Analytic Topology: Problem sheet 2

1. (i) X is said to be *countably compact* if every countable open covering has a finite subcovering. Prove that a T_1 space X is countably compact if and only if every infinite subset has a limit point in X.

(ii) X has a *countable basis at* x if there is a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets, each containing x, such that, for each open $V \ni x$, there exists n such that $x \in U_n \subseteq V$, and X is *first countable* if it has a countable basis at every point $x \in X$. (So, for example, metric spaces are first countable.) Prove that a countably compact, first countable, Hausdorff space is regular.

2. Show that a metric space is Lindelöf if and only if it is separable, if and only if it is second countable.

3. (i) $\mathscr{B}$ is a *basis* for a filter $\mathscr{F}$ on a set X if and only if

$$\mathscr{F} = \{F \subseteq X : (\exists B \in \mathscr{B})(B \subseteq F)\}.$$

$\mathscr{B}$ is a *filter-basis* on X if and only if
(a) $\varnothing \notin \mathscr{B}$ and $\mathscr{B} \neq \varnothing$,
(b) if B_1, $B_2 \in \mathscr{B}$, then $\exists B_3 \in \mathscr{B}$ such that $B_3 \subseteq B_1 \cap B_2$.
Prove that a family of subsets of a set X is a filter-basis if and only if it is the basis of some filter on X.

(ii) Suppose that $\mathscr{N}_x$ is the filter of all neighbourhoods of a point x. A filter-basis $\mathscr{D}$ *converges to* y if U open, $U \ni y$ implies $\exists D \in \mathscr{D}$ with $D \subseteq U$. Prove that $f : X \to Y$ is continuous if and only if, for every $x \in X$, $f(\mathscr{N}_x)$ converges to $f(x)$.

(iii) Prove that the following are equivalent:
(a) X is Hausdorff,
(b) no filter on X converges to more than one point,
(c) if a filter $\mathscr{F}$ on X converges to x, then x is the only cluster point of $\mathscr{F}$.

(iv) Suppose that f is a function from X onto Y, and that $x \in X$. Prove that f is continuous at x if and only if, for every ultrafilter $\mathscr{U}$ on X which converges to x, the ultrafilter $f(\mathscr{U})$ converges to $f(x)$.

4. Suppose M, N, X, Y are topological spaces, $\pi_X : X \times Y \to X$ is the usual projection.

(i) Prove that $f : M \to N$ is closed (ie. $f(C)$ is closed in N, for each C closed in M) if and only if, for each $n \in N$ and each open $U \supseteq f^{-1}(n)$, there is an open $V \ni n$ such that $f^{-1}(V) \subseteq U$.

(ii) If Y is compact, prove that π_X is closed.

(iii) If Y is compact Hausdorff, prove that $g : X \to Y$ is continuous if and only if its graph is closed in $X \times Y$.

5. (i) For f, $g \in \prod_{n\in\mathbb{N}}[0, 1]$, define $D(f, g) = \sum_{n=0}^{\infty} 2^{-n}|f(n) - g(n)|$. Show that the Tychonoff topology on the product $\prod_{n\in\mathbb{N}}[0, 1]$, and the topology generated by the metric D, are the same.

(ii) If $\{X_\lambda : \lambda \in \Lambda\}$ is a family of non-empty topological spaces. Show that $\prod_{\lambda\in\Lambda} X_\lambda$ is Hausdorff in the Tychonoff topology if and only if each X_λ is Hausdorff.

(iii) Suppose that $\{X_\lambda : \lambda \in \Lambda\}$ is a family of non-empty topological spaces and that $\prod_{\lambda\in\Lambda} X_\lambda$ is given the Tychonoff topology. Fix a point $f \in \prod_{\lambda\in\Lambda} X_\lambda$ and, for any fixed $\mu \in \Lambda$, let Y_μ be that subset defined by

$$Y_\mu = \{g : g(\lambda) = f(\lambda) \text{ whenever } \lambda \neq \mu\}.$$

Prove that the restriction of the projection mapping π_μ to Y_μ is a homeomorphism from Y_μ to X_μ.

(iv) Suppose that for each $\lambda \in \Lambda$, X_λ is non-empty and connected. For any given $f \in \prod_{\lambda\in\Lambda} X_\lambda$, show that

$$D = \{g : g(\lambda) = f(\lambda) \text{ for all but finitely many } \lambda\}$$

is connected. Show that $\prod_{\lambda\in\Lambda} X_\lambda$ is connected, in the Tychonoff topology.

6. Find the generalization of Tychonoff's Theorem to locally compact spaces, and prove it.

7. (i) Suppose that $\{X_\lambda : \lambda \in \Lambda\}$ is a family of non-empty topological spaces. We define the *box topology* on the cartesian product $\prod_{\lambda\in\Lambda} X_\lambda$ to be the topology with basis consisting of *all* products $\prod_{\lambda\in\Lambda} T_\lambda$, where each T_λ is open in X_λ.

Show that, in the box topology, a product of infinitely many Hausdorff spaces, each of which has at least two points, is not compact.

[Hint: consider the case where each X_λ has exactly two points. What is the box topology like then?]

(ii) (Optional) For each natural number n, let $X_n = [0,1]$. In the cartesian product $\prod_{n\in\mathbb{N}} X_n$, let

$$U = \left\{ g : (\exists r > 0)(\forall n)\left(g(n) < \frac{r}{n}\right) \right\}.$$

Show that, in the box topology, U is clopen (ie. closed and open). Deduce that a box product of connected spaces need not be connected.