C2.1a Lie algebras

Mathematical Institute, University of Oxford
Michaelmas Term 2018

Problem Sheet 0 (not for handing in)

1. Let V be a finite dimensional vector space and let $A \subset \mathrm{End}(V)$ denote a subspace consisting of commuting diagonalisable endomorphisms. Show that we may find a basis of V in which each element of A is represented by a diagonal matrix.

2. Let k be a field and let V be a k-vector space. If $x \in \mathrm{End}(V)$, and $V = \bigoplus_\lambda V_\lambda$ is the decomposition of V into a direct sum of generalised eigenspaces of x, we define $x_s \in \mathrm{End}(V)$ to be the linear map given by $x_s(v) = \lambda.v$ for $v \in V_\lambda$. It is called the *semisimple* part of x. Clearly it is diagonalisable.

i) Show that the element $x_n = x - x_s$ is nilpotent, and check that x_s and x_n commute.

ii) Show that if $x, y \in \mathrm{End}(V)$ commute, and y is nilpotent, then the generalised eigenspaces of x and $x + y$ coincide.

3. Let k be an infinite field (not necessarily algebraically closed or of characteristic zero), and suppose that V is a finite dimensional k-vector space. If $U_1, U_2, \ldots, U_r$ are proper subspaces of V, show that $V \neq U_1 \cup U_2 \cup \ldots \cup U_r$.

There is an algebraic way to think about the idea of "infinitesimals". The next two questions of the sheet explore this idea a little. Let k be a field and let $D_\mathsf{k} = \mathsf{k}[t]/(t^2)$. Write ε for the image of t in D_k, so that $\varepsilon^2 = 0$. We want to consider $\mathrm{Mat}_n(D_\mathsf{k})$ the space of $n \times n$ matrices over D_k.

4. Show that $\mathrm{GL}_n(D_\mathsf{k})$, the group of invertible matrices over D_k is exactly the set:

$$\{A + \varepsilon B : A \in \mathrm{GL}_n(\mathsf{k}), B \in \mathrm{Mat}_n(\mathsf{k})\}.$$

The natural homomorphism $\mathrm{e}\colon D_\mathsf{k} \to \mathsf{k}$ given by $\epsilon \mapsto 0$ induces a homomorphism of groups $\mathrm{e}_n\colon \mathrm{GL}_n(D_\mathsf{k}) \to \mathrm{GL}_n(\mathsf{k})$. Deduce that the kernel can be identified with $\mathrm{Mat}_n(\mathsf{k})$, *i.e.* $\mathfrak{gl}_n(\mathsf{k})$.

5. i) The determinant is defined for a matrix with entries in any commutative ring. For $X \in \mathrm{Mat}_n(D_\mathsf{k})$ find $\det(X)$ in terms of the column vectors of A, B where $X = A + \varepsilon B$, $A, B \in \mathrm{Mat}_n(\mathsf{k})$. In particular, show that if $X = I + \varepsilon B$ then $\det(X) = 1$ if and only if $\mathrm{tr}(B) = 0$.

ii) The special orthogonal group is defined to be

$$\mathrm{SO}_n(\mathsf{k}) = \{A \in \mathrm{GL}_n(\mathsf{k}) : \det(A) = 1, A.A^t = I\}.$$

Show that the kernel of the map $\mathrm{SO}_n(D_\mathsf{k}) \to \mathrm{SO}_n(\mathsf{k})$ can be identified with

$$\mathfrak{so}_n(\mathsf{k}) = \{X \in \mathfrak{gl}_n(\mathsf{k}) : X + X^t = 0\}.$$

6. Read Appendix 1 in the lecture notes for a review of the relevant facts about symmetric bilinear forms needed for this lecture course.