Homological Algebra

Solutions to sheet 4

Ex 20: The module $k[x,x^{-1}]$ is a union (colimit) of free modules $x^n \cdot k[x]$ (as $n \to -\infty$) and is therefore flat.

The following is a flat resolution of M:

$$M = k[x,x^{-1}]/_{x\cdot k[x]} \twoheadleftarrow k[x,x^{-1}] \xleftarrow{\cdot x} k[x] \leftarrow 0$$

To compute $\mathrm{Tor}(M,M)$ we apply $- \otimes_{k[x]} M$ to the above resolution:

$$\underbrace{k[x,x^{-1}] \otimes_{k[x]} M}_{\substack{= M[x^{-1}] \\ = 0}} \longleftarrow \underbrace{k[x] \otimes_{k[x]} M}_{= M} \longleftarrow 0$$

Hence $\mathrm{Tor}_0(M\,M) = 0$ and $\mathrm{Tor}_1(M\,M) = M$

$\mathrm{Tor}_i(M\,M) = 0 \quad \forall i \geq 2.$

Ex 21: The following is a projective resolution:

$$
k \longleftarrow \underbrace{R \xleftarrow{[x\ y]} R^2 \xleftarrow{\left[\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right]} R^2 \xleftarrow{\left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]} R^2 \xleftarrow{\left[\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right]} R^2 \longleftarrow \cdots}_{=P_\bullet}
$$

Applying $\operatorname{Hom}_R(-,k)$ and using that x and y act by zero on k, we get

$$
k \xrightarrow{0} k^2 \xrightarrow{0} k^2 \xrightarrow{0} k^2 \xrightarrow{0} \cdots
$$

So $\operatorname{Ext}^n_R(k,k) = \begin{cases} k & n=0 \\ k^2 & n \geq 1 \end{cases}$.

The standard basis element $1 \in k$ of Ext^0 is represented by the identity map $P_\bullet \to P_\bullet$

The two basis elements of $\operatorname{Ext}^1 = k^2$ are represented by the following two maps $P_\bullet \to P_{\bullet - 1}$:

$$
a = \left(
\begin{tikzcd}
R & R^2 \arrow[l, "{[x\ y]}"'] \arrow[d, "{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}"'] & R^2 \arrow[l, "{\left[\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right]}"'] \arrow[d, "{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]}"'] & R^2 \arrow[l, "{\left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]}"'] \arrow[d, "{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]}"'] & R^2 \arrow[l, "{\left[\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right]}"'] \arrow[d, "{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]}"'] & \cdots \arrow[l, "{\left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]}"'] \\
& R & R^2 \arrow[l, "{[x\ y]}"'] & R^2 \arrow[l, "{\left[\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right]}"'] & R^2 \arrow[l, "{\left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]}"'] & \cdots \arrow[l, "{\left[\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right]}"']
\end{tikzcd}
\right)
$$

and

$$
b = \left(
\begin{tikzcd}
R & R^2 \arrow[l, "{[x\ y]}"'] \arrow[d, "{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}"'] & R^2 \arrow[l, "{\left[\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right]}"'] \arrow[d, "{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]}"'] & R^2 \arrow[l, "{\left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]}"'] \arrow[d, "{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]}"'] & R^2 \arrow[l, "{\left[\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right]}"'] \arrow[d, "{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]}"'] & \cdots \arrow[l] \\
& R & R^2 \arrow[l, "{\left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]}"'] & R^2 \arrow[l, "{\left[\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right]}"'] & R^2 \arrow[l, "{\left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]}"'] & \cdots \arrow[l]
\end{tikzcd}
\right)
$$

To compute ab, we compose the above chain maps:

$$
\left(
\begin{tikzcd}
{} & {} \arrow[l] \arrow[d, "{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}"'] & {} \arrow[l] \arrow[d, "{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]}"'] & {} \arrow[l] \arrow[d, "{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]}"'] & {} \arrow[l] \arrow[d, "{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]}"'] & {} \arrow[l] \\
& {} & {} \arrow[l] \arrow[d, "{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}"'] & {} \arrow[l] \arrow[d, "{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]}"'] & {} \arrow[l] \arrow[d, "{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]}"'] & {} \arrow[l] \\
& & {} & {} \arrow[l] & {} \arrow[l] & {} \arrow[l]
\end{tikzcd}
\right) =
$$

$$
= \left(
\begin{tikzcd}
R & R^2 \arrow[l, "{[x\ y]}"'] & R^2 \arrow[l, "{\left[\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right]}"'] \arrow[d, "{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}"'] & R^2 \arrow[l, "{\left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]}"'] \arrow[d, "{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]}"'] & \cdots \arrow[l] \\
& & R & R^2 \arrow[l, "{\left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]}"'] & \cdots \arrow[l]
\end{tikzcd}
\right)
$$

