

## Problem sheet 2

## General Relativity II, Hilary Term 2020

Questions marked with a star have lowest priority to be discussed during class. Any comments or corrections please to Jan.Sbierski@maths.ox.ac.uk.

1) Let  $(M, g)$  be a *static* spacetime, i.e., there exists a timelike and hypersurface orthogonal Killing vector field  $V$ . Show that one can locally choose coordinates  $\{x_0, x_1, \dots, x_n\}$  such that

- $V = \frac{\partial}{\partial x_0}$
- $g_{\mu\nu}$  is independent of  $x_0$
- $g_{0i} = 0$  for  $i = 1, \dots, n$ .

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2) Consider the vector fields  $V = \partial_x + y\partial_z$  and  $W = \partial_y + x\partial_z$  in  $\mathbb{R}^3$  with the standard Cartesian coordinates  $(x, y, z)$ . Show that  $\text{span}\{V, W\}$  is integrable and construct global coordinates  $(u, v, w)$  on  $\mathbb{R}^3$  such that the integral manifolds are given as level sets of  $w$ .

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3) \* **(Revision)** Let  $(M, g)$  be a Lorentzian manifold. Show that the Bianchi identity  $\nabla_{[a}R_{bc]de} = 0$  implies the divergence freeness of the Einstein tensor, i.e.

$$\nabla^a(R_{ab} - \frac{1}{2}g_{ab}R) = 0.$$

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4) Let  $(M, g)$  be a Lorentzian manifold and let  $T$  be a symmetric 2-covariant tensor field satisfying the conservation equation  $\nabla^a T_{ab} = 0$ . Let  $K$  be a Killing vector field on  $(M, g)$ .

(a) Show that the one-form  $J(\cdot) = T(\cdot, K)$  is divergence free, i.e.  $\nabla^a J_a = 0$ .

(b) Let now  $(M, g)$  be 3+1 dimensional Minkowski spacetime with canonical  $\{t, x, y, z\}$  coordinates and assume that  $T$  vanishes for  $r = \sqrt{x^2 + y^2 + z^2}$  large enough. Show that for each Killing vector field  $K$  the corresponding charge

$$Q[K] := \int_{t=t_0} J_0(t_0, x, y, z) dx dy dz$$

is conserved, i.e., independent of time.

Express the conserved charges corresponding to the 10 linearly independent Killing vector fields found in problem 6b) from the first problem sheet in terms of the components of the stress-energy tensor. Give a physical interpretation of each of these charges.

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5) \* Let  $(M, g)$  be a Lorentzian manifold.

(a) Let  $\phi \in C^\infty(M)$ . We say that  $\phi$  satisfies the wave equation iff  $\square_g \phi := \nabla^a \nabla_a \phi = 0$ . Define the symmetric 2-covariant tensor field  $T$  associated to  $\phi$  by

$$T(X, Y) := (X\phi)(Y\phi) - \frac{1}{2}g(X, Y) \cdot g^{-1}(d\phi, d\phi).$$

Show that  $T$  is divergence free, i.e.  $\nabla^a T_{ab} = 0$  if, and only if,  $\phi$  satisfies the wave equation. Find the expression for  $T_{00}$  in terms of  $\phi$  in the special case that  $(M, g)$  is the Minkowski spacetime.

(b) Let  $F$  be a two-form and define the associated symmetric 2-covariant tensor field

$$T_{ab} = \frac{1}{4\pi} (F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd}).$$

Show that  $T$  satisfies  $\nabla^a T_{ab} = 0$  if  $F$  satisfies the Maxwell equations  $dF = 0$  and  $\nabla^a F_{ab} = 0$  (the other direction does in general not hold true). In the special case that  $(M, g)$  is the Minkowski spacetime find the expression for  $T_{00}$  in terms of the electric field  $E_i = -F_{0i}$  and the magnetic field  $B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk}$ .

6) Let  $(M, g)$  be the 3 + 1-dimensional Minkowski spacetime with Cartesian coordinates  $\{x^0 = t, \underline{x}\}$ ,  $\underline{x} = (x^1, x^2, x^3)$ . Let  $T_{ab}$  be a symmetric 2-covariant tensor field which satisfies the conservation equation  $\partial_a T^{ab} = 0$  and such that for every  $t \in R$  we have that  $T_{ab}(t, \cdot)$  is compactly supported in space (i.e. in  $\mathbb{R}^3$ ). Define

$$P^i(t) := \int_{\mathbb{R}^3} T^{0i}(t, \underline{x}) d\underline{x}, \quad D^i(t) := \int_{\mathbb{R}^3} T^{00}(t, \underline{x}) x^i d\underline{x}, \quad Q^{ij}(t) := \int_{\mathbb{R}^3} T^{00} x^i x^j(t, \underline{x}) d\underline{x},$$

where  $i, j = 1, 2, 3$ . Show that the following holds:

$$\frac{d}{dt} D^i = P^i, \quad \frac{d}{dt} P^i = 0, \quad \frac{d^2}{dt^2} Q^{ij} = 2 \int_{\mathbb{R}^3} T^{ij}(\cdot, \underline{x}) d\underline{x}.$$

7) Let  $(M, g)$  be the 3 + 1-dimensional Minkowski spacetime with Cartesian coordinates  $\{x^0 = t, \underline{x}\}$ ,  $\underline{x} = (x^1, x^2, x^3)$ . Let  $T_{ab}$  be a symmetric 2-covariant tensor field which satisfies the conservation equation  $\partial_a T^{ab} = 0$  and such that for every  $t \in R$  we have that  $T_{ab}(t, \cdot)$  is compactly supported in space (i.e. in  $\mathbb{R}^3$ ). Moreover assume that the stress-energy distribution is stationary, i.e., that  $\partial_t T_{ab} = 0$ . Verify the following identities which are used in the lectures to derive the far field of an isolated gravitational body in the linear approximation.

(a)  $(T^{lm} x^i x^j)_{,lm} = 2T^{ij}$  and show that this implies  $\int_{\mathbb{R}^3} T^{ij}(t, \underline{x}) d\underline{x} = 0$ .

(b) Show that  $\int_{\mathbb{R}^3} T^{0j}(t, \underline{x}) d\underline{x} = 0$ .

(c)  $(T^i{}_j x^j)_{,i} = T^i{}_i$  and show that this implies  $\int_{\mathbb{R}^3} T^i{}_i(t, \underline{x}) d\underline{x} = 0$ .

(d)  $(T^{0i} x^j x^k)_{,i} = T^{0j} x^k + T^{0k} x^j$  and show that this implies  $\int_{\mathbb{R}^3} (T^{0j} x^k + T^{0k} x^j)(t, \underline{x}) d\underline{x} = 0$ .

(e)  $(T^k{}_i x^i x^j - \frac{1}{2} T^{jk} x_i x^i)_{,k} = T^i{}_i x^j$  and show that this implies  $\int_{\mathbb{R}^3} T^i{}_i(t, \underline{x}) x^j d\underline{x} = 0$ .

(f)  $\frac{1}{|\underline{x} - \underline{x}'|_{\mathbb{R}^3}} = \frac{1}{r} + \frac{\underline{x} \cdot \underline{x}'}{r^3} + \mathcal{O}\left(\frac{1}{r^3}\right)$  as a function of  $\underline{x}$ , where  $r^2 = \underline{x} \cdot \underline{x}$ .

Note that the space indices  $i, j, k, l, m, \dots$  run from 1 to 3.