C2.1 Lie algebras
Solutions to problem sheet 4

Throughout this sheet we assume that all Lie algebras and all representations discussed are finite dimensional unless the contrary is explicitly stated, and we work over a field k *which is algebraically closed of characteristic zero.*

1. Let $\mathfrak{g}$ be a simple Lie algebra. Show that any nonzero trace form on $\mathfrak{g}$ is a multiple of the Killing form. (*Hint*: Show that the form can be used to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ as a $\mathfrak{g}$-representation. See Problem Sheet 3.)

Solution: Since $\mathfrak{g}$ is simple, the adjoint representation is irreducible (recall from a previous problem sheet). Now a symmetric bilinear form $t\colon \mathfrak{g}\times\mathfrak{g} \to \mathsf{k}$ induces a linear map $\theta\colon \mathfrak{g} \to \mathfrak{g}^*$ via $\theta(x)(y) = t(x, y)$. If t is nondegenerate, then this map is an isomorphism of vector spaces (this is just the definition of nondegeneracy, as $\mathfrak{g}$ is finite dimensional). Now we claim that if the form is invariant, then θ is an isomorphism of $\mathfrak{g}$-representations: indeed if $x, y, z \in \mathfrak{g}$ then

$$x(\theta(y))(z) = -\theta(y)([x, z]) = -t(y, [x, z]) = -t([y, x], z) = t(\mathrm{ad}(x)(y), z) = \theta(\mathrm{ad}(x)(y))(z).$$

Note this is an equivalence, that is, a symmetric bilinear form is invariant if and only if the associated linear map from $\mathfrak{g}$ to $\mathfrak{g}^*$ is a $\mathfrak{g}$-homomorphism. Now if V is an irreducible representation, V^* is also (since if U is a subrepresentation of V, U^0 is a subrepresentation of V^*). Thus Schur's Lemma shows that there is, up to a scalar, a unique isomorphism of $\mathfrak{g}$-representations from V to V^* if V and V^* are isomorphic, and no nonzero such map otherwise. Translating this via the map $\theta \mapsto t$ we see that, up to scalars, there can be at most one nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Since κ is certainly one such, $\mathfrak{g} \cong \mathfrak{g}^*$ and so the space of invariant symmetric bilinear forms on $\mathfrak{g}$ is one-dimensional as claimed. □

2. Show that homomorphisms between semisimple Lie algebras are compatible with the Jordan decomposition, that is, if $\mathfrak{g}_1, \mathfrak{g}_2$ are semisimple Lie algebras, and $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism, then if $x = s + n$ is the Jordan decomposition of $x \in \mathfrak{g}_1$, $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$ in $\mathfrak{g}_2$. (*For this part you may assume the fact, stated in lectures, that if* $x = s + n$ *is the Jordan decomposition of* x *and* $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ *is a representation, then* $\rho(s)$ *is semsimple and* $\rho(n)$ *is nilpotent.*)

Solution: Given an arbitrary homomorphism $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$, we obtain a representation of $\mathfrak{g}_1$ on $\mathfrak{g}_2$ via the composition $\rho = \mathrm{ad}_{\mathfrak{g}_2} \circ\phi\colon \mathfrak{g}_1 \to \mathfrak{gl}(\mathfrak{g}_2)$ (where $\mathrm{ad}_{\mathfrak{g}_2}$ denotes the adjoint representation of $\mathfrak{g}_2$). The compatibility of the Jordan decomposition with representations implies that if $x = s+n$ is the Jordan decomposition of $x \in \mathfrak{g}_1$, then $\mathrm{ad}_{\mathfrak{g}_2}(\phi(s))$ is semisimple and $\mathrm{ad}_{\mathfrak{g}_2}(\phi(n))$ is nilpotent. Since clearly and $[\phi(s), \phi(n)] = \phi([s, n]) = 0$, it follows by uniqueness that $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$ as required. □

3. Use Weyl's theorem to give an alternative proof of the fact that any derivation of a semisimple Lie algebra $\mathfrak{g}$ is inner. (*Hint*: Suppose that δ is a derivation, show that $V = \mathsf{k} \oplus \mathfrak{g}$ has the structure of a $\mathfrak{g}$ representation via $x(a, y) = (0, a\delta(x) + [x, y])$, and consider a complement to the subrepresentation $\mathfrak{g}$.)

