Final Honour School of Mathematics Part C

C2.2 Homological Algebra
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1. Let $\mathcal{A}$ be an abelian category.

(a) [6 marks] (i) Define the *kernel* of a morphism $f : M \to N$ in $\mathcal{A}$.

[2pt] A morphism $\iota : K \to M$ exhibits K as the kernel of f if $f\iota = 0$ and for all $\iota' : K' \to M$ with $f\iota' = 0$ there $\exists$ a unique factorisation of ι' through ι. I.e. $\iota' = \iota g$ for some unique map $g : K' \to K$. **(B)**

(ii) Define the notion of a *short exact sequence* in an abelian category.

[2pt] A sequence $0 \to A \to B \to C \to 0$ is exact if $A \to B$ exhibits A as the kernel of $B \to C$, and $B \to C$ exhibits C as the cokernel of $A \to B$. **(B)**

(iii) Let R be a ring, and let us assume that $\mathcal{A}$ is the category of R-modules. Prove that, in that case, $\ker(f) = \{x \in M \mid f(x) = 0\}$.

[2pt] Let $K := \{x \in M \mid f(x) = 0\}$. The inclusion $\iota : K \to M$ satisfies $f\iota = 0$. Given $\iota' : K' \to M$ with $f\iota' = 0$, then for every $k' \in K'$ we have $\iota'(k') \in K$. The image of ι' is therefore contained in K, and we get a factorisation $\iota' = \iota g$. The map ι is injective so if $\iota' = \iota g$ and $\iota' = \iota g'$ we must have $g = g'$. The factorisation is therefore unique. **(S)**

From now on, we fix a ring R and assume that $\mathcal{A} = R$-Mod is the category of R-modules.

(b) [8 marks] (i) Explain what is meant by a *short exact sequence of chain complexes.*

[1pt] A short exact sequence of chain complexes is a diagram of chain complexes $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ such that for every $n \in \mathbb{Z}$ the sequence $0 \to A_n \to B_n \to C_n \to 0$ is exact. **(B)**

(ii) Given a short exact sequence of chain complexes $0 \to P_\bullet \xrightarrow{f_\bullet} Q_\bullet \xrightarrow{g_\bullet} R_\bullet \to 0$, there is an associated long exact sequence in homology; define the connecting homomorphism $\partial : H_n(R_\bullet) \to H_{n-1}(P_\bullet)$ and prove that it is well defined.

[4pt] Given $x \in R_n$ representing a class $[x] \in H_n(R_\bullet)$, pick a preimage $y \in Q_n$ of x. The element $dy \in Q_{n-1}$ has the property that its image is zero in R_{n-1}; it is therefore in the image of the map f_{n-1}. We set $\partial([x]) := [f_{n-1}^{-1}(dy)]$. We need to check that: (a) $d(f_{n-1}^{-1}(dy)) = 0$ (b) The class $[f_{n-1}^{-1}(dy)]$ is independent of the choice of preimage y of x (c) The class $[f_{n-1}^{-1}(dy)]$ is independent of the choice of representative x in its homology class. **(S)**

(a) Since f_{n-2} is injective, it's enough to check that $f_{n-2}d(f_{n-1}^{-1}(dy)) = 0$. We compute: $f_{n-2}d(f_{n-1}^{-1}(dy)) = d(f_{n-1}(f_{n-1}^{-1}(dy))) = d(dy) = 0$.

(b) Two preimages $y, y' \in Q_n$ of x differ by an element $f_n(z)$ for some $z \in P_n$. Replacing y by y' has the effect of adding $[f_{n-1}^{-1}(d(f_n(z)))]$ to $[f_{n-1}^{-1}(dy)]$. But $[f_{n-1}^{-1}(d(f_n(z)))] = [f_{n-1}^{-1}(f_{n-1}(d(z)))] = [dz] = 0$.

(c) It's enough to show that when $x = d\hat{x}$ for some $\hat{x} \in R_{n+1}$, the recipe for ∂ yields zero in $H_{n-1}(P_\bullet)$. Pick a preimage $\hat{y} \in Q_{n+1}$ of $\hat{x}$, and set $y := d\hat{y}$ (this is allowed by part (b)). Then $\partial([d\hat{x}]) = [f_{n-1}^{-1}(dd\hat{y})] = 0$.

