Homological algebra

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Sheet 3

Exercise 16. Prove that for any $\mathbb{Z}$-modules A and B we have $\mathrm{Ext}^i_{\mathbb{Z}}(A,B)=0$ and $\mathrm{Tor}_i^{\mathbb{Z}}(A,B)=0$ for all $i\geq 2$.

Exercise 17. Compute all the terms and all the maps in the long exact sequences

$$\begin{aligned}0 &\to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/4,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z})\\ &\to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4,\mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z})\\ &\to \mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) \to \dots\end{aligned}$$

and

$$\begin{aligned}0 &\to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/2) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/4) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/2)\\ &\to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/2) \to \dots\end{aligned}$$

associated to the short exact sequence $0\to\mathbb{Z}/2\to\mathbb{Z}/4\to\mathbb{Z}/2\to 0$.

Compute all the terms and all the maps in the long exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}_{\mathbb{Z}/8}(\mathbb{Z}/2,\mathbb{Z}/4) \to \mathrm{Hom}_{\mathbb{Z}/8}(\mathbb{Z}/4,\mathbb{Z}/4) \to \mathrm{Hom}_{\mathbb{Z}/8}(\mathbb{Z}/2,\mathbb{Z}/4)\\ &\to \mathrm{Ext}^1_{\mathbb{Z}/8}(\mathbb{Z}/2,\mathbb{Z}/4) \to \mathrm{Ext}^1_{\mathbb{Z}/8}(\mathbb{Z}/4,\mathbb{Z}/4) \to \mathrm{Ext}^1_{\mathbb{Z}/8}(\mathbb{Z}/2,\mathbb{Z}/4)\\ &\to \mathrm{Ext}^2_{\mathbb{Z}/8}(\mathbb{Z}/2,\mathbb{Z}/4) \to \dots\end{aligned}$$

associated to the short exact sequence $0\to\mathbb{Z}/2\to\mathbb{Z}/4\to\mathbb{Z}/2\to 0$.

Let $a>b\geq 2$. Compute all the terms and all the maps in the long exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/2,\mathbb{Z}/2^b) \to \mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/4,\mathbb{Z}/2^b) \to \mathrm{Hom}_{\mathbb{Z}/2^a}(\mathbb{Z}/2,\mathbb{Z}/2^b)\\ &\to \mathrm{Ext}^1_{\mathbb{Z}/2^a}(\mathbb{Z}/2,\mathbb{Z}/2^b) \to \mathrm{Ext}^1_{\mathbb{Z}/2^a}(\mathbb{Z}/4,\mathbb{Z}/2^b) \to \mathrm{Ext}^1_{\mathbb{Z}/2^a}(\mathbb{Z}/2,\mathbb{Z}/2^b)\\ &\to \mathrm{Ext}^2_{\mathbb{Z}/2^a}(\mathbb{Z}/2,\mathbb{Z}/2^b) \to \dots\end{aligned}$$

associated to the short exact sequence $0\to\mathbb{Z}/2\to\mathbb{Z}/4\to\mathbb{Z}/2\to 0$.

Exercise 18. Consider the functor $A\mapsto\prod_{i=1}^{\infty}A : \mathbb{Z}\text{-modules}\to\mathbb{Z}\text{-modules}$.
Is this functor left exact? Is this functor right exact? What are its derived functors?

Exercise 19. Let k be a field, let $R=k[x,y]$, and let $M:=R/(x^2,xy,y^2)$. Consider the following short exact sequences of R-modules:

$$0\to k\oplus k\to M\to k\to 0$$

$$0\to k\to \mathrm{Hom}_k(M,k)\to k\oplus k\to 0$$

$$0\to k^{\oplus 3}\to M\oplus M/\langle(y,-x)\rangle\to k^{\oplus 2}\to 0$$

Compute all the terms and maps in the associated long exact sequences of Tor groups:

$$\begin{aligned}&\dots\to\mathrm{Tor}_2^R(M,k)\to\\ \mathrm{Tor}_1^R(M,k\oplus k)\to\mathrm{Tor}_1^R&(M,M)\to\mathrm{Tor}_1^R(M,k)\to\\ M\otimes_R(k\oplus k)\longrightarrow M&\otimes_R M\longrightarrow M\otimes_R k\to 0\end{aligned}$$

and

$$\ldots \to \mathrm{Tor}_1^R(M, k \oplus k) \to$$
$$M \otimes_R k \longrightarrow M \otimes_R \mathrm{Hom}_k(M, k) \longrightarrow M \otimes_R (k \oplus k) \to 0$$

and

$$\ldots \to \mathrm{Tor}_1^R(k, k \oplus k) \to$$
$$k \otimes_R k \longrightarrow k \otimes_R \mathrm{Hom}_k(M, k) \longrightarrow k \otimes_R (k \oplus k) \to 0$$

and

$$\ldots \to \mathrm{Tor}_1^R(k, k^{\oplus 2}) \to$$
$$k \otimes_R k^{\oplus 3} \longrightarrow k \otimes_R (M \oplus M/\langle (y, -x) \rangle) \longrightarrow k \otimes_R k^{\oplus 2} \to 0$$

Hand in the Monday before the exercise class, at 12:00.