Solution: We first check that V is a representation: for any $a \in \mathsf{k}, x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned}
(x.y - y.x)(a, z) &= x(0, a.\delta(y) + [y, z]) - y(0, a.\delta(x) + [x, z]) \\
&= (0, [x, a.\delta(y) + [y, z]]) - (0, [y, a.\delta(x) + [x, z]) \\
&= (0, a[x, \delta(y)] + [x, [y, z]] + a[\delta(x), y] - [y, [x, z]]) \\
&= (0, a\delta([x, y]) + [[x, y], z]) \\
&= [x, y](a, z).
\end{aligned}$$

where in the second last line we use the Jacobi identity and the definition of a derivation. Now it is clear that $M = \{(0, x) : x \in \mathfrak{g}\}$ is a subrepresentation of V (isomorphic to the adjoint representation) and the quotient V/M is isomorphic to the trivial representation. By Weyl's theorem M has a complementary subrepresentation L, which is the trivial representation. But then if $(a, z) \in V$ is a nonzero element of M, we may scale it so that $a = -1$, and then for all $x \in \mathfrak{g}$ we have $x(-1, z) = 0$, which implies $-\delta(x) + [x, z] = 0$, that is $\delta = \mathrm{ad}(z)$. $\square$

4. Let $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ be the symplectic Lie algebra. Show that $\mathfrak{h}$, the space of matrices in $\mathfrak{g}$ which are diagonal, is a Cartan subalgebra, and find the roots of $\mathfrak{sp}_{2n}(\mathbb{C})$.

Solution: $\mathfrak{sp}_{2n}$*:* For a matrix A , in this question we will use the notation ${}^tA$ to denote the matrix obtained by flipping the entries along the "anti-diagonal", so that if $A = (a_{ij})$ then the (i, j)-th entry of ${}^tA$ is $a_{n+1-j,n+1-i}$. The Lie algebra $\mathfrak{sp}_{2n}$ then consists of block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. such that $-A = {}^tD$ and ${}^tB = B$, ${}^tC = C$. Again, let $\mathfrak{h}$ denote the intersection of $\mathfrak{sp}_{2n}$ with the diagonal matrices. Then

$$\mathfrak{h} = \left\{ \sum_{i=1}^{2n} \lambda_i E_{ii} \;\middle|\; \text{and } \lambda_i = -\lambda_{2n+1-i} \right\}.$$

Hence we may identify

$$\mathfrak{h}^* = \bigoplus_{i=1}^{n} e_i^*$$

where $e_i^*(h) = \lambda_i$. We can write

$$\begin{aligned}
\mathfrak{sp}_{2n} = \mathfrak{h} \oplus &\bigoplus_{i \neq j \leq n} \mathbb{C}(E_{ij} - E_{2n+1-i,2n+1-j}) \oplus \\
\oplus &\bigoplus_{1 \leq i+j \leq n} \mathbb{C}(E_{i,n+j} + E_{n+1-j,2n+1-i}) \oplus \bigoplus_{i=1}^{n} E_{i,2n+1-i} \\
\oplus &\bigoplus_{1 \leq i+j \leq n} \mathbb{C}(E_{n+i,j} + E_{2n+1-j,n+1-i}) \oplus \bigoplus_{i=1}^{n} E_{2n+1-i,i}
\end{aligned}$$

and one calculates the weights in each case to be $e_i^* - e_j^*$, $e_i^* + e_j^*$, $2e_i^*$, $-e_i^* - e_j^*$ and $-2e_i^*$. Hence $\mathfrak{h}$ is a Cartan subalgebra and R is of type C_n. □

5. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. If $\Phi \subset \mathfrak{h}^*$ is the corresponding root system find an expression for the dimension of $\mathfrak{g}$ in terms of Φ. (In particular, the dimension of $\mathfrak{g}$ is determined by the root system).

Solution: In lectures we have seen the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

We also saw that $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$. Hence

$$\dim \mathfrak{g} = \dim \mathfrak{h} + |\Phi| = \text{rank}(\Phi) + |\Phi|.$$

□

6. Suppose that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$. Show that if V is irreducible as a $\mathfrak{g}$-representation and $\text{tr}(\rho(x)) = 0$ for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is semisimple.

Solution: Let $\mathfrak{s}$ be the radical of $\mathfrak{g}$. Then since $\mathfrak{s}$ is solvable, by Lie's theorem there is a nonzero vector v and a linear map $\lambda\colon \mathfrak{s}/D(\mathfrak{s}) \to \mathsf{k}$ such that $\rho(x)(v) = \lambda(x).v$. But then as $\mathfrak{s}$ is an ideal, we see that for all $x \in \mathfrak{g}$, $s \in \mathfrak{s}$ we have

$$\begin{aligned} sx(v) &= [s,x] + xs(v) \\ &= \lambda(s)x(v) \end{aligned}$$

by Lie's Lemma. It follows the set of vectors $\{v \in V : s(v) = \lambda(s).v\}$ is a nonzero $\mathfrak{g}$-subrepresentation of V, so that since V is irreducible it must be all of V. But then the $\mathfrak{s} \subseteq \mathfrak{g} \cap \mathsf{k}.\text{id}_V$, and since we assume that $\text{tr}(x) = 0$ for all $x \in \mathfrak{g}$ this is zero so that $\mathfrak{s} = 0$ as required. □

