C2.1a Lie algebras

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Problem Sheet 4

Throughout this sheet we assume that all Lie algebras and all representations discussed are finite dimensional unless the contrary is explicitly stated, and we work over a field k *which is algebraically closed of characteristic zero.*

1. Let $\mathfrak{g}$ be a simple Lie algebra. Show that any nonzero trace form on $\mathfrak{g}$ is a multiple of the Killing form. (*Hint*: Show that the form can be used to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ as a $\mathfrak{g}$-representation. See Problem Sheet 3.)

2. Show that homomorphisms between semisimple Lie algebras are compatible with the Jordan decomposition, that is, if $\mathfrak{g}_1, \mathfrak{g}_2$ are semisimple Lie algebras, and $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism, then if $x = s + n$ is the Jordan decomposition of $x \in \mathfrak{g}_1$, $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$ in $\mathfrak{g}_2$. (*You may assume the fact, stated in lectures, that if $x = s + n$ is the Jordan decomposition of x and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, then $\rho(s)$ is semsimple and $\rho(n)$ is nilpotent.*)

3. Use Weyl's theorem to give an alternative proof of the fact that any derivation of a semisimple Lie algebra $\mathfrak{g}$ is inner. (*Hint*: Suppose that δ is a derivation, show that $V = \mathsf{k} \oplus \mathfrak{g}$ has the structure of a $\mathfrak{g}$ representation via $x(a, y) = (0, a\delta(x) + [x, y])$, and consider a complement to the subrepresentation $\mathfrak{g}$.)

4. Let $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ be the symplectic Lie algebra. Show that $\mathfrak{h}$, the space of matrices in $\mathfrak{g}$ which are diagonal, is a Cartan subalgebra, and find the roots of $\mathfrak{sp}_{2n}(\mathbb{C})$.

5. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. If $\Phi \subset \mathfrak{h}^*$ is the corresponding root system find an expression for the dimension of $\mathfrak{g}$ in terms of Φ. (In particular, the dimension of $\mathfrak{g}$ is determined by the root system).

6. Suppose that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$. Show that if V is irreducible as a $\mathfrak{g}$-representation and $\mathrm{tr}(\rho(x)) = 0$ for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is semisimple.

7. Let k be a field and let $\mathfrak{s}_{\mathsf{k}}$ be the 3-dimensional k-Lie algebra with basis $\{e_0, e_1, e_2\}$ and structure constants $[e_i, e_{i+1}] = e_{i+2}$ (where we read the indices modulo 3, so that we have for example $[e_2, e_0] = e_1$).

i) Show that $\mathfrak{s}_{\mathsf{k}}$ is a simple Lie algebra.

ii) Show that $\mathfrak{s}_{\mathbb{R}}$ is isomorphic to the Lie algebra $(\mathbb{R}^3, \wedge)$, where $\wedge$ is the cross product.

iii) Show that $\mathfrak{s}_{\mathbb{R}}$ (equivalently, $(\mathbb{R}^3, \wedge)$) is not isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

iv) Show that $\mathfrak{s}_{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$.

8 (Optional for the vacation)**.** Complete the details in the classification of Dynkin diagrams and root systems of simple Lie algebras began in lectures.