Analytic Topology: Problem sheet 0

This sheet is revision of material that might be contained in an introductory course in topology. If you have difficulty, then consult a text such as *General Topology* by Willard (or, in desperation, look at the solution sheet). But recall that the main purpose of problem sheets is *training* in methods and appropriate modes of thought, so it's good to make a substantial effort before looking up the answers.

1. (i) Prove that every compact subset of a Hausdorff space is closed.

(ii) Give an example of a space X with a compact subset K which is not closed.

2. (i) Prove that every closed subset of a compact space is compact.

(ii) Give an example of a space X with a closed subspace A which is not compact.

3. Prove that the image of any compact space under a continuous function is compact.

4. (i) Prove that if X is a compact space, Y is a Hausdorff space, and $f : X \to Y$ is bijective and continuous, then it is a homeomorphism.

(ii) Give examples to show that the hypotheses that X is compact and that Y is Hausdorff cannot be omitted.

5. Let (X, d) be a metric space.

(i) Show that a subset A of X is closed if and only if every accumulation point a of a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A, is itself an element of A.

[An *accumulation point* of a sequence $(y_n)_{n\in\mathbb{N}}$ is a point x having the property that every open set U containing x contains y_n for infinitely many values of n.]

(ii) Show that if a is an accumulation point of a sequence $(a_n)_{n\in\mathbb{N}}$, then there is a subsequence of $(a_n)_{n\in\mathbb{N}}$ which converges to a.

(iii) Deduce that in a metric space, the topology can be completely described in terms of convergent sequences.

6. Let X be a Hausdorff space, let x be an element of X, and let C be a compact subset of X such that $x \notin C$.

Prove that there exist disjoint open sets U and V such that $x \in U$ and $C \subseteq V$.

(This is an instance of an informal metatheorem that compact sets behave in many ways like points.)

[Hint: For each $y \in C$ there exist disjoint open sets $U_y \ni x$ and $V_y \ni y$. (Why?) Now use compactness. The tricky bit is finding suitable sets U and V which are *disjoint*.]

7. *[Harder]* Prove that a product of two compact spaces is compact.