Problem sheet 4 General Relativity II, Hilary Term 2020

Questions marked with a star have lowest priority to be discussed during class. Any comments or corrections please to Jan.Sbierski@maths.ox.ac.uk.

1) Let (M, g) be a Lorentzian manifold and let $\tilde{g} = \Omega^2 g$ be a Lorentzian metric on M that is conformal to g, where Ω is a smooth function with $\Omega(x) \neq 0$ for all $x \in M$.

   (a) Show that the Christoffel symbols $\tilde{\Gamma}^\mu_{\nu\kappa}$ of $\tilde{g}$ are given by

   $$\tilde{\Gamma}^\mu_{\nu\kappa} = \Gamma^\mu_{\nu\kappa} + \partial_\kappa \log\Omega \cdot \delta^\mu{}_\nu + \partial_\nu \log\Omega \cdot \delta^\mu{}_\kappa - \partial_\lambda \log\Omega \cdot g^{\mu\lambda} g_{\nu\kappa} \; .$$

   (b) Let $\gamma : \mathbb{R} \supseteq I \to M$ be a null geodesic with respect to g. Show that it is also a null geodesic with respect to $\tilde{g}$ (but not necessarily affinely parametrised).

   (c) * Give a counterexample to the above for timelike/spacelike geodesics, i.e., give an explicit example of a Lorentzian manifold (M, g) together with a conformal metric $\tilde{g}$ and a timelike/spacelike geodesic $\gamma : I \to \mathbb{R}$ with respect to g which, however, is not a timelike/spacelike geodesic with respect to $\tilde{g}$.

---

2) This question introduces the deSitter spacetime. Consider $4+1$-dimensional Minkowski spacetime, i.e., $\mathbb{R}^5$ with standard Cartesian coordinates $\{v, w, x, y, z\}$ and metric $m = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2$. Let $M \subseteq \mathbb{R}^5$ denote the level set

   $$-v^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2 \; ,$$

   with $\alpha > 0$. Check that this is a timelike hypersurface. Can you sketch it (suppressing some dimensions)?

   By restricting the Minkowski metric to the tangent spaces of M we obtain a Lorentzian metric g on M. In fact, the Ricci curvature of the Lorentzian metric g on M satisfies $R_{\mu\nu} = \frac{3}{\alpha^2} g_{\mu\nu}$. The Einstein equations with cosmological constant Λ read

   $$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \; .$$

   It thus follows that (M, g) is a solution to the Einstein equations with cosmological constant $\Lambda = \frac{3}{\alpha^2}$ and $T_{ab} = 0$. It is called the *deSitter* spacetime.

   We now introduce coordinates on M by $(t, \chi, \theta, \varphi) \overset{\iota}{\mapsto} \big(v, w, x, y, z\big)$ with

   $$\begin{aligned} v &= \alpha \sinh(\frac{t}{\alpha}) \\ w &= \alpha \cosh(\frac{t}{\alpha}) \cos\chi \\ x &= \alpha \cosh(\frac{t}{\alpha}) \sin\chi \cos\theta \\ y &= \alpha \cosh(\frac{t}{\alpha}) \sin\chi \sin\theta \cos\varphi \\ z &= \alpha \cosh(\frac{t}{\alpha}) \sin\chi \sin\theta \sin\varphi \; . \end{aligned}$$

   What is the range of these coordinates? Do they cover all of M? Show that in these coordinates the metric g is given by

   $$g = -dt^2 + \alpha^2 \cosh^2(\frac{t}{\alpha})(d\chi^2 + \sin^2\chi[d\theta^2 + \sin^2\theta \, d\varphi^2]) \; .$$

Draw the hypersurfaces of constant t in your above sketch. What is their topology, how does their geometry change with coordinate time t?

We now construct the Penrose diagram. Choose a new time-coordinate $\lambda(t)$ which satisfies $\frac{d\lambda}{dt} = \frac{1}{\alpha \cosh(\frac{t}{\alpha})}$. Write the metric in the coordinates $(\lambda, \chi, \theta, \varphi)$ and show that the deSitter spacetime is conformal to part of the Einstein static universe. Which boundary surfaces would you call past/future null infinity? Draw the Penrose diagram. Explain why an observer, even if she observes for an infinite time, cannot observe the entire spacetime. How does this compare to the situation in Minkowski spacetime?

