Analytic Topology: Problem sheet 1

Some of the problems on this sheet may have the character of revision, or be extensions of problems on sheet 0.

1. Prove that the following are equivalent:

(i) X is Hausdorff;

(ii) if $p \in X$, then for each $q \neq p$, there is an open set $U \ni p$ such that $q \notin \overline{U}$;

(iii) for each $p \in X$, $\bigcap\{\overline{U} : U \text{ open and } U \ni p\} = \{p\}$;

(iv) the diagonal

$$\Delta = \{(x, x) \in X \times X : x \in X\}$$

is closed in $X \times X$.

2. (i) If X is regular, $C \subseteq X$, $D \subseteq X$, C compact, D closed, and $C \cap D = \varnothing$, find disjoint open U, V such that $C \subseteq U$ and $D \subseteq V$, and hence show that a compact Hausdorff space is normal.

(ii) Show that X is normal if and only if, for each closed C and open $U \supseteq C$, there exists open V such that $C \subseteq V \subseteq \overline{V} \subseteq U$.

(iii) X is said to be *completely normal* if, for each pair of subsets A, B such that $\overline{A} \cap B = \varnothing = A \cap \overline{B}$, there exist disjoint open U, V such that $A \subseteq U$, $B \subseteq V$. Prove that a topological space is completely normal if and only if every subspace is normal.

3. Let (X, d) be a metric space with its usual topology, $\varnothing \neq A \subseteq X$. Define, for $x \in X$, $D(x, A) = \inf\{d(x, y) : y \in A\}$. Prove that:

(i) $D(x, A) : X \to \mathbb{R}$ is continuous (x varies, A is fixed),

(ii) $D(x, A) = 0$ if and only if $x \in \overline{A}$,

(iii) if C is closed in X, there exists an infinite sequence (V_n) of sets V_n open in X with $C = \bigcap_{n \in \mathbb{N}} V_n$,

(iv) X is completely normal.

A topological space X is *first countable* if and only if there is a countable local basis at every point (that is, for all $x \in X$, there exists a countable family $\{U_n : n \in \mathbb{N}\}$ such that for all open $V \ni x$, $U_n \subseteq V$. Suppose X is first countable and $f : X \to Y$. Prove that:

(v) if $A \subseteq X$, then $x \in \overline{A}$ if and only if there is a sequence *on* A converging to x;

(vi) f is continuous at x_0 if and only if $f(x_n) \to f(x_0)$ for each sequence (x_n) for which $x_n \to x_0$.

4. X is *extremally disconnected* if the closure of every open set is open. Subsets A and B are *functionally separated* if there is a continuous function $f : X \to [0, 1]$ such that $f[A] \subseteq \{0\}$, $f[B] \subseteq \{1\}$. ($[0, 1]$ has its subspace topology inherited from $\mathbb{R}$. We write $\subseteq$ for $=$ in case A or B is empty.) Prove that the following are equivalent:

(i) X is extremally disconnected,

(ii) every two disjoint open sets in X have disjoint closures,

(iii) every two disjoint open sets in X are functionally separated.

5. Suppose that X and Y are topological spaces, $A \subseteq X$, $B \subseteq Y$. Prove that $\overline{A} \times \overline{B} = \overline{A \times B}$. Prove that if X and Y are regular, then $X \times Y$ is regular.

6. Suppose that X is arbitrary, Y is Hausdorff, and $f : X \to Y$, $g : X \to Y$ are both continuous. Prove that:

(i) $\{x \in X : f(x) = g(x)\}$ is closed in X,

(ii) if $D \subseteq X$ is dense (that is, $\overline{D} = X$), and $f|_D = g|_D$ (that is, if $f(x) = g(x)$ for each $x \in D$), then $f = g$,

(iii) the set $G_f = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$ (G_f is known as the *graph* of f. People who have done set theory will note that it has become conventional to identify a function with its graph),

(iv) if $Z \subseteq Y$ and the continuous function $h : Y \to Z$ is such that $h(y) = y$ for each $y \in Z$, then Z is closed in Y. (Such an h is called a *retraction*.)