(iii) Prove that

$$\ker\big(\partial : H_n(R_\bullet) \to H_{n-1}(P_\bullet)\big) \;=\; \operatorname{im}\big(H_n(Q_\bullet) \to H_n(R_\bullet)\big).$$

[3pt] (im $\subset$ ker): If $[x] \in H_n(R_\bullet)$ is the image of $[y] \in H_n(Q_\bullet)$, then $\partial([x]) = [f_{n-1}^{-1}(dy)] = 0$ because $dy = 0$. **(S)**

(ker $\subset$ im): If $\partial([x]) = [f_{n-1}^{-1}(dy)] = 0$, then $f_{n-1}^{-1}(dy) = dz$ for some $z \in P_n$. The element $y' := y - f_n(z) \in Q_n$ is also a preimage of x, and satisfies $dy' = 0$. It therefore represents an element in $H_n(Q_\bullet)$. By construction, $[y'] \mapsto [x]$.

(c) [5 marks] (i) Define the *Tor groups* $\operatorname{Tor}_*(-,-)$.

[1pt] Given a projective resolution $P_\bullet \to A$, the Tor groups $\operatorname{Tor}_*^R(A, B)$ are the homology groups of the complex $P_\bullet \otimes_R B$. **(B)**

(ii) Given a short exact sequence of right R-modules $0 \to A \to B \to C \to 0$ and a left R-module M, write down the associated long exact sequence of Tor groups;
[1pt] The sequence reads $\ldots \to \mathrm{Tor}_2(A, M) \to \mathrm{Tor}_2(B, M) \to \mathrm{Tor}_2(C, M) \to \mathrm{Tor}_1(A, M) \to \mathrm{Tor}_1(B, M) \to \mathrm{Tor}_1(C, M) \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ **(B)**

(iii) Explain the main steps involved in defining this long exact sequence and in proving that it is indeed exact (you may rely on the results stated in part (b.ii)).
[3pt] Given a short exact sequence $0 \to A \to B \to C \to 0$, by the horseshoe lemma, one may find projective resolutions $P_\bullet \to A$, $Q_\bullet \to B$, $R_\bullet \to C$ which assemble into a short exact sequence of chain complexes $0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0$, compatibly with the augmentations to A, B, C. Since each R_n is projective, for each n, the short exact sequence $0 \to P_n \to Q_n \to R_n \to 0$ is split. The sequences $0 \to P_n \otimes_R M \to Q_n \otimes_R M \to R_n \otimes_R M \to 0$ are therefore also (split) exact. So we get a short exact sequence of chain complexes $0 \to P_\bullet \otimes_R M \to Q_\bullet \otimes_R M \to R_\bullet \otimes_R M \to 0$. Applying homology, we get a long exact sequence of Tor groups. **(S)**

(d) [6 marks] Let k be a field, and let $R := k[x]/(x^n)$. Consider the short exact sequence of R-modules

$$0 \to R/(x^a) \to R/(x^{a+b}) \to R/(x^b) \to 0, \qquad (\star)$$

and let $M := R/x^c$, where $a < b < c < a + b < n/2$. Then there is an associated long exact sequence of Tor groups, obtained by applying the derived functors of $M \otimes_R -$ to the terms in the short exact sequence $(\star)$.

