much harder. Given an R-module M, let S denote the set of all pairs (I, f), where I is an ideal of R, and $f : I \to M$ is an R-module homomorphism.

We write M' for the following pushout:

$$\begin{array}{ccc} M & \longrightarrow & M' \\ {\scriptstyle \bigoplus f}\uparrow & & \uparrow \\ \bigoplus_{(I,f)\in S} I & \longrightarrow & \bigoplus_{(I,f)\in S} R \end{array}$$

Write $M_0 := M$ and $M_{n+1} := (M_n)'$. If every ideal is finitely generated, then $M_\infty := \mathrm{colim}(M_0 \to M_1 \to M_2 \to \ldots)$ is an injective module. It obviousely contains M as a submodule. To show that M_∞ is injective, we use Baer's criterion. Using the fact that every ideal is finitely generated, every map $f : I \to M_\infty$ factors through some finite stage of the colimit, let's say $f : I \to M_n$. The extension problem will then admit a solution at the next stage: $\begin{array}{ccc} M_{n+1} & & \\ {\scriptstyle f}\uparrow & \overset{\exists}{\nwarrow} & \\ I & \longrightarrow & R \end{array}$. Here, the map $R \to M_{n+1}$ comes from $\begin{array}{ccc} M_n & \longrightarrow & M_{n+1} \\ {\scriptstyle \bigoplus f}\uparrow & & \uparrow \\ \bigoplus_{(I,f)} I & \longrightarrow & \bigoplus_{(I,f)} R \\ \uparrow & & \uparrow \\ I & \longrightarrow & R \end{array}$, where the bottom vertical maps $I \to \bigoplus_{(I,f)} I$ and $R \to \bigoplus_{(I,f)} R$ are the inclusions of the summands indexed by (I, f).

For general rings, i.e. without the condition that every ideal is finitely generated, then a similar construction can be made to work, provided one replaces $\mathrm{colim}_{n\in\mathbb{N}} M_n$ by a colimit indexed over all ordinals which are small than a suitably chosen cardinal. Let λ be the smallest cardinal which is bigger than the cardinality of R. For every ordinal α with $|\alpha| < \lambda$, define inductively $M_0 := M$, $M_\alpha := (M_\beta)'$ if $\alpha = \beta + 1$, and $M_\alpha := \mathrm{colim}_{\beta<\alpha} M_\beta$ if α is a limit ordinal. Then $\mathrm{colim}_{|\alpha|<\lambda} M_\alpha$ is an injective that contains M as a submodule.

Recall that a ring is called Noetherian if every ideal is finitely generated. Using Bear's criterion, one can prove:

Lemma 11 (exercise)**.** *Let R be a Noetherian ring, and let $\{I_i\}_{i\in\mathcal{I}}$ be a collection of injective modules. Then $\bigoplus_{i\in\mathcal{I}} I_i$ is injective.*

In the absence of the Noetherian condition, one can still show that $\prod_{i\in\mathcal{I}} I_i$ is injective.

Proposition. A $\mathbb{Z}$-module is injective if and only if it is a direct sum of the following groups: $\mathbb{Q}$, and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, for p a prime.

Proof. Let I be an injective $\mathbb{Z}$-module. Consider the collection of submodules M equipped with a direct sum decomposition into pieces isomorphic to $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. This is a poset under inclusion respecting the direct sum decompositions. By an application of Zorn's lemma, this poset admits a maximal element. If the maximal element is I, we're done.

Assume by contradiction that the maximal element M is not I. Since M is injective, the short exact sequence $0 \to M \to I \to I/M \to 0$ splits. So it's enough to find a submodule of $N := I/M$ which is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. Note that N is injective as it's a direct summand of an injective module.

