C8.6 Limit Theorems and Large Deviations in Probability

Sheet 2 HT 2020

1. Let (S,d) be a complete metric space, and $x \in \mathbb{C}(S)$ the space of all continuous paths in S. Let $\lambda > 0$ and $T > 0$. Define $\tau_0 = 0$, and τ_n inductively by

$$\tau_n = \inf\left\{t \geq \tau_{n-1} : d(x(t), x(\tau_{n-1})) \geq \frac{\lambda}{4}\right\}$$

for $n = 1, 2, \ldots$, where the convention that $\inf \emptyset = \infty$ is applied. The entropy number is defined to be

$$N(x,\lambda,T) = \min\{n : \tau_{n+1} > T\}$$

and the minimal gap

$$\delta(x,\lambda,T) = \min\{\tau_n - \tau_{n-1} : 1 \leq n \leq N(x,\lambda,T)\}.$$

Show that for every $0 < \delta \leq \delta(x,\lambda,T)$ we have

$$\omega(x,\delta,T) \leq \lambda.$$

2. Let $T > 0$ and $x : [0,T] \to \mathbb{R}^d$ be a continuous path in $\mathbb{R}^d$. For $0 \leq s < t \leq T$ we use $x_{s,t}$ to denote the increment $x(t) - x(s)$. For each $m = 1, 2, \cdots$, $t_j^m = \frac{j}{2^m}T$ are called dyadic points of degree m, where $j = 0, 1, \cdots, 2^m$. The collection of all these dyadic points of degree m is denoted by D_m, and $D = \bigcup_{m=1}^{\infty} D_m$. Then D is countable and dense in $[0,T]$.

(i) Let $0 \leq s < t \leq T$. There is a sequence of dyadic points $s_i \in D$, such that for each i, there is $m \in \mathbb{N}$ and some $j = 0, \cdots, 2^m$ (so m and j depend on i) such that if $s_i \neq s_{i+1}$ then $s_i = t_j^m$, $s_{i+1} = t_{j+1}^m$ and

$$[s,t) = \bigcup_{i=-\infty}^{\infty} [s_i, s_{i+1})$$

is an disjoint union, where for each m there are at most two different i_1 and i_2 appearing in the previous union such that $s_{i_1}, s_{i_{1+1}}, s_{i_2}, s_{i_{2+1}}$ belong to the same D_m. Moreover $s_i \uparrow t$ as $i \to \infty$ and $s_i \downarrow s$ as $i = -\infty$. Therefore $x_{s,t} = \sum_{i=-\infty}^{\infty} x_{s_i,s_{i+1}}$.

[Hint. The idea is to select at each step a dyadic interval $[t_j^m, t_{j+1}^m]$ lying inside $[s,t]$ with the maximum length (i.e. the least m). For example, as the first step, choose the least m_0 and some j_{m_0}, such that $[t_{j_{m_0}}^{m_0}, t_{j_{m_0}+1}^{m_0}] \subset [s,t]$ and

$$t_{j_{m_0}}^{m_0} - s < \frac{1}{2^{m_0}}T, \quad t - t_{j_{m_0}+1}^{m_0} < \frac{1}{2^{m_0}}T.$$

Therefore, if $|t-s| = \delta$, then $\frac{1}{2^{m_0}}T \simeq \delta$, that is $m_0 = \left[\log_2 \frac{T}{\delta}\right]$.*]*

(ii) Let $p > 1$ and $\gamma > p - 1$. There is a constant C depending only on p and γ such that

$$\sup_P \sum_l \left|x_{t_l,t_{l+1}}\right|^p \leq C \sum_{m=1}^{\infty} m^\gamma \sum_{j=0}^{2^m-1} \left|x_{t_j^m, t_{j+1}^m}\right|^p$$

where $\sup_P$ runs over all possible finite partitions $0 = t_0 < \cdots < t_k = T$. *[Hint.* Use the notations in (i) to $[s,t] = [t_l, t_{l+1}]$*]*.