Compute all the terms and all the maps in the above long exact sequence.
[6pt] We first claim that, when $p, q \leqslant n/2$, we have **(N)**

$$\mathrm{Tor}_n^R\big(R/x^p, R/x^q\big) = R/x^{\min(p,q)}$$

for all $n \geqslant 0$. By the symmetry of Tor, it's enough to prove the case $q \leqslant p$: a projective resolution of R/x^p is given by $0 \leftarrow R \xleftarrow{x^p} R \xleftarrow{x^{n-p}} R \xleftarrow{x^p} R \xleftarrow{x^{n-p}} \ldots$. Since $q \leqslant p$ and $q \leqslant n - p$, applying $- \otimes_R R/x^q$ gives $0 \leftarrow R/x^q \xleftarrow{0} R/x^q \xleftarrow{0} R/x^q \xleftarrow{0} \ldots$. So all the Tor groups are R/x^q.
The long exact sequence of Tor groups associated to $0 \to R/(x^a) \to R/(x^{a+b}) \to R/(x^b) \to 0$ and R/x^c therefore reads

$$\ldots \to R/(x^b) \to R/(x^a) \to R/(x^c) \to R/(x^b) \to R/(x^a) \to R/(x^c) \to R/(x^b) \to 0$$

The only possible pattern of R-module maps which makes this into a long exact sequence can be computed inductively starting from the very right. It is 6-periodic, and given by

$$\cdots R/(x^c) \overset{1}{\twoheadrightarrow} R/(x^b) \xrightarrow{0} R/(x^a) \overset{x^{c-a}}{\rightarrowtail} R/(x^c) \xrightarrow{x^{a+b-c}} R/(x^b) \xrightarrow{x^{c-b}} R/(x^a) \xrightarrow{x^b} R/(x^c) \overset{1}{\twoheadrightarrow} R/(x^b) \to 0$$

2. Let R be a ring. Unless otherwise stated, we always work in the category of R-modules.

   (a) [8 marks] (i) Define what it means for an R-module to be *injective*.
   [2pt] A module E is injective if $\forall$ monomorphism $i : A \to B$ and every map $f : A \to E$, one can factorise f as $f = gi$ for some map $g : B \to E$. **(B)**

   (ii) State Baer's criterion for injectivity.
   [2pt] An R-module E is injective iff $\forall$ ideal $I \subset R$ and every R-module map $f : I \to E$, one can factorise f as $f = gi$ for some R-module map $g : R \to E$. Here, i denotes the inclusion of I into R. **(B)**

   (iii) Prove that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.
   [2pt] All the ideals of $\mathbb{Z}$ are principal. For every ideal $n\mathbb{Z} \subset \mathbb{Z}$ and every map $f : n\mathbb{Z} \to \mathbb{Q}$, we need to extend f to a map $f' : \mathbb{Z} \to \mathbb{Q}$. If $n = 0$, we let $f' = 0$. If $n \neq 0$, we let $f'(a) := \frac{1}{n} \cdot f(na)$. **(B)**

   (iv) State the classification theorem of injective $\mathbb{Z}$-modules (make sure to define all the terms that you use).
   [2pt] Both of the following answers are acceptable: **(S)**
   (a) A $\mathbb{Z}$-module is injective iff it is divisible. Here, A divisible means $\forall a \in A$, $\forall n \in \mathbb{N}$, $\exists b \in A$ s.t. $nb = a$.
   (b) A $\mathbb{Z}$-module is injective iff it is a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

   (b) [7 marks] (i) Define $\mathrm{Ext}^*(A, B)$ using injective resolutions.
   [1pt] Let $B \to I^\bullet$ be an injective resolution. Then $\mathrm{Ext}^n(A, B)$ is the n-th cohomology group of the cochain complex $\mathrm{Hom}(A, I^\bullet)$. **(B)**

   (ii) Let $R := \mathbb{Z}/4$.
   Prove that $\mathbb{Z}/4$ is an injective $\mathbb{Z}/4$-module.
   Compute $\mathrm{Ext}^*_R(\mathbb{Z}/2, \mathbb{Z}/2)$ using injective resolutions.
   [3pt] One first checks that $\mathbb{Z}/4$ is injective using Baer's criterion. The only non-trivial ideal is $I := 2R \subset R$; it is isomorphic to $\mathbb{Z}/2$. There are exactly two maps $\mathbb{Z}/2 \to \mathbb{Z}/4$ (one zero, one non-zero), and it's easy to see that they both factor through the inclusion $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$. **(S)**
   An injective resolution of $\mathbb{Z}/2$ as $\mathbb{Z}/4$-module is given by

$$\mathbb{Z}/2 \overset{\cdot 2}{\rightarrowtail} \mathbb{Z}/4 \overset{\cdot 2}{\to} \mathbb{Z}/4 \overset{\cdot 2}{\to} \mathbb{Z}/4 \overset{\cdot 2}{\to} \dots$$