Pick $x \in N$, non-zero, and let C_0 be the cyclic subgroup generated by x. Let $C \subset C_0$ be a subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}$. Let $D := \mathbb{Z}[\frac{1}{p}]/p\mathbb{Z} \cong \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ if $C \cong \mathbb{Z}/p\mathbb{Z}$, and $D := \mathbb{Q}$ if $C \cong \mathbb{Z}$. Since N is injective, the map $C \to N$ extends to a map $D \to N$.

It remains to show that the map $D \to N$ is injective. Indeed, for every non-zero element $d \in D$, there exists $n \in \mathbb{N}$ such that $nd \in C$. The map $D \to N$ is injective when restricted to C. So it's injective on all of D. □

Similarly, if k is an algebraically closed field, a $k[x]$-module is injective if and only if it is a direct sum of copies of the fraction field $k(x)$, and of the modules $k[\tilde{x}, \tilde{x}^{-1}]/k[\tilde{x}]$ for $\tilde{x} := x - a$ and $a \in k$.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence. Let $P_\bullet \to A$ be a projective resolution of A, and let $Q_\bullet \to C$ is a projective resolution of C. In the above situation, the horseshoe lemma says that there exists a projective resolution $R_\bullet \to B$ which fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & P_\bullet & \to & R_\bullet & \to & Q_\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

where each row $0 \to P_n \to R_n \to Q_n \to 0$ is short exact.

The horseshoe lemma is proven by postulating that, for each $n \in \mathbb{N}$, the short exact sequence $0 \to P_n \to R_n \to Q_n \to 0$ is given by $0 \to P_n \xrightarrow{\iota} P_n \oplus Q_n \xrightarrow{\pi} Q_n \to 0$, where ι is the inclusion of the first summand, and π is the projection onto the second summand. One then inductively constructs the maps $d_0^R : P_0 \oplus Q_0 \to B$, and then $d_n^R : P_n \oplus Q_n \to P_{n-1} \oplus Q_{n-1}$ for every $n \in \mathbb{N}$ so as to have everything fit into a diagram

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & P_2 & \to & P_2 \oplus Q_2 & \to & Q_2 & \to & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & P_1 & \to & P_1 \oplus Q_1 & \to & Q_1 & \to & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & P_0 & \to & P_0 \oplus Q_0 & \to & Q_0 & \to & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

The key step is to ensure that $d_{n+1}^R \circ d_n^R = 0$. Indeed, once we have the commutativity of the above diagram, and the relation $d_{n+1}^R \circ d_n^R = 0$, it automatically follows as an application of the homology long exact sequence that $\ker(d_n^R) = \operatorname{im}(d_{n+1}^R)$. So $R_\bullet := (P_n \oplus Q_n, d_n^R)$ is indeed a resolution of B.

Recall that given projective resolutions $A \leftarrow P_\bullet$ and $B \leftarrow Q_\bullet$, the cochain complex

$$\underline{\mathrm{Hom}}(C_\bullet, D_\bullet) := \mathrm{Tot}^{\Pi}\Big(\mathrm{Hom}(P_\bullet, Q_\bullet)\Big)$$

computes $\mathrm{Ext}(A, B)$. (By this we mean that the nth cohomology group of this complex is canonically isomorphic to $\mathrm{Ext}(A, B)$.)

Using this fact, composition of homomorphisms $\circ : \mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ induces a well-defined map $\mathrm{Ext}^i(A, B) \otimes \mathrm{Ext}^j(B, C) \to \mathrm{Ext}^{i+j}(A, C)$. In particular, this equips the graded abelian group

$$\mathrm{Ext}^*(A, A) := \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i(A, A)$$

with the structure of a ring.

