Problem sheet 1 General Relativity II, Hilary Term 2020

Questions marked with a star have lowest priority to be discussed during class. Any comments or corrections please to Jan.Sbierski@maths.ox.ac.uk.

1) * **(Revision)** Let M be a smooth manifold and recall that $\mathfrak{X}^\infty(M)$ denotes the space of vector fields and $\Omega^1(M)$ the space of covector fields (1-forms). Show that a map

$$\tau : \underbrace{\mathfrak{X}^\infty(M) \times \cdots \times \mathfrak{X}^\infty(M)}_{\ell \text{ times}} \times \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k \text{ times}} \to C^\infty(M)$$

is induced by a (k, ℓ)-tensor field if, and only if, it is multilinear over $C^\infty(M)$.

Similarly a map

$$\tau : \underbrace{\mathfrak{X}^\infty(M) \times \cdots \times \mathfrak{X}^\infty(M)}_{\ell \text{ times}} \times \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k \text{ times}} \to \mathfrak{X}^\infty(M)$$

is induced by a $(k+1, \ell)$-tensor field if, and only if, it is multilinear over $C^\infty(M)$.

[*This is nearly trivial, just be careful with unravelling the definitions.*]

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2) Let $X, Y, Z \in \mathfrak{X}^\infty(M)$ be smooth vector fields on a Lorentzian manifold (M, g). Show that

$$(\nabla\nabla Z)(X, Y) - (\nabla\nabla Z)(Y, X) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

holds, i.e., the left and the right hand side are equivalent definitions of the Riemann curvature tensor $R(X, Y)Z$.

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3) Let (M, g) be a Lorentzian (or Riemannian) manifold, consider a point $p \in M$ and let x^μ be a local coordinate system centred at p (i.e. $x^\mu(p) = 0$). Let $X, Y, Z \in T_pM$ be three tangent vectors. Let $0 < \varepsilon, \delta \ll 1$ be very small. We first parallely propagate Z along the curve γ, i.e., first from 0 along the straight coordinate line to εX^μ and then along the straight coordinate line to $\varepsilon X^\mu + \delta Y^\mu$ to obtain the vector $Z_\gamma(\varepsilon X^\mu + \delta Y^\mu)$. Now, we parallely propagate Z along the curve γ', i.e., first from 0 to δY^μ and then to $\varepsilon X^\mu + \delta Y^\mu$, both along straight coordinate lines. Denote the resulting vector by $Z_{\gamma'}(\varepsilon X^\mu + \delta Y^\mu)$. Show that to leading order in ε and δ we have

$$Z^\rho_\gamma(\varepsilon X^\mu + \delta Y^\mu) - Z^\rho_{\gamma'}(\varepsilon X^\mu + \delta Y^\mu) = -\varepsilon\delta R^\rho{}_{\kappa\alpha\beta}(0) Z^\kappa X^\alpha Y^\beta \ ,$$

thus giving another interpretation of curvature.

[*Hint: Can you justify that the parallel transport of Z from 0 to εX^μ is to leading order $Z^\mu - \varepsilon\Gamma^\mu_{\kappa\sigma}(0)X^\kappa Z^\sigma$?*]

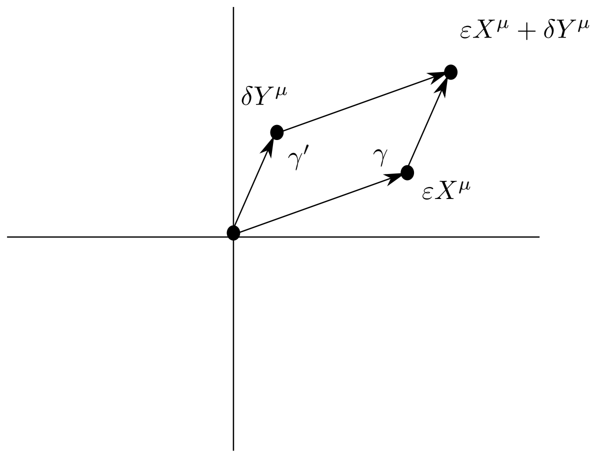

Figure 1: For problem 3

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4) * **(Revision)** Let M be a smooth manifold.