   Applying $\mathrm{Hom}_R(\mathbb{Z}/2, -)$ to the (truncated) resolution, we get $\mathbb{Z}/2 \overset{0}{\to} \mathbb{Z}/2 \overset{0}{\to} \mathbb{Z}/2 \overset{0}{\to} \mathbb{Z}/2 \overset{0}{\to} \dots$ Therefore $\mathrm{Ext}^n_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$, $\forall n \geqslant 0$.

   (iii) Explain the main steps of the proof that $\mathrm{Ext}^*(A, B)$ is independent of the choice of injective resolution.
   [3pt] Any two injective resolutions $I^\bullet$, $J^\bullet$ of B are chain homotopy equivalent. Applying a linear functor, such as $\mathrm{Hom}(A, -)$, sends chain homotopy equivalent complexes to chain homotopy equivalent complexes. Therefore $\mathrm{Hom}(A, I^\bullet)$ and $\mathrm{Hom}(A, J^\bullet)$ are chain homotopy equivalent. To finish the argument, one uses the fact that chain homotopy equivalent cochain complexes have isomorphic cohomology groups. **(S)**

   (c) [5 marks] (i) Prove that the quotient of an injective $\mathbb{Z}$-module is again injective.
   [2pt] We prove that the quotient of a divisible group is again divisible. Let A be divisible and let $q : A \twoheadrightarrow B$ be a quotient map. Given $b \in B$ and $n \in \mathbb{N}$, pick a preimage $a \in A$ and an element a' s.t. $na' = a$. Then $b' := q(a')$ satisfies $nb' = b$. **(S)**

   (ii) Prove that, when R is an arbitrary ring, the quotient of an injective R-module is in general not injective. *Hint:* argue by contradiction.

[3pt] If it were true that the quotient of an injective R-module is always injective, then every R-module A would admit an injective resolution of length 1, namely $A \rightarrowtail I_0 \to I_0/A \to 0$. It would follow that $\mathrm{Ext}^n_R(B, A) = 0$ for all $n \geqslant 2$. Contradiction. **(N)**

(d) [5 marks] (i) Let k be a field.
Let R be a commutative k-algebra, and let P be a projective R-module.
Prove that $\mathrm{Hom}_k(P, k)$ is always an injective R-module.
[3pt] An R-module map $A \to \mathrm{Hom}_k(P, k)$ is the same thing as a k-linear map $A \otimes_R P \to k$. Given a monomorphism $A \rightarrowtail B$, we need to show that the map $\mathrm{Hom}_R(B, \mathrm{Hom}_k(P, k)) \to \mathrm{Hom}_R(A, \mathrm{Hom}_k(P, k))$ is surjective. That's equivalent to the map $\mathrm{Hom}_k(B \otimes_R P, k) \to \mathrm{Hom}_k(A \otimes_R P, k)$ being surjective. That in turn is equivalent to the map $A \otimes_R P \to B \otimes_R P$ being injective (using that k is a field). The latter holds true since P is projective, and hence flat. **(N)**

(ii) Let $R := k[x, y]$, and let $M := k[[x^{-1}, y^{-1}]]$ be the ring of formal power series in two variables, called x^{-1} and y^{-1}. Explain how to equip M with the structure of a $k[x, y]$-module, and prove that it is an an injective module.
[2pt] M is the k-linear dual of $k[x, y]$ and is therefore an injective module by part (i). The R-module structure on M is given by $x^a y^b \cdot x^n y^m = x^{a+n} y^{b+m}$ if both $a + n$ and $b + m$ are non-positive, and $x^a y^b \cdot x^n y^m = 0$ otherwise. **(N)**

3. (a) [6 marks] (i) Define the notion of a *projective object* in an abelian category.

