C2.1a Lie algebras

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Problem Sheet 3

1. Let κ denote the Killing form on $\mathfrak{gl}_n(\mathbb{C})$ and let $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-$ denote the subspaces of diagonal, strictly upper triangular and strictly lower triangular matrices respectively.

i) Show that $\mathfrak{h}$ is orthogonal to $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ and that the restriction of κ to $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ is nondegenerate. (*Hint*: It is probably useful to calculate the values of the Killing form on matrix coefficients).

ii) Calculate $\mathfrak{n}_+^{\perp}$.

iii) Describe the radical of the restriction of κ to $\mathfrak{h}$ and conclude that the restriction of κ to $\mathfrak{sl}_n(\mathbb{C})$ is nondegenerate.

2. Suppose $\mathfrak{g}$ is a Lie algebra and that β is an invariant symmetric bilinear form of $\mathfrak{g}$. (Invariant means $\beta([x,y],z) = \beta(x,[y,z])$ for all $x, y, z \in \mathfrak{g}$.) Show that β induces a linear map

$$\tau : \mathfrak{g} \to \mathfrak{g}^*, \quad x \mapsto \beta(x, -).$$

Viewing both $\mathfrak{g}$ and $\mathfrak{g}^*$ as $\mathfrak{g}$-modules, show that τ is a $\mathfrak{g}$-module homomorphism. Deduce that if β is nondegenerate, then $\mathfrak{g}$ and $\mathfrak{g}^*$ are isomorphic as $\mathfrak{g}$-modules.

3. Show that the Killing form for $\mathfrak{sl}_n$ is given by:

$$\kappa(x,y) = 2n.\mathrm{tr}(xy), \quad x, y \in \mathfrak{sl}_n.$$

The next few questions of this exercise sheet classify all the irreducible finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

Recall that if we let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then e, f and h give a basis of $\mathfrak{sl}_2$ with relations

$$[h,e] = 2e, [h,f] = -2f \quad \text{and} \quad [e,f] = h.$$

Hence, a representation of $\mathfrak{sl}_2(\mathbb{C})$ consists of a vector space V over $\mathbb{C}$ together with three endomorphisms E, F and H satisfying

$$HE - EH = 2E, HF - FH = -2F \quad \text{and} \quad EF - FE = H.$$

(We recover the representation $\phi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ by setting $\phi(e) = E$, $\phi(f) = F$ and $\phi(h) = H$.)

We will also need a partial ordering on k: since k has characteristic zero it contains a copy of $\mathbb{Q}$, and we will say that $a < b$ if $b - a \in \mathbb{Q}_{>0}$. If $I \subset \mathsf{k}$ is a finite subset of k we say $\lambda \in I$ is *maximal* if $\lambda < \mu$ implies $\mu \notin I$.

In the rest of this problem set we always assume that V is *finite dimensional*.

4. a) Show that the endomorphisms E and H satisfy the relation

$$(H - (\lambda + 2))^k E = E(H - \lambda)^k.$$

(Here $\lambda \in \mathbb{C}$ and we write λ instead of $\lambda \cdot \mathrm{id}_V$.) Deduce that if $v \in V$ belongs to the generalised λ-eigenspace of H, then Ev belongs to the generalised $(\lambda + 2)$-eigenspace.

b) Deduce a similar statement for the action of F on the generalised eigenspaces of H.

c) Let λ be an eigenvalue for H which is a maximal element of the set of eigenvalues of H in the sense described above. Use a) to show that $EV_\lambda = 0$.

d) Use b) to deduce that for large enough n we have $F^n(v) = 0$.

5. a) Show the relation (for $n \geq 1$)

$$HF^n = F^n H - 2nF^n.$$

b) Show ($n \geq 1$ as before)

$$EF^n = F^n E + nF^{n-1}H - n(n-1)F^{n-1}.$$

c) Deduce that, if $v \in V$ is a vector such that $Ev = 0$ then

$$E^n F^n v = nE^{n-1}F^{n-1}(H - (n-1))v = n! \prod_{i=1}^{n} (H - (i-1))v.$$

6. Let λ be a maximal eigenvalue of H (in the above sense) and let V_λ denote the generalised λ-eigenspace. Use 4(d) and 5(c) to deduce that H acts diagonalisably on V_λ and that λ is a non-negative integer.

7. a) Let λ be a maximal eigenvalue of H as in the previous question, and choose a non-zero vector $v \in V_\lambda$. We know by Questions 4,5 and 6 that $Ev = 0$ and that λ is an non-negative integer. Show the relations:

$$HF^k v = (\lambda - 2k)F^k v,$$
$$EF^k v = k(\lambda - (k-1))F^{k-1} v.$$

Deduce that $F^{\lambda+1}v = 0$ and that the $F^i v$ for $0 \leq i \leq \lambda$ are linearly independent and span a simple submodule of V.

b) Check that the above relations define an $\mathfrak{sl}_2(\mathbb{C})$-module for any non-negative integer λ. Deduce that there is (up to isomorphism) a unique simple module $V(\lambda)$ of dimension $\lambda + 1$ for all non-negative integers λ.