C2.1a Lie algebras

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Problem Sheet 2

1. Let $\mathfrak{g}$ be a complex Lie algebra. Show that $\mathfrak{g}$ is nilpotent if and only if every 2-dimensional subalgebra of $\mathfrak{g}$ is abelian.

2. Let V be a finite dimensional complex vector space and let $x, y \in \mathfrak{gl}(V)$ be linear maps. Suppose that x and y both commute with $z = [x, y]$. Show that z is a nilpotent endomorphism of V.

3. Let V be a finite dimensional complex vector space. If $x \in \mathrm{End}(V)$, and $V = \bigoplus_\lambda V_\lambda$ is the decomposition of V into a direct sum of generalised eigenspaces of x, we define $x_s \in \mathrm{End}(V)$ to be the linear map given by $x_s(v) = \lambda.v$ for $v \in V_\lambda$. It is called the *semisimple* part of x. Clearly it is diagonalisable.

i) Show that x is regular if and only if x_s is regular.

ii) When is a semisimple (*i.e.* diagonalisable) element of $\mathfrak{gl}(V)$ regular?

iii) Exhibit a Cartan subalgebra of $\mathfrak{gl}(V)$, and describe the set of all regular elements of $\mathfrak{gl}(V)$.

Terminology: Note the following definitions: if (V, ϕ) is a representation of a Lie algebra $\mathfrak{g}$, then we say a subspace $U < V$ is a *subrepresentation* if $\phi(x)(U) \subseteq U$ for all $x \in \mathfrak{g}$. Note that this implies that ϕ restricts to give a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(U)$. A nonzero representation is said to be *irreducible* or *simple* if it has no non-zero proper subrepresentation.

4. Let $\mathfrak{g}$ be a Lie algebra. Suppose that the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is irreducible. What can you say about $\mathfrak{g}$?

5. Let $\mathfrak{g}$ be the set of complex matrices of the form $\begin{pmatrix} \alpha & \beta & \lambda \\ \gamma & \delta & \mu \\ 0 & 0 & 0 \end{pmatrix}$ where $\alpha + \delta = 0$. Show that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_3(\mathbb{C})$. Find the radical of $\mathfrak{g}$ and show that $\mathfrak{g}$ contains a subalgebra isomorphic to $\mathfrak{g}/\mathrm{rad}\mathfrak{g}$. Prove that the only ideal of $\mathfrak{g}$ strictly contained in $\mathrm{rad}\mathfrak{g}$ is $\{0\}$.

6. Let $\mathfrak{b}_n$ be the Lie algebra of upper triangular matrices in $\mathfrak{gl}_n(\mathsf{k})$. This is a solvable but not nilpotent Lie algebra. Find a Cartan subalgebra of $\mathfrak{b}_n$.