# Homological algebra

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**Sheet 1**

**Exercise 1.** Let $A$, $B$, $C$ be $R$-modules. Show that there exists canonical $R$-module isomorphisms $\mathrm{Hom}(A \oplus B, C) \cong \mathrm{Hom}(A, C) \oplus \mathrm{Hom}(B, C)$ and $\mathrm{Hom}(A, B \oplus C) \cong \mathrm{Hom}(A, B) \oplus \mathrm{Hom}(A, C)$.

More generally, in a category which admits sums and products, prove that $\mathrm{Hom}(\bigoplus_{i \in \mathcal{I}} M_i, N) = \prod_{i \in \mathcal{I}} \mathrm{Hom}(M_i, N)$ and $\mathrm{Hom}(M, \prod_{i \in \mathcal{I}} N_i) = \prod_{i \in \mathcal{I}} \mathrm{Hom}(M, N_i)$.

**Exercise 2.** A *monomorphism* is a morphism $f$ that satisfies $(f \circ g_1 = f \circ g_2) \Rightarrow (g_1 = g_2)$. Equivalently, it is a morphism $f : X \to Y$ with the property that whenever two morphisms $g_1, g_2 : Z \to X$ are distinct, they remain distinct after composing them with $f$. Dually, an *epimorphism* is a map $f$ that satisfies $(g_1 \circ f = g_2 \circ f) \Rightarrow (g_1 = g_2)$.

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. Prove, using the language of category theory, that $f$ is always a monomorphism, and that $g$ is always an epimorphism.

**Exercise 3.** Consider a commutative diagram of the following form, where the rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

Prove that the map $B \to B'$ is an isomorphism.

**Exercise 4.** Let $R := k[x, y]$ where $k$ is a field. Let $M_1 := R^2 / \langle (x, 0), (y^2, -x), (0, y) \rangle$ and $M_2 := R / \langle x^2, xy, y^3 \rangle$. (Here, the symbol $\langle \ldots \rangle$ means 'submodule generated by'.) Provide examples of non-split short exact sequences of $R$-modules

$$0 \to M_1 \to\ ??\ \to M_2 \to 0.$$

**Exercise 5.** Prove that every short exact sequence of abelian groups $0 \to A \to B \to \mathbb{Z} \to 0$ splits. Prove that every short exact sequence of abelian groups $0 \to \mathbb{Q} \to B \to C \to 0$ splits.

**Exercise 6.** Prove that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p:\mathrm{prime}} \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

**Exercise 7.** Prove that, in general, there is no isomorphism $\mathrm{Hom}(M, \bigoplus_{i \in \mathcal{I}} N_i) = \bigoplus_{i \in \mathcal{I}} \mathrm{Hom}(M, N_i)$.

**Exercise 8** (super-hard)**.** Prove that, surprisingly, the natural inclusion $\bigoplus_{i \in \mathbb{N}} \mathbb{Z} \to \mathrm{Hom}(\prod_{i \in \mathbb{N}} \mathbb{Z}, \mathbb{Z})$ is an isomorphism.

*(Hand in Monday Oct 21st)*