Upon identifying cycles in $\underline{\mathrm{Hom}}(C_\bullet, D_\bullet)$ with chain maps $C_\bullet \to D_\bullet$, one get the following convenient description of Ext:

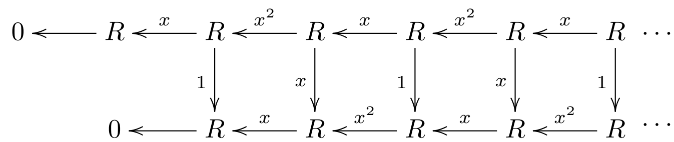
$$H^n\Big(\underline{\mathrm{Hom}}(C_\bullet, D_\bullet)\Big) = \frac{\text{degree } (-n) \text{ chain maps } C_\bullet \to D_\bullet}{\text{chain maps which are chain-homotopic to zero}}$$

Here, a degree $(-n)$ chain map $C_\bullet \to D_\bullet$ is a chain map $f_\bullet : C_{\bullet+n} \to D_\bullet$ i.e. a collection of maps $f_i : C_{i+n} \to D_i$ satisfying $f_i \circ d^C = d^D \circ f_{i+1}$. Two chain maps $f_\bullet, g_\bullet : C_{\bullet+n} \to D_\bullet$ are chain homotopic if there exists a collection of maps $h_i : C_{i+n} \to D_{i+1}$ satisfying $f_i - g_i = h_{i-1} \circ d^C + d^D \circ h_i$. Given two chain maps $f_\bullet : D_{\bullet+n} \to E_\bullet$ and $g_\bullet : C_{\bullet+m} \to D_\bullet$ representing element $\alpha \in \mathrm{Ext}^n$ and $\beta \in \mathrm{Ext}^m$, the product $\alpha\beta \in \mathrm{Ext}^{n+m}$ is represented by the composite chain map $f_\bullet \circ g_\bullet : C_{\bullet+(n+m)} \to E_\bullet$.

Here are some examples of Ext-ring computations:
- $\mathrm{Ext}_{k[x]}(k,k) = k[y]/y^2$, with y in degree 1.
- $\mathrm{Ext}_{k[x]/(x^2)}(k,k) = k[y]$, with y in degree 1.
- $\mathrm{Ext}_{k[x]/(x^3)}(k,k) = k[y,z]/(y^2)$, with y in degree 1 and z in degree 2.

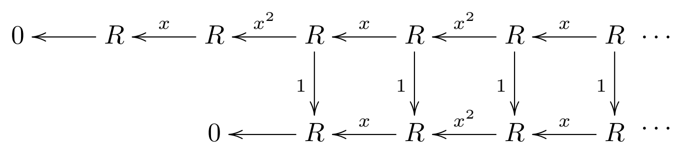
Let's work out the last example in detail. Let $R := k[x]/(x^3)$ and let $P_\bullet := \Big(R \xleftarrow{x} R \xleftarrow{x^2} R \xleftarrow{x} R \dots\Big)$ be a resolution of k. Then the generator y of $\mathrm{Ext}^1(k,k)$ is given by

$y :=$

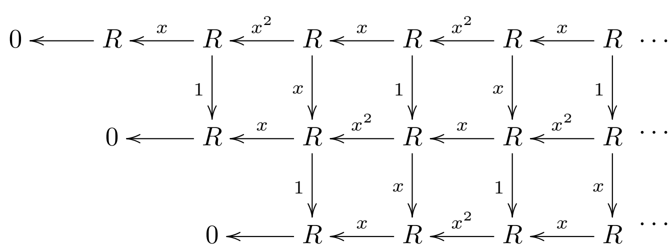


and the generator z of $\mathrm{Ext}^2(k,k)$ is given by

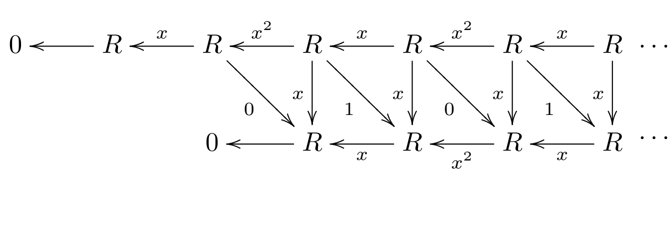
$z :=$



To check that $y^2 = 0$ in the ring $\mathrm{Ext}^*(k,k)$, one composes the chain maps as follows:



This gives $x \cdot z$, which is zero in the Ext ring (because $\mathrm{Ext}^2(k,k) = k$ as an R-module). Alternatively, one can construct an explicit null-homotopy of the above composite:



Exercise 1. *Let k be a field, and let $R := k[x,y]$. Write k for the R-module $R/(x,y)$. Let $n_1 > n_2 > \ldots > n_s = 0$, and $0 = m_1 < m_2 < \ldots < m_s$ be integers.*
Compute $\mathrm{Tor}_*^R(R/(x^{n_1}y^{m_1}, x^{n_2}y^{m_2}, \ldots, x^{n_s}y^{m_s}), k)$
Solution: A projective resolution of $R/(x^{n_1}y^{m_1}, x^{n_2}y^{m_2}, \ldots, x^{n_s}y^{m_s})$ is given by

$$R^{s-1} \xrightarrow{\begin{pmatrix} y^{m_2-m_1} & 0 & \ldots & 0 \\ x^{n_1-n_2} & y^{m_3-m_2} & \ldots & 0 \\ 0 & x^{n_2-n_3} & y^{m_4-m_3} & \ldots \\ & \ldots & & \ddots \end{pmatrix}} R^s \xrightarrow{\begin{pmatrix} x^{n_1}y^{m_1} \\ \vdots \\ x^{n_s}y^{m_s} \end{pmatrix}} R$$

After tensoring by k, the differentials become zero and we get $\mathrm{Tor}_0 = k$, $\mathrm{Tor}_1 = k^s$, $\mathrm{Tor}_2 = k^{s-1}$.
Let $R = k[x,y]$ be as above, and let $a > n > 0$ be integers.
Compute $\mathrm{Tor}_*^R(R/(x^n, y^n), R/(x^a, xy, y^a))$.

Solution: $R \xrightarrow{(\, y^n \ -x^n \,)} R^2 \xrightarrow{\binom{x^n}{y^n}} R$ is a projective resolution of $R/(x^n, y^n)$. After tensoring with $R/(x^a, xy, y^a)$, this becomes

$$R/(x^a, xy, y^a) \xrightarrow{d_2 = (\, y^n \ -x^n \,)} \left[R/(x^a, xy, y^a)\right]^2 \xrightarrow{d_1 = \binom{x^n}{y^n}} R/(x^a, xy, y^a).$$

The homology in degree zero is $\mathrm{Tor}_0 = \mathrm{coker}(d_1) = R/(x^n, xy, y^n)$. The kernel of d_1 has a k-basis given by $\{x^{a-n}, x^{a-n+1}, \ldots, x^{a-1}, y, y^2, \ldots, y^{a-1}\}$ in the first copy of $R/(x^a, xy, y^a)$ and by $\{x, x^2, \ldots, x^{a-1}, y^{a-n}, y^{a-n+1}, \ldots, y^{a-1}\}$ in second first copy of $R/(x^a, xy, y^a)$. Let us write

$$x_1^{a-n}, x_1^{a-n+1}, \ldots, x_1^{a-1}, y_1, y_1^2, \ldots, y_1^{a-1} \quad \text{and} \quad x_2, x_2^2, \ldots, x_2^{a-1}, y_2^{a-n}, y_2^{a-n+1}, \ldots, y_2^{a-1}$$

to distinguish them. The elements in the image of d_2 are $y_1^n - x_2^n$, y_1^{n+1}, y_1^{n+2}, …, y_1^{a-1}, x_2^{n+1}, x_2^{n+2}, …, x_2^{a-1}. So a k-basis of $\mathrm{Tor}_1 = \ker(d_1)/\mathrm{im}(d_2)$ is given by

$$x_1^{a-n}, x_1^{a-n+1}, \ldots, x_1^{a-1}, y_1, y_1^2, \ldots, y_1^n = x_2^n, x_2^{n-1}, \ldots, x_2^3, x_2^2, x_2, y_2^{a-n}, y_2^{a-n+1}, \ldots, y_2^{a-1}$$