To compute ba, we compose these same maps in the other order:

$$
\left[
\left(
\begin{tikzcd}
{} & {} \arrow[l] \arrow[d, "{\begin{bmatrix}1\\0\end{bmatrix}}"'] & {} \arrow[l] \arrow[d, "{\begin{bmatrix}0&0\\1&0\end{bmatrix}}"'] & {} \arrow[l] \arrow[d, "{\begin{bmatrix}0&0\\1&0\end{bmatrix}}"'] & {} \arrow[l] \arrow[d, "{\begin{bmatrix}0&0\\1&0\end{bmatrix}}"'] & {} \arrow[l] \\
 & {} & {} \arrow[l] \arrow[d, "{\begin{bmatrix}0\\1\end{bmatrix}}"'] & {} \arrow[l] \arrow[d, "{\begin{bmatrix}0&1\\0&0\end{bmatrix}}"'] & {} \arrow[l] \arrow[d, "{\begin{bmatrix}0&1\\0&0\end{bmatrix}}"'] & {} \arrow[l] \\
 & & {} & {} \arrow[l] & {} \arrow[l] & {} \arrow[l]
\end{tikzcd}
\right) =
\right.
$$

$$
\left.
= \left(
\begin{tikzcd}
R & R^2 \arrow[l, "{[x,y]}"'] & R^2 \arrow[l, "{\begin{bmatrix}y&0\\0&x\end{bmatrix}}"'] \arrow[d, "{\begin{bmatrix}1\\0\end{bmatrix}}"'] & R^2 \arrow[l, "{\begin{bmatrix}x&0\\0&y\end{bmatrix}}"'] \arrow[d, "{\begin{bmatrix}1&0\\0&0\end{bmatrix}}"'] & \cdots \arrow[l] \\
 & & R & R^2 \arrow[l, "{[xy]}"] & \cdots \arrow[l]
\end{tikzcd}
\right)
\right]
$$

In particular, $ab \neq ba$ in $\mathrm{Ext}_R^2(k,k)$.

Similarly, one shows that $a^2 = 0$ and $b^2 = 0$.

For every $n \in \mathbb{N}$, $n \geq 1$, consider the following two words $\underbrace{ababab\ldots}_{n \text{ letters}}$ and $\underbrace{bababa\ldots}_{n \text{ letters}}$.

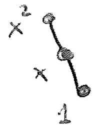
By composing the chain maps $a : P_\bullet \to P_{\bullet-1}$ and $b : P_\bullet \to P_{\bullet-1}$ as we did before, one checks that

$ababab\ldots$ and $bababa\ldots$

form a basis of $\mathrm{Ext}_R^n(k,k) \cong k^2$. All other words of length n will contain either a^2 or b^2 as a subword, and are thus zero in Ext^n. So we don't need any other relations than $a^2=0$ and $b^2=0$ to present our Ext-ring:

$$\bigoplus_{n \in \mathbb{N}} \mathrm{Ext}_R^n(k,k) = k\langle a, b\rangle \Big/ {\scriptstyle \begin{matrix} a^2=0 \\ b^2=0 \end{matrix}}$$

