Analytic Topology: Problem sheet 4

1. (i) Let E be that subset of $\mathbb{R}^2$ defined by

$$E = \left[\bigcup_{n=1}^{\infty} \left([0,1] \times \left\{\frac{1}{n}\right\}\right)\right] \cup \{(0,0)\} \cup \{(1,0)\}.$$

Identify all the components and all the quasi-components of E. (E has the topology induced by the usual topology on $\mathbb{R}^2$.)

(ii) Let F be that subset of $\mathbb{R}^2$ defined by

$$F = \left\{(x,y) : y = \sin\frac{\pi}{x},\ x \in (0,1]\right\} \cup A, \text{ where } A = \{(0,y) : -1 \leq y \leq 1\},$$

with the topology induced by the usual topology on $\mathbb{R}^2$. Prove that F is a connected, but not locally connected, topological space.

2. (i) Suppose that A and B are closed subsets of a compact Hausdorff space X such that no component of X meets both A and B. Show that there exists a clopen subset C of X such that $A \subseteq C \subseteq X \setminus B$.

(ii) Suppose that D is a non-empty compact proper subset of a connected Hausdorff space Y. Show that every component of D meets the boundary of D.

3. (i) Prove that a paracompact regular space is normal.

(ii) Prove that a closed subspace of a paracompact space is paracompact.

4. Prove that the following three conditions on a regular space X are equivalent:

(i) Every open covering of X has a locally finite open refinement (that is, X is paracompact);

(ii) Every open covering of X has a locally finite refinement (the elements of the refinement not being necessarily open or closed),

(iii) Every open covering of X has a locally finite closed refinement (the elements of the refinement being closed sets).

[For (iii)$\Rightarrow$(i), it may help to do the following: Let $\mathscr{V}$ be an open cover witnessing local finiteness of a refinement. Now find a locally finite closed refinement of $\mathscr{V}$, and try to use it to build a locally finite open refinement of the original cover. Or, look it up in Willard!]

5. Let $\mathbb{B}$ be a Boolean algebra. Prove that the following hold, for all $a, b, c \in \mathbb{B}$:

(i) $a \leq b$ if and only if $a \vee b = b$,

(ii) $b = \neg a$ if and only if $b \wedge a = \mathbb{0}$ and $b \vee a = \mathbb{1}$,

(iii) $\neg(a \wedge b) = (\neg a) \vee (\neg b)$.

6. (Optional) Let $\mathbb{A}$, $\mathbb{B}$ be Boolean algebras, and let $\phi : \mathbb{A} \to \mathbb{B}$ be a homomorphism. Prove that $\mathscr{S}\phi$ is one-to-one if and only if ϕ is onto, and is onto if and only if ϕ is one-to-one.

7. Let $\mathbb{B}$ be a Boolean algebra. Prove that $\eta_{\mathbb{B}} : \mathbb{B} \to \mathscr{B}\mathscr{S}\mathbb{B}$ is an isomorphism.

8. (i) What is the Stone dual of the one-point topological space?

(ii) Let $*$ be the one-point topological space. Let X be a compact Hausdorff zero-dimensional space. One can think of a point of X as being the range of a function from $*$ to X. Let $f : * \to X$, and let p be the unique point in the range of f. (f is automatically continuous. Why?) Describe $\mathscr{B}f$ completely in terms of p.

(iii) A *product* of two Boolean algebras $\mathbb{A}$ and $\mathbb{B}$ is the cartesian product with pointwise operations, that is, $\mathbb{1}_{\mathbb{A}\times\mathbb{B}} = (\mathbb{1}_{\mathbb{A}}, \mathbb{1}_{\mathbb{B}})$, $\mathbb{O}_{\mathbb{A}\times\mathbb{B}} = (\mathbb{O}_{\mathbb{A}}, \mathbb{1}_{\mathbb{B}})$, $\neg(a,b) = (\neg a, \neg b)$, $(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2)$, $(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee a_2, b_1 \vee b_2)$, and $(a_1, b_1) \leq (a_2, b_2)$ if and only if $a_1 \leq b_1$ and $a_2 \leq b_2$. Suppose that Z is the disjoint union of two compact zero-dimensional Hausdorff spaces X and Y. Show that $\mathscr{B}Z$ is isomorphic to $\mathscr{B}X \times \mathscr{B}Y$.