Homological algebra

André Henriques

Sheet 2

Exercise 9. Prove that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

Exercise 10. Write down an injective resolution of $\mathbb{Z}$ as a $\mathbb{Z}$-module.
(You may invoke without proof the fact that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.)

Exercise 11. Write down free resolutions for:
- $\mathbb{Z}/2$ as a $\mathbb{Z}$-module.
- $\mathbb{Z}/2$ as a $(\mathbb{Z}/2)[x]$-module.
- $\mathbb{Z}/2$ as a $\mathbb{Z}[x]$-module.
- $\mathbb{Z}/2$ as a $\mathbb{Z}[x]/(2x)$-module.

Exercise 12. Let R be a principal ideal domain, and let M be a free R-module.
Prove that any submodule of M is free.

Exercise 13. Let R be a commutative ring, and let $r \in R$ be an element. Then

$$R[r^{-1}] := R[x]/(rx-1) = \mathrm{coker}\big(R[x] \xrightarrow{rx-1} R[x]\big).$$

Let M be an R-module, and let $M[r^{-1}] := \mathrm{coker}(M[x] \xrightarrow{rx-1} M[x])$, (where $M[x] = \big\{ \sum_i m_i x^i \big\}$ with $m_i \in M$). Prove that

$$M \otimes_R R[r^{-1}] = M[r^{-1}].$$

Exercise 14. Prove the general Frobenius reciprocity formula:

$$\mathrm{Hom}_S(A, \mathrm{Hom}_R(B, C)) \cong \mathrm{Hom}_R(A \otimes_S B, C).$$

(Here, A is a right S-module, B is an S-R-bimodule, and C is a right R-module.)

Exercise 15. Compute the following Ext and Tor groups:
- $\mathrm{Tor}_*^{k[x]}(k[x]/(x-a), k[x]/(x-b))$, for $a, b \in k$.
- $\mathrm{Tor}_*^{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}/b)$, for $a, b \in \mathbb{Z}$.
- $\mathrm{Ext}^*_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2)$.
- $\mathrm{Ext}^*_{\mathbb{Z}/2^a}(\mathbb{Z}/2^b, \mathbb{Z}/2^c)$, for $a \geq b \geq c$.
- $\mathrm{Ext}^*_{k[x,y]/(x^2,xy,y^2)}(k, k)$. .

Hand in the Monday before the exercise class, at 12:00.