C8.5 Introduction to SLE

Sheet 2

Problem 1.

Scaling property of Loewner Evolution. Let K_t be a growing hull with the standard capacity parametrization and let $u(t)$ be the corresponding driving function. Let $\tilde{K}_s = \lambda K_{s/\lambda^2}$. Show that $\tilde{K}_s$ has the standard capacity parametrization and compute its driving function.

Problem 2.

Loewner Evolution of inverse map. Let g_t be the solution of Loewner Evolution with driving function u_t. Show that the inverse map $f_t(z) = g_t^{-1}(z)$ satisfies the equation

$$\partial_t f_t(z) = -f_t'(z)\frac{2}{z - u_t}, \qquad z \in \mathbb{H}. \tag{1}$$

Problem 3.

Suppose $0 < \alpha < 1$.

(1) Consider
$$f(z) = (z + \alpha)^{1-\alpha}(z + \alpha - 1)^{\alpha},$$
where the branches of powers are chosen so that f is positive for real $z > 1 - \alpha$. Show that $f = f_K$ where K is the interval $[0, \alpha^{\alpha}(1-\alpha)^{1-\alpha}e^{i\alpha\pi}]$. Show that it maps $z = 1 - 2\alpha$ to the endpoint of K.

(2) Let γ be a straight interval in $\mathbb{H}$ growing from the origin and forming the angle $\pi\alpha$ with the positive real line. Parametrize it by capacity, write the corresponding maps f_t and verify that they satisfy the Loewner Differential equation (1).

Problem 4.

Solve the radial Loewner evolution driven by the family of measures μ_t, there μ_t is the uniform measure on the unit circle, namely $\mu_t(\mathrm{d}\theta) = \mathrm{d}\theta/2\pi$.

Problem 5.

Let K_t be a growing family of $\mathbb{H}$-hulls and $u(t)$ be the corresponding driving function. Let $\tilde{u}(t) = -u(t)$ and $\tilde{K}_t$ be the corresponding family of hulls. How $\tilde{K}_t$ is related to K_t?

Problem 6.

Show that SLE(κ) is scale invariant. Namely that if $\gamma(t)$ is an SLE(κ) curve, then $\tilde{\gamma}(t) = \lambda\gamma(t/\lambda^2)$ has the same distribution as $\gamma(t)$.