[1pt] An object P is projective if the functor $\mathrm{Hom}(P, -)$ sends epimorphisms to epimorphisms. I.e. for every epimorphism $f : A \twoheadrightarrow B$, the map $f \circ - : \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B)$ is surjective. **(B)**

(ii) Prove that free R-modules are projective in the abelian category of R-modules.

[2pt] If P is free on some set X, then $\mathrm{Hom}(P, A) = A^X$. If $A \to B$ is surjective then, clearly, so is the induced map $A^X \to B^X$. **(B)**

(iii) Define what is meant by a *projective resolution* of an R-module.

[1pt] A projective resolution of M is an exact sequence $\ldots \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to 0$, where all the P_n are projective. **(B)**

(iv) Prove that every R-module M admits a projective resolution.

[2pt] Pick a generating set $\{m_i\}_{i \in I_0}$ of M, and let $P_0 := R^{I_0}$. The map $\epsilon : P_0 \to M$ sends the ith basis element of P_0 to m_i. **(B)**

Pick a generating set $\{x_{i,1}\}_{i \in I_1}$ of $\ker(\epsilon)$, and let $P_1 := R^{I_1}$. The map $d_1 : P_1 \to P_0$ sends the ith basis element to $x_{i,1}$.

The construction continues by induction; the inductive step is as follows: Pick a generating set $\{x_{i,n}\}_{i \in I_n}$ of $\ker(d_{n-1})$, and let $P_n := R^{I_n}$. The map $d_n : P_n \to P_{n-1}$ sends the ith basis element to $x_{i,n}$.

From now on, unless otherwise stated, we always work in the category of R-modules, for some ring R.

(b) [11 marks] (i) Define what is meant by two chain maps (between chain complexes of R-modules) being *chain homotopic*.

[1pt] Two chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are chain homotopic if there exist maps $h_n : C_n \to D_{n+1}$ such that $dh_n + h_{n-1}d = f_n - g_n \; \forall n$. **(B)**

(ii) Prove that chain homotopic maps induce the same map at the level of homology.

[2pt] For $x \in C_n$ representing a class $[x] \in H_n(C_\bullet)$, we have $dx = 0$. So $dh_n(x) = f_n(x) - g_n(x)$. Hence $[f_n(x)] = [g_n(x)]$ in $H_n(D_\bullet)$. **(S)**

(iii) Prove that if two chain complexes are chain homotopy equivalent then they have isomorphic homology groups.

[2pt] A chain homotopy equivalence between $C_\bullet$ and $D_\bullet$ consists of chain maps $f_\bullet : C_\bullet \to D_\bullet$ and $g_\bullet : D_\bullet \to C_\bullet$ such that $f_\bullet \circ g_\bullet$ and $g_\bullet \circ f_\bullet$ are chain homotopic to the identity. By part (ii), at the level of homology, the maps $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)$ and $H_n(g_\bullet) : H_n(D_\bullet) \to H_n(C_\bullet)$ are such that $H_n(f_\bullet) \circ H_n(g_\bullet)$ and $H_n(g_\bullet) \circ H_n(f_\bullet)$ are equal to the identity. Hence $H_n(f_\bullet)$ and $H_n(g_\bullet)$ are isomorphisms (and each other's inverses). **(S)**

(iv) Provide an example of two chain complexes which are chain homotopy equivalent but not isomorphic. Explain why they have the stated properties.

[3pt] The zero complex and the complex $C_\bullet := (\ldots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to \ldots)$ are chain homotopy equivalent. The relevant maps are $0 : 0 \to C_\bullet$ and $0 : C_\bullet \to 0$. The only non-trivial thing to show is that the zero map $C_\bullet \to C_\bullet$ is chain homotopic to the identity. The chain homotopy which exhibits that fact has only one non-zero term, which is an iso $\mathbb{Z} \to \mathbb{Z}$. **(S)**

(v) Provide an example of two chain complexes which have isomorphic homology groups, but which are not chain homotopy equivalent. Explain why they have the stated properties.