---

3) Let (M, g) be a Lorentzian manifold and let $\Sigma \subseteq M$ be a Killing horizon of a Killing vector field T. Show that the surface gravity κ, given by $\nabla_T T|_\Sigma = \kappa T|_\Sigma$, satisfies

$$\kappa^2 = -\frac{1}{2}\big[(\nabla_a T_b)(\nabla^a T^b)\big]|_\Sigma \ .$$

Hint: Use that T is hypersurface orthogonal on Σ.

---

4) This problem guides you through the derivation of the laws of geometric optics in curved spacetime. Let (M, g) be a Lorentzian manifold and $F \in \Omega^2(M)$ a smooth two-form, the Faraday tensor. The source-free Maxwell equations read

$$dF = 0 \qquad \text{and} \qquad \nabla^\mu F_{\mu\nu} = 0 \ . \tag{1}$$

Since $dF = 0$, one can locally[1] find a potential $A \in \Omega^1(M)$ such that $dA = F$.

(a) Show that F satisfies (1) iff A satisfies

$$\nabla^\mu \nabla_\mu A_\nu - \nabla_\nu \nabla^\mu A_\mu - R_{\kappa\nu} A^\kappa = 0 \ . \tag{2}$$

(b) Recall the gauge freedom $\tilde{A}_\mu = A_\mu + \partial_\mu \chi$. Show that any solution A_μ can be put into the Lorentz gauge $\nabla^\mu \tilde{A}_\mu = 0$ by solving an inhomogeneous wave equation for χ (note that $\Box_g \chi := \nabla^\mu \nabla_\mu \chi$ is the wave operator in curved spacetimes).

(c) We now construct approximate solutions of (2) in the Lorentz gauge, i.e., of

$$\nabla^\mu \nabla_\mu A_\nu - R_{\mu\nu} A^\mu = 0 \qquad \text{and} \qquad \nabla^\mu A_\mu = 0 \ . \tag{3}$$

We make the *geometric optics ansatz*

$$A_\nu^{\text{approx}} = \frac{1}{\lambda} a_\nu e^{i\lambda\phi} \ , \tag{4}$$

where $a_\nu \in \Omega^1(M)$, $\phi \in C^\infty(M)$, and $\lambda > 0$ is a large parameter. Compute $\nabla^\mu \nabla_\mu A_\nu^{\text{approx}} - R^\mu{}_\nu A_\mu^{\text{approx}}$ and $\nabla^\mu A_\mu^{\text{approx}}$, group the terms according to their power in λ, and show that the equations (3) are satisfied by (4) up to order $\mathcal{O}(\frac{1}{\lambda})$ iff a_μ and ϕ satisfy

$$\nabla^\mu \phi \cdot a_\mu = 0 \ , \qquad \nabla^\mu \phi \cdot \nabla_\mu \phi = 0 \ , \qquad \nabla^\mu \phi \cdot \nabla_\mu a_\nu + \frac{1}{2} \Box_g \phi \cdot a_\nu = 0 \ . \tag{5}$$

Also infer that if the large parameter λ is large compared to covariant derivatives of a_ν and the spacetime curvature $R_{\mu\nu}$, then (4) with a_ν and ϕ satisfying (5) is a good approximate solution of (3).

[1] Or in fact in any *simply connected* domain – so for example in particular in all of the Schwarzschild spacetime.

(d) The vector $k := (d\phi)^\sharp$ is called the *wave vector*. Can you justify this terminology?

Consider an observer following a timelike curve γ parametrised by proper time who carries with himself an orthonormal basis $\{E_0 = \dot{\gamma}, E_1, \ldots, E_n\}$ of the tangent space which forms his local reference frame. Show that he would interpret the quantity $-\frac{1}{2\pi}\lambda \cdot E_0\phi|_p = -\frac{1}{2\pi}\lambda \cdot g(E_0, k)|_p$ as the frequency of the electromagnetic wave (4) at a point p on his worldline.