Ex 22 : We represent graphically our ring $R = k[x]/x^3$ by



, and the two modules

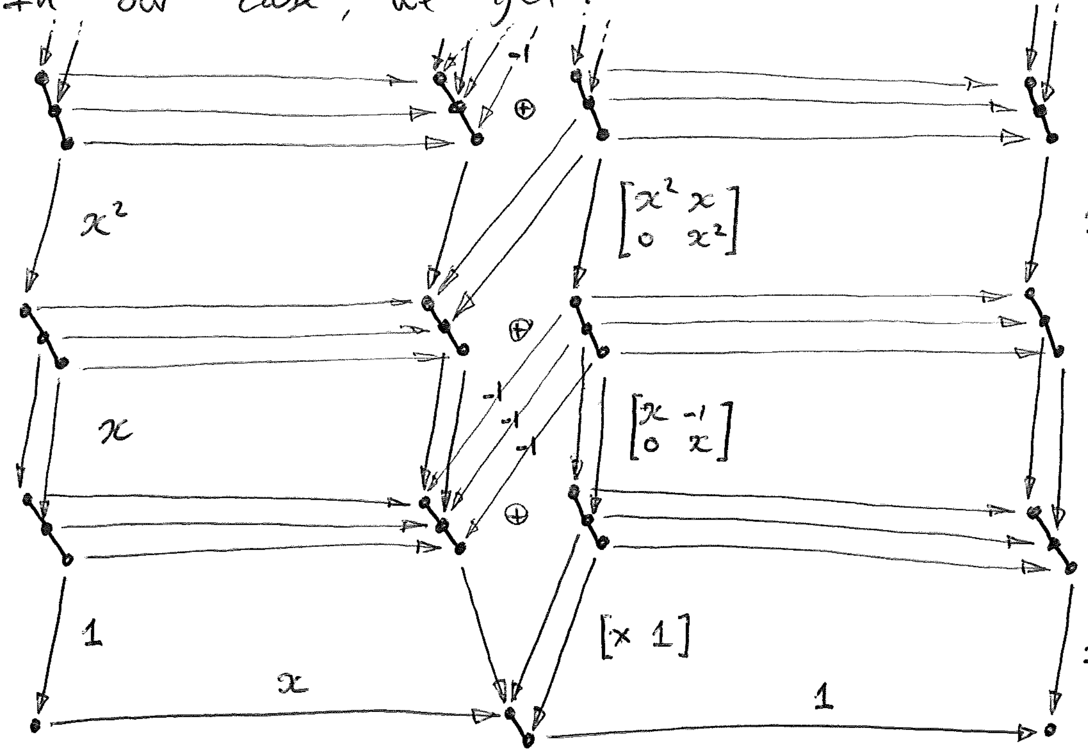
$M =$  , $N =$ 

For any SES $0 \to A \to B \to C \to 0$ and any projective resolutions $P_\bullet \to A$ and $Q_\bullet \to C$, by the horseshoe Lemma, we can find a SES of resolutions of the form

$$\boxed{\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_2 \arrow[r] \arrow[d] & P_2 \oplus Q_2 \arrow[r] \arrow[d] & Q_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}}$$

where the only "difficult" maps are $Q_{n+1} \to P_n$ (and $Q_0 \to B$)

In our case, we get :



and the pattern repeats 2-periodically.

Ex 23 We first check that F is left exact.

Given a SES $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, we consider $0 \to F(A) \to F(B) \to F(C)$ and want to show it is exact:

- $F(f) = f|_{F(A)}$ hence $F(f)$ is injective.
- $F(g) \circ F(f) = 0$ is obvious
- Pick $b \in F(B)$ s.t. $F(g)\, b = 0$

  Since $F(g) = g|_{F(B)}$ $g(b) = 0$

  Hence $\exists\, a \in A$ s.t. $f(a) = 0$

  Since f is injective and $2^n b = 0$ for some $n \in \mathbb{N}$
  we also have $2^n a = 0$.

  So $a \in F(A)$ and $F(f)\, a = b$ ✓

To compute $R^iF(\mathbb{Z})$ and $R^iF(\mathbb{Z}/2)$, we use the following injective resolutions:

$$\mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \ldots$$

$$\mathbb{Z}/2 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \ldots$$

Apply F: to the 1st one

$$F(\mathbb{Q}) = 0 \longrightarrow F(\mathbb{Q}/\mathbb{Z}) = \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z} \longrightarrow 0 \longrightarrow \ldots$$

hence $R^0F(\mathbb{Z}) = 0$
$R^1F(\mathbb{Z}) = \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z}$

Apply F to the 2^{nd} one:

$$\mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z} \longrightarrow 0 \ldots$$

hence

$$R^0F(\mathbb{Z}/2) = \mathbb{Z}/2$$
$$R^1F(\mathbb{Z}/2) = \mathbb{Z}/2$$

(and all the higher R^iF are zero)

The long exact sequence of derived functors reads as follows:

$$0 \longrightarrow F(\mathbb{Z})=0 \longrightarrow F(\mathbb{Z})=0 \longrightarrow F(\mathbb{Z}/2)=\mathbb{Z}/2$$

$$\xrightarrow[\;1 \mapsto 1/2\;]{\delta}$$

because that's the only non-zero map

$$R^1F(\mathbb{Z}) = \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z} \xrightarrow{\times 2} R^1F(\mathbb{Z}) = \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z} \longrightarrow 0$$

$$\xrightarrow{\delta}$$

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \ldots$$