Analytic Topology: Problem sheet 3

1. (i) If X is a locally compact Hausdorff space, show that X has a basis of open sets with compact closure.

(ii) Prove that a locally compact subset A of a Hausdorff space X is of the form $V \cap F$, where V is open and F is closed in X. [Hint: what might F be?]

2. Prove that the following properties of a locally compact Hausdorff space X are equivalent.

(i) X is σ-compact (that is, X is a union of a countable family of compact subsets),

(ii) X can be represented as $X = \bigcup_{i=1}^{\infty} U_i$, where each U_i is an open set with compact closure, and $\overline{U_i} \subseteq U_{i+1}$ for each $i \in \mathbb{N}$,

(iii) X is Lindelöf.

3. Suppose that f is a proper mapping from X onto Y. Suppose $U \subseteq X$ is open. Defining $f^*(U) = Y \setminus f(X \setminus U)$, show that $f^*(U)$ is open, that $f^*(U) \subseteq f(U)$; and that $f^{-1}(A) \subseteq U$ implies $A \subseteq f^*(U)$. Prove that:

(i) if X is Hausdorff (respectively regular), then Y is Hausdorff (respectively regular).

(ii) if Y is Lindelöf (respectively countably compact), then X is Lindelöf (respectively countably compact).

(iii) Assuming X to be Hausdorff, X is locally compact iff Y is.

4. Suppose $\mathscr{U}$ is an ultrafilter on a (non-empty) set X. We say that $\mathscr{U}$ is *fixed*, or is a *principal ultrafilter*, if $\bigcap_{U \in \mathscr{U}} U \neq \varnothing$; otherwise it is *free*.

(i) Show that if $\mathscr{U}$ is fixed, then it has the form $\{U \subseteq X : x \in U\}$, for some $x \in X$. Deduce that it has a basis consisting of one set.

(ii) Show that if $\mathscr{U}$ is free, then it does not have a *countable* basis.

5. (i) Suppose X is a T_4 space and that $h : X \to \beta X$ is the "canonical embedding" of X into its Stone-Čech compactification. If A and B are disjoint closed subsets of X, prove that $\overline{h(A)}^{\beta X} \cap \overline{h(B)}^{\beta X} = \varnothing$.

(ii) Let E denote the set of even natural numbers. Show that $\overline{h(E)}^{\beta \mathbb{N}}$ is homeomorphic to βE, and hence that $\beta \mathbb{N}$ may be represented as the disjoint union of two homeomorphs of itself. (Here $\mathbb{N}$ has its usual discrete topology.)

6. (i) If $\mathscr{N}_p$ is the neighbourhood filter in $\beta\mathbb{N}$ of a point $p \in \beta\mathbb{N} \setminus h(\mathbb{N})$, and $\mathscr{U} = \{N \cap \mathbb{N} : N \in \mathscr{N}_p\}$, show that $\mathscr{U}$ is a free ultrafilter on $\mathbb{N}$, and deduce that $\mathscr{N}_p$ does not have a countable basis.

(ii) Show that every compact metric space is a continuous image of $\beta\mathbb{N}$.

(iii) Is $\beta\mathbb{N}$ metrisable? (If so, prove it; if not, show why not.)