   (a) Let $X, Y, Z \in \mathfrak{X}^\infty(M)$. Prove the Jacobi identity

   $$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \; .$$

   (b) Let g be a Lorentzian (or Riemannian) metric on M and ∇ the associated Levi-Civita connection. Show that the torsion $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ is a $(1,2)$-tensor field. Here $X, Y \in \mathfrak{X}^\infty(M)$.

   (c) Is the Lie bracket $[\cdot,\cdot] : \mathfrak{X}^\infty(M) \times \mathfrak{X}^\infty(M) \to \mathfrak{X}^\infty(M)$ a connection?

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5) Let M be a smooth manifold and let X, Y be smooth vector fields. Show that for a general (k,ℓ)-tensor field T we have the following identity for the Lie derivative:

$$\mathcal{L}_X(\mathcal{L}_Y T) - \mathcal{L}_Y(\mathcal{L}_X T) = \mathcal{L}_{[X,Y]} T \; .$$

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6) (a) Let (M,g) be an n-dimensional Lorentzian (or Riemannian) manifold. Use the equation

   $$\nabla_a \nabla_b K_c = R^d{}_{abc} V_d \qquad (1)$$

   from the lectures to show that the maximum number of linearly independent Killing vector fields on M is $\frac{n(n+1)}{2}$.

   [*Hint: Derive a system of ODEs with initial data given at a point p in M.*]

   (b) Consider 4-dimensional Minkowski spacetime. Write down equation (1) in standard Cartesian coordinates and derive the 10-dimensional space of Killing vectors on Minkowski spacetime. Choose the basis vectors such that they form the infinitesimal generators of translations and Lorentz transformations.

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7) *(**Optional**) Let (M, g) be an n-dimensional *Riemannian* manifold.

(a) Let $p \in M$ and denote with $\Lambda^2 T_p^* M$ the space of all 2-covectors at p that are antisymmetric, i.e., all $\omega \in T_p^* M \otimes T_p^* M$ such that $\omega_{ab} = -\omega_{ba}$. Moreover, for $\alpha, \beta \in T_p^* M$ we define the wedge product $\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha \in \Lambda^2 T_p^* M$.

Let $\alpha^1, \ldots, \alpha^n$ be an orthonormal basis for $T_p^* M$. Show that $\alpha^i \wedge \alpha^j$ with $1 \leq i < j \leq n$ is a basis of $\Lambda^2 T_p^* M$ and thus $\Lambda^2 T_p^* M$ is $\frac{n(n-1)}{2}$ dimensional.

(b) Show that for $\alpha, \beta, \gamma, \delta \in T_p^* M$ the mapping

$$< \alpha \wedge \beta, \gamma \wedge \delta >:= \det \begin{pmatrix} g^{-1}(\alpha, \gamma) & g^{-1}(\alpha, \delta) \\ g^{-1}(\beta, \gamma) & g^{-1}(\beta, \delta) \end{pmatrix}$$

induces an inner product on $\Lambda^2 T_p^* M$ with respect to which $\alpha^i \wedge \alpha^j$, $1 \leq i < j \leq n$, is an orthonormal basis. Also show that for $\omega, \rho \in \Lambda^2 T_p^* M$ one has $< \omega, \rho >= g^{ik} g^{jl} \omega_{ij} \rho_{kl}$.

(c) Consider the Riemann curvature tensor as a $(2,2)$-tensor ${R_{ij}}^{kl}$ and show that it is a self-adjoint linear map $\mathbb{R} : \Lambda^2 T_p^* M \to \Lambda^2 T_p^* M$ with respect to the inner product $< \cdot, \cdot >$.

(d) Show that if (M, g) is connected and has $\frac{n(n+1)}{2}$ linearly independent Killing vector fields that then the Riemannian curvature tensor is of the form $R_{ijkl} = 2C g_{k[i} g_{j]l} = C(g_{ki} g_{jl} - g_{kj} g_{il})$ with C being a constant.

[Hint: You may use that an isometry $\phi : M \to M$ preserves the Riemann tensor, i.e., $(\phi^* R)_{ijkl} = R_{ijkl}$.]

Where in GR I have you encountered such spaces? One can indeed show further that manifolds whose Riemann tensor is of the above form with the same constant C are locally isometric.