(iii) For $m = 1, 2, \cdots$, let $\triangle_m = \{(s,t) : s, t \in D_m$ with $|t-s| = 2^{-m}T\}$, and $K_m = \sup_{(s,t)\in\triangle_m} |x_t - x_s|$. Then

$$K_m \leq \sup_j |x_{t_j^m, t_{j+1}^m}|.$$

(iv) Let $\alpha > 0$ and consider $z_{s,t} = |x_{s,t}|/(t-s)^\alpha$ where $0 \le s < t \le T$. Show that

$$\sup_{s\neq t} \frac{|x_{s,t}|}{|t-s|^\alpha} \le 2^{(\alpha+1)} \sum_{m=0}^{\infty} 2^{m\alpha} K_m \le 2^{(\alpha+1)} \sum_{m=0}^{\infty} 2^{m\alpha} \max_{j=0,\cdots,2^m-1} |x_{t_j^m, t_{j+1}^m}|.$$

[*Hint.* Using (i) $x_{s,t} = \sum_{i=-\infty}^{\infty} x_{s_i, s_{i+1}}$, and the fact that if $|t-s| \le 2^{-m}T$ then $s_i \in \bigcup_{n\ge m} D_n$.]
(v) By using (ii) to show that if $B = (B_t)_{t\ge 0}$ is a Brownian motion in $\mathbb{R}^d$, then

$$\mathbb{E}\left[\sup_P \sum_l |B_{t_{l+1}} - B_{t_l}|^p\right] < \infty \qquad \text{for any } p > 2.$$

3. Let (X_t) be a continuous stochastic process with values in $\mathbb{R}^d$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $T > 0$. Suppose there are $p > 0$, $\varepsilon > 0$ and a constant $C > 0$ such that

$$\mathbb{E}\left[|X_t - X_s|^p\right] \le C|t-s|^{1+\varepsilon} \qquad \text{for all } 0 \le s,t \le T.$$

Show that, if $\alpha < \frac{\varepsilon}{p}$ then

$$\mathbb{E}\left[\left|\sup_{s,t\le T, s\neq t} \frac{|X_t - X_s|}{|t-s|^\alpha}\right|^p\right] < \infty.$$

Hence prove that Brownian motion is α-Hölder continuous for $\alpha < 1/2$. [*Hint.* Q2 (iii) may be helpful.]

4. (*Kolmogorov's criterion for compactness*) Let $(X^{(n)})$ be a sequence of $\mathbb{R}^d$-valued continuous stochastic processes, such that
(i) the family of initial distributions of $X_0^{(n)}$ is tight, and
(ii) there are constants $p > 0$, $\varepsilon > 0$ and $C > 0$ such that

$$\mathbb{E}\left[\left|X_t^{(n)} - X_s^{(n)}\right|^p\right] \le C|t-s||^{1+\varepsilon}$$

for all $s, t \ge 0$ and for $n = 1, 2, \cdots$.
Then the laws of $(X^{(n)})$ are relatively compact.

5. Let (S, d) be a metric space. For each $n = 1, 2, \ldots$, $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_j^{(n)} < \cdots$, where $t_j^{(n)} \uparrow \infty$ as $j \to \infty$, such that $t_j = \lim_{n\to\infty} t_j^{(n)}$, $t_j < t_{j+1}$ and $t_k \uparrow \infty$. Let $w^{(n)}(t) = a_{n,j}$ for $t \in [t_j^{(n)}, t_{j+1}^{(n)})$ for each j, so that $w^{(n)} \in \mathbb{D}(S)$ and each $w^{(n)}$ is a step function (with countable many jumps). Suppose $a_j = \lim_{n\to\infty} a_{n,j}$ exists for each $j = 0, 1, \ldots$. Define $w(t) = a_j$ if $t \in [t_j, t_{j+1})$ for some j. Show that $w \in \mathbb{D}(S)$ and $D(w^{(n)}, w) \to \infty$ as $n \to \infty$.
[*Hint.* Let $T > 0$ and choose k such that $t_k \le T < t_{k+1}$. Let

$$\sigma^{(n)}(t) = \begin{cases} t_j + \frac{t_{j+1} - t_j}{t_{j+1}^{(n)} - t_j^{(n)}}(t - t_j^{(n)}), & \text{if } j \le k-1, t \in [t_j^{(n)}, t_{j+1}^{(n)}), \\ t_k + (t - t_k^{(n)}), & \text{if } t \ge t_k^{(n)}. \end{cases}$$

Show that $\sigma^{(n)} \in \Lambda$ for each n and show that for n large enough it holds that

$$\sup_{t\le T} |\sigma^{(n)}(t) - t| \le \max_{1\le j\le k} |t_j^{(n)} - t_j|$$

and

$$\sup_{t\le T} d(w^{(n)}(t), w(\sigma^{(n)}(t))) \le \max_{1\le j\le k} d(a_{n,j}, a_j).]$$