Homological algebra

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Sheet 1

Exercise 1. Show that if p and q are distinct prime numbers, then $\mathbb{Z}/p \otimes_{\mathbb{Z}} \mathbb{Z}/q = 0$.

Exercise 2. Let $R = \mathbb{R}[x]/x^n$. Prove that the obvious inclusion of R-modules $R/x \hookrightarrow R$ is not split. Compute the quotient, and write down the short exact sequence formed by those three modules.

Exercise 3. Let R be ring, let $\{A_i\}_{i\in\mathcal{I}}$ be a collection of right R-modules, and let B be a left R-module. Show that $\big(\bigoplus A_i\big) \otimes_R B \cong \bigoplus (A_i \otimes_R B)$.

Exercise 4. Let $R := \mathbb{Z}[x]$. Compute $\mathrm{Hom}_R\big(R/(2x), R/(4)\big)$ as an R-module. Show that it is isomorphic to R/I for some ideal $I \subset R$.

Exercise 5. Let $R = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b, c \in k\}$ be the ring of upper triangular 2×2 matrices with coefficients in some field k. Show that R is a direct sum of two smaller R-modules: $R = P \oplus Q$. Compute $\mathrm{Hom}_R(P, Q)$ and $\mathrm{Hom}_R(Q, P)$.

Exercise 6. Let k be a field. Find an exact sequence of $k[x]$-modules $0 \to A \to B \to C \to 0$ such that the induced sequence $A \otimes_{k[x]} k[x]/(x) \to B \otimes_{k[x]} k[x]/(x) \to C \otimes_{k[x]} k[x]/(x)$ is not exact.

Exercise 7. Find an example of a ring R and two modules M and N such that the abelian group $M \otimes_R N$ does not carry the structure of an R-module. *Hint:* try a 2×2 matrix algebra.

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(Hand in Monday Oct 15th at 5pm)