C8.6 Limit Theorems and Large Deviations in Probability

Sheet 1 HT 2020

1. Let $\mathbb{C}(\mathbb{R}^d)$ be the space of all continuous paths in $\mathbb{R}^d$ equipped with the metric ρ given in the lecture notes, page 3, so that $(\mathbb{C}(\mathbb{R}^d), \rho)$ is a Polish space. Let $\{X_t : t \geq 0\}$ be the coordinate process defined as $X_t(w) = w(t)$ for every $w \in \mathbb{C}(\mathbb{R}^d)$, where $t \geq 0$. Show that $\mathscr{B}(\mathbb{C}(\mathbb{R}^d)) = \sigma\{X_t : t \geq 0\}$.

2. Let (E, ρ) be a metric space. If $A \subset E$ is a subset, then $\rho(x, A) = \inf_{y \in A} \rho(x, y)$.

(a) Show that the closure $\bar{A} = \{x \in E : \rho(x, A) = 0\}$, hence show that A is closed then $\rho(x, A) = 0$ if and only if $x \in A$.

(b) Let F be a closed subset of E and define

$$f_n(x) = \left(\frac{1}{1 + \rho(x, F)} \right)^n$$

for each $n = 1, 2, \ldots$, and $x \in E$. Show that $f_n \in U_\rho(E)$, i.e. f_n is uniformly continuous on E, and show that $f_n \downarrow 1_F$.

(c) Suppose $f \in C_b(E)$ and choose $M > 0$ such that $-M < f(x) < M$ for every $x \in E$. Let μ be a finite measure on $(E, \mathscr{B}(E))$. Show that for every $\varepsilon > 0$ there is a partition

$$-M = a_0 < a_1 < \cdots < a_k < a_{k+1} = M$$

such that $a_j - a_{j-1} < \varepsilon$, $\mu[f = a_j] = 0$ for $j = 1, \ldots, k$, and

$$\sup_{x \in E} \left| f(x) - \sum_j a_j 1_{B_j}(x) \right| < \varepsilon,$$

where $B_j = \{x : a_{j-1} \leq f(x) < a_j\}$ and $j = 1, \cdots, k+1$.

3. Let (E, ρ) be a metric space. Let $U_\rho(E)$ denote the space of all bounded and uniformly continuous functions on E. Let P_n, P be probability measures on $(E, \mathscr{B}(E))$, where $n = 1, 2, \ldots$. Show that $P_n \to P$ weakly, if and only if one of the followings (and therefore all of them) holds:

(i) For every $f \in U_\rho(E)$,

$$\int_E f dP_n \to \int_E f dP;$$

(ii) For every closed subset F,

$$\limsup_{n\to\infty} P_n(F) \le P(F);$$

(iii) For every open subset G,

$$\liminf_{n\to\infty} P_n(G) \ge P(G);$$

(iv) For every $B \in \mathscr{B}(E)$ which is continuous with respect to P, i.e. $P(\partial B) = 0$, where $\partial B = \bar{B} \setminus B^0$ and $B^0 = \cup\{U : \text{ open and } U \subset B\}$,

$$\lim_{n\to\infty} P_n(B) = P(B).$$

Hint. First show that $P_n \to P$ weakly by using definition implies (i). Then show (ii) and (iii) are equivalent. Next show that (ii) and (ii) together imply (iv). Using Q2 (b) show that (i) yields (ii) (so (ii) as well) and therefore (iv) as well. Finally, by using Q2 (c), prove that (iv) implies that $P_n \to P$ weakly.

4. Let (E_i, ρ_i) be two metric spaces, where $i = 1, 2$. Let P_n and P $(n = 1, 2, \ldots)$ be probability measures on $(E_1, \mathscr{B}(E_1))$ and $X : E_1 \to E_2$ be measurable. Let $Q_n = P_n \circ X^{-1}$ be the pull-back measure of P_n under X, that is

$$Q_n(B) = P_n\left(X^{-1}(B)\right) \quad \text{for every } B \in \mathscr{B}(E_2),$$

and $Q = P \circ X^{-1}$. Assume that the following two conditions are satisfied:

(i) $P_n \to P$ weakly as $n \to \infty$, and

(ii) X is P-continuous in the sense that $P(D(X)) = 0$, where $D(X)$ is the subset of discontinuous points of X.

Prove that $Q_n \to Q$ weakly as $n \to \infty$.

5. Let P_n and P be probability measures on $(E, \mathscr{B}(E))$, where (E, ρ) is a metric space and $n = 1, 2, \ldots$. Suppose $P_n \to P$ weakly as $n \to \infty$. Show that

$$\lim_{n\to\infty} \int_E f(x) P_n(dx) = \int_E f(x) P(dx)$$

for every bounded, Borel measurable and P-continuous function f.