Homological algebra

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<u>**Sheet 4**</u>

Exercise 20. Let k be a field, and let $R = k[x]$. Consider the module $M := k[x, x^{-1}]/x{\cdot}k[x]$. A typical element of M can be written as a Laurent polynomial $\sum_{n\leq 0} a_n x^n$. Compute

$$\mathrm{Tor}_*^R(M, M).$$

Exercise 21. Let k be a field, and let $R := k[x, y]/(xy)$. View k as an R-module by letting x and y act by zero. Compute the ring structure of

$$\mathrm{Ext}_R^*(k, k) = \bigoplus_n \mathrm{Ext}_R^n(k, k).$$

Is this ring commutative?

Exercise 22. Let k be a field, and let $R = k[x]/x^3$. Let $M := k[x]/x$ and $N := k[x]/x^2$. According to the horseshoe lemma, the short exact sequence $0 \to M \to N \to M \to 0$ can be extended a sort exact sequence of projective resolutions

$$\begin{array}{ccccccccc} 0 & \to & P_\bullet & \to & R_\bullet & \to & Q_\bullet & \to & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & N & \to & M & \to & 0 \end{array}$$

Write down explicitly all the modules and all the maps in the above commutative diagram.

Exercise 23. Consider the functor which assigns to an abelian group A its set of 2-torsion points:

$$F(A) = \{a \in A : \exists n \in \mathbb{N} \text{ s.t. } 2^n a = 0\}.$$

Prove that this functor left exact. Compute all the group and all the maps in the long exact sequence of derived functors

$$\begin{aligned} 0 &\to R^0F(\mathbb{Z}) \to R^0F(\mathbb{Z}) \to R^0F(\mathbb{Z}/2\mathbb{Z}) \\ &\to R^1F(\mathbb{Z}) \to R^1F(\mathbb{Z}) \to R^1F(\mathbb{Z}/2\mathbb{Z}) \\ &\to R^2F(\mathbb{Z}) \to \dots \end{aligned}$$

associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.