[3pt] The complex $C_\bullet := (\ldots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to \ldots)$ and the complex $D_\bullet := (\ldots \to 0 \to 0 \to \mathbb{Z}/2 \to 0 \to \ldots)$ have isomorphic homology groups but are not chain homotopy equivalent. Indeed, there are no non-zero maps $D_\bullet \to C_\bullet$. **(S)**

(c) [3 marks] Given two chain complexes $P_\bullet$ and $Q_\bullet$, we let

$$\underline{\mathrm{Hom}}(P_\bullet, Q_\bullet) := \mathrm{Tot}^{\Pi}(\mathrm{Hom}(P_\bullet, Q_\bullet)). \tag{$*$}$$

Explain how one can identify cocycles in $Z^n(\underline{\mathrm{Hom}}(P_\bullet, Q_\bullet))$ with chain maps $P_\bullet \to Q_\bullet$ of degree $-n$.

[3pt] An element $(f_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} \mathrm{Hom}(P_{i+n}, Q_i)$ is a cocycle of the cochain complex $(*)$ iff **(N)**

$$(-1)^i f_i \circ d + d \circ f_{i+1} = 0 \qquad \forall i \in \mathbb{Z}.$$

Let $\tilde{f}_i := \varepsilon_i f_i$, where $\varepsilon_i := 1$ when $i = 0,3$ mod(4) and $\varepsilon_i := -1$ when $i = 1,2$ mod(4). Then the new maps $\tilde{f}_i$ satisfy $\tilde{f}_i \circ d = d \circ \tilde{f}_{i+1}$, $\forall i \in \mathbb{Z}$. So, collectively, they form a chain map $\tilde{f}_\bullet : P_\bullet \to Q_\bullet$ of degree $-n$.

(d) [5 marks] Recall that $\mathrm{Ext}^*(A, B)$ can be computed as the cohomology of the cochain complex $(*)$, where $P_\bullet$ and $Q_\bullet$ are projective resolutions of A and B, respectively.
Recall also that two cocycles $f, g \in Z^n(\underline{\mathrm{Hom}}(P_\bullet, Q_\bullet))$ represent the same element in $H^n(\underline{\mathrm{Hom}}(P_\bullet, Q_\bullet))$ if and only if the corresponding chain maps are chain homotopic.

(i) Let k be a field.
Compute $\mathrm{Ext}^*_{k[x]/x^2}(k, k)$ according to the above recipe
(i.e. by choosing a projective resolution $P_\bullet \to k$ and computing the cohomology of $\underline{\mathrm{Hom}}(P_\bullet, P_\bullet)$, equivalently the set of chain homotopy classes of maps $P_\bullet \to P_\bullet$).
[3pt] Let $R := k[x]/x^2$. A projective resolution of k is given by $R \xleftarrow{x} R \xleftarrow{x} R \xleftarrow{x} \ldots$. A general degree $(-n)$ chain map $P_\bullet \to P_\bullet$ is of the form **(N)**

$$\begin{array}{ccccccccccccccc} 0 & \leftarrow & R & \xleftarrow{x} & \ldots & \xleftarrow{x} & R & \xleftarrow{x} & R & \xleftarrow{x} & R & \xleftarrow{x} & R & \xleftarrow{x} & \ldots \\ & & & & & & & & & & \downarrow & & \downarrow & & \\ & & & & & & & & 0 & \leftarrow & R & \xleftarrow{x} & R & \xleftarrow{x} & \ldots \end{array} \tag{$\dagger$}$$

where the jth vertical map is given by $1 \mapsto a + b_j x$ for $a, b_j \in k$. Such a map is nulhomotopic iff $a = 0$. It follows that $\mathrm{Ext}^n_R(k, k) = k$ for every $n \in \mathbb{N}$.

(ii) Use this to compute the ring structure on $\mathrm{Ext}^*_{k[x]/x^2}(k, k)$.
[2pt] The generator y_n of $\mathrm{Ext}^n(k, k)$ is represented by the chain map $(\dagger)$, where all the vertical maps are identity maps. One easily checks that $y_n \circ y_m = y_{n+m}$. It follows that $\mathrm{Ext}^*_{k[x]/x^2}(k, k) = k[y_1]$, with $y_n = y_1^n$. **(N)**