C8.5 Introduction to SLE

Sheet 1

Problem 1.

Recall that a continuous complex martingale Z_t is called a conformal martingale if $\langle Z, Z\rangle_t = 0$. Verify that the complex Brownian motion is a conformal martingale.

Show that if f is an analytic function and Z_t is a conformal martingale, then

$$f(Z_t) - f(Z_0) = \int_0^t f'(Z_s)\mathrm{d}Z_s.$$

Problem 2.

An explicit example of $\mathbb{H}$*-version of the Carathéodory convergence.* For $0 \leq \theta \leq \pi$ let $\Omega = \Omega_\theta = \mathbb{H} \setminus \gamma([0,\theta])$, where $\gamma(t) = e^{it}$. Find the thermodynamically normalized map $g_\theta : \Omega \to \mathbb{H}$ and find the half-plane capacity of $K_\theta = \gamma([0,\theta])$.

Verify explicitly that as $\theta \to \pi$ the domains Ω_θ converge to $\mathbb{H} \setminus \mathbb{D}$ and that $g_\theta(z) \to z + 1/z$ uniformly on compact subsets.

Problem 3.

Proof of the Schwarz formula in $\mathbb{H}$. Use the Cauchy formula to prove the Schwarz formula in the upper half-plane. Namely, show that

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Im} f(t)}{t-z}\mathrm{d}t, \quad z \in \mathbb{H}.$$

You may assume that the function decays sufficiently fast at infinity.

Problem 4.

Existence of the mapping-out function. Let K be a half-plane hull.

(1) Prove that there is a conformal map $g : \mathbb{H} \setminus K \to \mathbb{H}$ such that $g(z) \to \infty$ as $z \to \infty$. [You may assume the standard version of the Riemann mapping theorem. You may assume that all relevant maps are continuous up to the boundary.]

(2) Prove that there is $R > 0$ such that this conformal map g satisfies

$$g(z) = a_1 z + a_0 + \sum_{n=1}^{\infty} b_n z^{-n}, \qquad |z| > R.$$

(3) Prove that there is a conformal map $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ as in (i) such that

$$\lim_{z\to\infty} g_K(z) - z = 0$$

or, equivalently, the power series as in (iii) is of the form

$$g(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}, \qquad |z| > R.$$

Problem 5.

Basic properties of the half-plane capacity. Let K be a half-plane hull. Show that

(1)

$$\operatorname{hcap}(\lambda K) = \lambda^2 \operatorname{hcap}(K), \quad \lambda > 0$$

(2)

$$\operatorname{hcap}(K + x)\operatorname{hcap}(K), \quad x > 0$$

Problem 6.

Probabilistic interpretation of the half-plane capacity. Let B_t be the standard Brownian motion in the plane started at $z \in \Omega$. Let τ be the first exit time from Ω, namely $\tau = \inf\{t > 0 : B_t \notin \Omega\}$. You can assume without proof that for any function h which is bounded and harmonic in Ω and continuous up to the boundary we have

$$h(z) = \mathbb{E}^z[h(B_\tau)],$$

where $\mathbb{E}^z$ is the expectation with respect to the law of the Brownian motion started from z.

Let K be a half-plane hull and $g = g_K$ be the corresponding conformal map. Let τ be the first exit time from $\mathbb{H} \setminus K$. Show that

(1)
$$\operatorname{Im} z = \operatorname{Im} g(z) + \mathbb{E}^z[\operatorname{Im} B_\tau], \quad \text{for all} z \in \mathbb{H} \setminus K$$

(2)
$$\operatorname{hcap}(K) = \lim_{y \to \infty} y \mathbb{E}^{iy}[\operatorname{Im} B_\tau]$$

Problem 7.

Recall that the map $g(z) = (z^2 + t^2)^{1/2}$ is a conformal map from $\mathbb{H} \setminus K = \mathbb{H} \setminus [0, it]$. Verify that $g = g_K$, namely that it has the correct normalization at infinity. Compute $\operatorname{hcap}(K)$.