7. Let k be a field and let $\mathfrak{s}_\mathsf{k}$ be the 3-dimensional k-Lie algebra with basis $\{e_0, e_1, e_2\}$ and structure constants $[e_i, e_{i+1}] = e_{i+2}$ (where we read the indices modulo 3, so that we have for example $[e_2, e_0] = e_1$).

i) Show that $\mathfrak{s}_\mathsf{k}$ is a simple Lie algebra.

ii) Show that $\mathfrak{s}_\mathbb{R}$ is isomorphic to the Lie algebra $(\mathbb{R}^3, \wedge)$, where $\wedge$ is the cross product.

iii) Show that $\mathfrak{s}_\mathbb{R}$ (equivalently, $(\mathbb{R}^3, \wedge)$) is not isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. (*Hint*: You may show that $(\mathbb{R}^3, \wedge)$ does not have any nonzero elements x such that $\text{ad}(x)$ is diagonalisable.

iv) Show that $\mathfrak{s}_\mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})$.

Solution: To see that $\mathfrak{s}_{\mathsf{k}}$ is simple, suppose that I is a nonzero ideal and let $x = ae_0 + be_1 + ce_2$ be a nonzero element of I. Then $[e_1, [e_0, x]] = [e_1, be_2 - ce_1] = be_0 \in I$, and similarly we find $ae_2, ce_1 \in I$ also. Thus since $x \neq 0$, we must have some e_i in I, but then clearly all of $\{e_0, e_1, e_2\}$ lie in I so that $I = \mathfrak{s}_{\mathsf{k}}$ as required.

By direct calculation we see that the image of the adjoint representation $\mathrm{ad}\colon \mathfrak{s}_{\mathsf{k}} \to \mathfrak{gl}_3(\mathsf{k})$ (where we use the basis $\{e_0, e_1, e_3\}$ to identify $\mathfrak{s}_{\mathsf{k}}$ with k^3) is exactly the Lie algebra of skew-symmetric matrices, indeed we have:

$$\mathrm{ad}(ae_0 + be_1 + ce_2) = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

which is clearly injective, so it follows $\mathfrak{s}_{\mathsf{k}}$ is in fact isomorphic to $(\mathbb{R}^3, \wedge)$.

The characteristic polynomial of a skew-symmetric matrix as above is $\lambda(\lambda^2 + a^2 + b^2 + c^2)$, thus when $\mathsf{k} = \mathbb{R}$, a non-zero skew-symmetric 3×3 matrix over $\mathbb{R}$ has exactly one real eigenvalue. On the other hand, recall that $\mathfrak{sl}_2(\mathsf{k})$ has a basis $\{e, f, h\}$ with structure constants $[e, f] = h, [h, e] = 2e, [h, f] = -2f$, thus the action of $\mathrm{ad}(h)$ on $\mathfrak{sl}_2(\mathbb{R})$ is diagonalisable with 3 distinct eigenvalues. It follows that we cannot have $\mathfrak{s}_{\mathbb{R}} \cong \mathfrak{sl}_2(\mathbb{R})$. When we take $\mathsf{k} = \mathbb{C}$ however, we can easily find a skew-symmetric matrix H with the required eigenvalues, and then find the ± 2-eigenspaces of H to determine matrices E and F (given H, the equation $[E, F] = H$ will normalize E, F up to a constant). Then we can define an isomorphism from $\mathfrak{sl}_2(\mathbb{C})$ via $h \mapsto H$, $e \mapsto E$, and $f \mapsto F$.

For example, if you take $H = 2ie_0$, and then we may take $E = e_1 + ie_2$, and $F = -e_1 + ie_2$. There are many other options however: you can take e.g. $H = i\sqrt{2}(e_0 - e_2)$, and then $E = \frac{1}{\sqrt{2}}e_0 - ie_1 + \frac{1}{\sqrt{2}}e_2$, and $F = -(\frac{1}{\sqrt{2}}e_0 + ie_1 + \frac{1}{\sqrt{2}}e_2)$.

Note that this shows the classification of simple Lie algebras over characteristic zero fields which are not algebraically closed is more delicate than the algebraically closed case. □

Solution: Question 8: For the classification of the Dynkin diagrams see

James Humphreys, Introduction to Lie Algebras and Representation Theory, Springer, 1972, end of Chapter III. □