(e) The equation $\nabla^\mu\phi \cdot \nabla_\mu\phi = 0$ is known as the *Eikonal equation*. It can be always solved locally. Show that it implies that the wave vector k is null and that it satisfies $\nabla_k k = 0$, i.e., it is propagated affinely along null geodesics.

(f) Let us now decompose the covector amplitude a_ν in (4) as $a_\nu = \alpha \cdot f_\nu$, with the *amplitude* $\alpha \in C^\infty(M)$ and the *polarisation covector* $f_\nu \in \Omega^1(M)$. It follows from the first equation in (5) that $f_\nu k^\nu = 0$, i.e., the polarisation vector is orthogonal to the wave vector, i.e., it must be tangent to the null hypersurfaces $\phi = const.$ Show that to leading order in λ the electric and magnetic fields do not change by adding a multiple of k_ν to f_ν.

Thus, only if f is spacelike do we have a non-vanishing electromagnetic field. Without loss of generality we can thus normalise the polarisation covector by $f_\nu f^\nu = 1$. Show that the third equation in (5) implies the propagation equation

$$\nabla_k \alpha + \frac{1}{2}\nabla^\mu k_\mu \cdot \alpha = 0 \tag{6}$$

for the amplitude along the integral curves of k and that the polarisation covector is parallely propagated along k, i.e.,

$$\nabla_k f = 0\ .$$

Note that (6) in particular implies that if α vanishes on some point on an integral curve of k (which are null geodesics by $\nabla_k k = 0$), then it vanishes along the whole curve. *This makes precise in which sense and under what conditions 'light propagates along null geodesics in general relativity'.*

(g) Consider now the Schwarzschild spacetime with an observer γ_A following a timelike curve of constant $r = r_A > 2M$, $\theta = \theta_0$, $\varphi = \varphi_0$ and another observer γ_B following a timelike curve of constant $r = r_B > r_A$, $\theta = \theta_0$, $\varphi = \varphi_0$. Make precise, using the laws of geometric optics derived in this exercise, that a high-frequency light signal of frequency f_A as measured by observer A, sent from A to B, arrives red-shifted at observer B with a frequency $f_B = \sqrt{\frac{1-\frac{2M}{r_A}}{1-\frac{2M}{r_B}}} f_A$.

---

5) Let $M = \mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2$ with the standard $\{t, r, \theta, \varphi\}$ coordinates where $r_+ = M + \sqrt{M^2 - a^2}$, $M > 0$, and $0 < a < M$. We define the Kerr metric g on M by

$$\begin{aligned} g = -(1 - \frac{2Mr}{\rho^2})\, dt^2 - \frac{2Mra\sin^2\theta}{\rho^2}\,(dt \otimes d\varphi + d\varphi \otimes dt) + \frac{\rho^2}{\Delta}\, dr^2 \\ + \rho^2\, d\theta^2 + \Big(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\Big)\sin^2\theta\, d\varphi^2\ , \end{aligned} \tag{7}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Consider a stationary observer A with velocity $u(\partial_t + \Omega\partial_\varphi)$ at some value of $r_0 \in (r_+, \infty)$ and some value of $\theta_0 \in (0, \pi)$, where $u > 0$ is chosen such that the velocity is normalised. Show that Ω corresponds to the angular frequency of A *as seen by an observer B with velocity ∂_t at infinity who is at rest with respect to the asymptotic Lorentz frame.*

Thus, an observer with $\Omega = 0$ appears static from infinity 'with respect to the fixed stars'.

(Hint: The movement of A as seen by B depends on the null geodesics connecting A's worldline with B's. Use the symmetries of the Kerr spacetime to answer this question without actually computing the null geodesics.)

---

6) * Show that the Kerr metric (7) from the last problem reduces to

   (a) the Schwarzschild metric for $a = 0$

   (b) the Minkowski metric in spheroidal coordinates for $M = 0$, but $a \neq 0$. Here, the spheroidal coordinates in Minkowski spacetime are given by $x = (r^2 + a^2)^{\frac{1}{2}} \sin\theta \cos\varphi$, $y = (r^2 + a^2)^{\frac{1}{2}} \sin\theta \sin\varphi$, $z = r\cos\theta$. Note that the surfaces $r = const$ are spheroids $\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$.