Homological algebra (Oxford, fall 2017)

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Problem sheet 4: (hand in Monday Nov. 27th at noon, or Monday Dec. 4th at noon)

Exercise 1. Prove that an exact functor between abelian categories (defined by the property that it sends short exact sequences to short exact sequences) sends acyclic complex to acyclic complexes.

Exercise 2. Let R be a ring. Show that for any left R-module N and any abelian group A, the abelian group $\mathrm{Hom}_{\mathrm{Ab}}(N, A)$ is naturally equipped with the structure of a right R-module. Prove that there is an adjunction

$$\mathrm{Hom}_{\mathrm{Ab}}(M \otimes_R N, A) \cong \mathrm{Hom}_{\mathrm{Mod}_R}(M, \mathrm{Hom}_{\mathrm{Ab}}(N, A)).$$

Exercise 3. Let $A \in \mathcal{A}$ be an object in an abelian category. Prove that the following are equivalent:

- A is projective.
- $\mathrm{Hom}(A, -)$ is exact.
- For every $B \in \mathcal{A}$ and every $n \geq 1$, we have $\mathrm{Ext}^n(A, B) = 0$.
- For every $B \in \mathcal{A}$, we have $\mathrm{Ext}^1(A, B) = 0$.

Exercise 4. Let R be a ring, and let M be a right R-module. Prove that the following are equivalent:

- $M \otimes_R -$ is exact.
- For every left R-module N and every $n \geq 1$, we have $\mathrm{Tor}_n^R(M, N) = 0$.
- For every left R-module N, we have $\mathrm{Tor}_1^R(M, N) = 0$.

A module with the above properties is called *flat*. Prove that every projective R-module is flat. Give an example of a $\mathbb{Z}$-module which is flat but not projective.

Exercise 5. Let R be a P.I.D. Prove that a submodule of a free R-module is always free. Deduce that, in the category of R-modules, we always have $\mathrm{Ext}^{\geq 2}(A, B) = 0$.

Exercise 6. Compute the following Ext-groups in the category of abelian groups first by using a projective resolution, and then by using an injective resolution:

$$\mathrm{Ext}^*(\mathbb{Z}/a, \mathbb{Z}/b) \qquad \mathrm{Ext}^*(\mathbb{Z}, \mathbb{Z}/b) \qquad \mathrm{Ext}^*(\mathbb{Z}/a, \mathbb{Z}) \qquad \mathrm{Ext}^*(\mathbb{Q}, \mathbb{Z})$$

(for the last one only use the injective resolution, because the projective resolution yields something way too messy).

Exercise 7. Let k be a field. Compute the following Ext-groups:

$$\mathrm{Ext}^*_{k[x,y]/(x,y,xy)}(k, k) \qquad \mathrm{Ext}^*_{k[x]}\big(k[x]/(x-a), k[x]/(x-b)\big) \qquad \mathrm{Ext}^*_{k[x,y]}\big(k[x,y]/(x,y^2), k[x,y]/(x^2,y)\big).$$

Exercise 8. Compute the following Tor-groups:

$$\mathrm{Tor}_*^{k[x,y]}(k, k) \qquad \mathrm{Tor}_*^{\Lambda_k(x,y)}(k, k) \qquad \mathrm{Tor}_*^{k[x,y]/(x^2-y^3)}(k, k),$$

where $\Lambda_k(x, y) := k\langle x, y\rangle/(x^2 = 0, y^2 = 0, xy = -yx)$ denotes an exterior algebra on two generators.