which decomposes as an R-module as

$$\underbrace{x_1^{a-n}, x_1^{a-n+1}, \ldots, x_1^{a-1}}_{\cong R/(y, x^n)}, \underbrace{y_1, y_1^2, \ldots, y_1^n = x_2^n, x_2^{n-1}, \ldots, x_2^3, x_2^2, x_2}_{\cong \frac{(x^{n-1}, y^{n-1})}{(x^n, y^n)}}, \underbrace{y_2^{a-n}, y_2^{a-n+1}, \ldots, y_2^{a-1}}_{\cong R/(y^n, x)}$$

Finally, $\mathrm{Tor}_2 = \ker(d_2)$ has a k-basis given by $x^{a-n}, x^{a-n+1}, \ldots, x^{a-1}$ and $y^{a-n}, y^{a-n+1}, \ldots, y^{a-1}$, and is isomorphic to $R/(y, x^n) \oplus R/(y^n, x)$.

Exercise 2. *Consider the abelian category whose objects are diagrams $(M_1 \xleftarrow{f_1} M_2 \xleftarrow{f_2} M_3 \xleftarrow{f_3} \ldots)$ of abelian groups indexed by $\mathbb{N}$, and whose morphisms are natural transformations between such diagrams. Show, that the functor which sends an object $(M_1 \xleftarrow{f_1} M_2 \xleftarrow{f_2} M_3 \xleftarrow{f_3} \ldots)$ to its inverse limit $\varprojlim M_i$ is not right exact.*

Hint: *Construct a suitable morphism between the object $(\mathbb{Z} \overset{id}{\leftarrow} \mathbb{Z} \overset{id}{\leftarrow} \mathbb{Z} \ldots)$ and the object $(\mathbb{Z}/2\mathbb{Z} \twoheadleftarrow \mathbb{Z}/4\mathbb{Z} \twoheadleftarrow \mathbb{Z}/8\mathbb{Z} \ldots)$, and analyse its properties.*

Solution: In order to show that a functor F is not right exact, it suffices to exhibit an epimorphism f such that $F(f)$ is not an epimorphism.
We consider the morphism

$$\begin{array}{ccccccccc} \mathbb{Z} & \xleftarrow{id} & \mathbb{Z} & \xleftarrow{id} & \mathbb{Z} & \xleftarrow{id} & \mathbb{Z} & \xleftarrow{id} & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{Z}/2\mathbb{Z} & \twoheadleftarrow & \mathbb{Z}/4\mathbb{Z} & \twoheadleftarrow & \mathbb{Z}/8\mathbb{Z} & \twoheadleftarrow & \mathbb{Z}/16\mathbb{Z} & \twoheadleftarrow & \ldots \end{array}$$

Its image under the functor $\varprojlim$ is the morphism of abelian groups $\mathbb{Z} \to \mathbb{Z}_2$ (the inclusion of the integers into the 2-adic integers). The latter is not be an epimorphism.

Consider the derived functors $\lim^i := R^i(\varprojlim)$ *of the inverse limit functor*

$$\varprojlim : (M_1 \xleftarrow{f_1} M_2 \xleftarrow{f_2} M_3 \xleftarrow{f_3} \dots) \mapsto (\varprojlim M_i).$$

[You may assume the knowledge that the inverse limit functor is left exact]
Assuming the knowledge that the functors $\lim^i$ *for* $i \geq 1$ *yield zero when evaluated on the object* $(\mathbb{Z} \overset{id}{\leftarrow} \mathbb{Z} \overset{id}{\leftarrow} \mathbb{Z} \dots)$, *compute the value of*

$$\lim{}^1\big(\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \dots\big).$$

Solution: The short exact sequence

$$0 \to (\mathbb{Z} \overset{\cdot 2}{\leftarrow} \mathbb{Z} \overset{\cdot 2}{\leftarrow} \mathbb{Z} \overset{\cdot 2}{\leftarrow} \dots) \to (\mathbb{Z} \overset{id}{\leftarrow} \mathbb{Z} \overset{id}{\leftarrow} \mathbb{Z} \dots) \to (\mathbb{Z}/2\mathbb{Z} \twoheadleftarrow \mathbb{Z}/4\mathbb{Z} \twoheadleftarrow \mathbb{Z}/8\mathbb{Z} \twoheadleftarrow \dots) \to 0$$

yields a long exact sequence of derived functors

$$\begin{aligned} 0 \to \varprojlim(\mathbb{Z} \overset{\cdot 2}{\leftarrow} \mathbb{Z} \overset{\cdot 2}{\leftarrow} \mathbb{Z} \overset{\cdot 2}{\leftarrow} \dots) &\to \varprojlim(\mathbb{Z} \overset{id}{\leftarrow} \mathbb{Z} \overset{id}{\leftarrow} \mathbb{Z} \dots) \\ &\to \varprojlim(\mathbb{Z}/2\mathbb{Z} \twoheadleftarrow \mathbb{Z}/4\mathbb{Z} \twoheadleftarrow \mathbb{Z}/8\mathbb{Z} \twoheadleftarrow \dots) \\ &\to \lim{}^1(\mathbb{Z} \overset{\cdot 2}{\leftarrow} \mathbb{Z} \overset{\cdot 2}{\leftarrow} \mathbb{Z} \overset{\cdot 2}{\leftarrow} \dots) \to 0 \end{aligned}$$

which reads

$$0 \to 0 \to \mathbb{Z} \to \mathbb{Z}_2 \to ? \to 0$$

It follows that $\lim^1(\mathbb{Z} \overset{\cdot 2}{\leftarrow} \mathbb{Z} \overset{\cdot 2}{\leftarrow} \mathbb{Z} \overset{\cdot 2}{\leftarrow} \dots) = \mathbb{Z}_2/\mathbb{Z}$.

Exercise 3. *Given a possibly non-abelian group* G, *the* n*th homology group of* G *with coefficients in an abelian group* A *is defined to be the* n*th Tor-group* $\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A)$. *(Here,* $\mathbb{Z}[G]$ *denotes the group algebra of* G *i.e., the free abelian group on the elements of* G, *equipped with the ring structure inherited from the multiplication in* G*).*

Here, both $\mathbb{Z}$ *and* A *are equipped with the action of* $\mathbb{Z}[G]$ *in which all the generators of* G *act trivially.*

Let G *be the cyclic group of order four, so that* $\mathbb{Z}[G] = \mathbb{Z}[x]/(x^4 - 1)$. *Compute the group homology* $H_i(G, \mathbb{Z})$ *for all* i.

Solution: The group algebra $\mathbb{Z}[G]$ is the same as the ring $\mathbb{Z}[x]/(x^4 - 1)$. So, by definition, $H_i(G, \mathbb{Z}) = \mathrm{Tor}_i^R(\mathbb{Z}, \mathbb{Z})$.
A free resolution of $\mathbb{Z}$ is given by

$$\dots R \xrightarrow{1 \mapsto 1+x+x^2+x^3} R \xrightarrow{1 \mapsto 1-x} R \xrightarrow{1 \mapsto 1+x+x^2+x^3} R \xrightarrow{1 \mapsto 1-x} R \to \mathbb{Z}$$

Removing the last term and tensoring by $\mathbb{Z}$, we get

$$\dots \mathbb{Z} \xrightarrow{1+x+x^2+x^3} \mathbb{Z} \xrightarrow{1-x} \mathbb{Z} \xrightarrow{1+x+x^2+x^3} \mathbb{Z} \xrightarrow{1-x} \mathbb{Z} \to 0$$

which is

$$\dots \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

So the homology is $\mathbb{Z}$ in degree zero, $\mathbb{Z}/4$ is odd degrees, and zero otherwise.