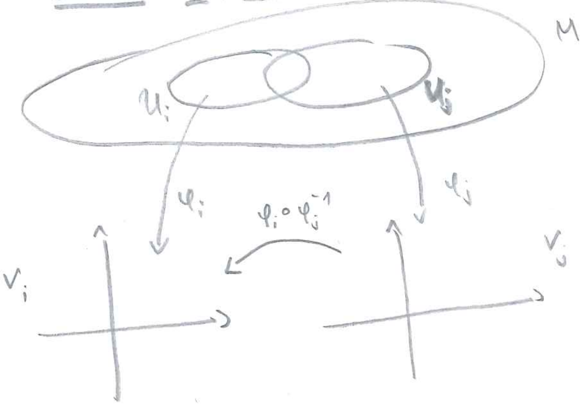
I: Review of GR I & Mathematical Preliminaries

- smooth Manifold M of dimension n $:\Leftrightarrow$ topological space M (2^{nd} countable, Hausdorff) together with collection $(\mathcal{U}_i, \varphi_i)$ of homeomorphisms $\varphi_i : \mathcal{U}_i \to V_i$, $\mathcal{U}_i \subset M$, $V_i \subset \mathbb{R}^n$, $\mathcal{U}_i, V_i$ open, s.t.

  i) every point $p \in M$ is contained in some $\mathcal{U}_i$

  ii) if $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$, then $\varphi_i \circ \varphi_j^{-1} : \varphi_j(\mathcal{U}_i \cap \mathcal{U}_j) \to \varphi_i(\mathcal{U}_i \cap \mathcal{U}_j)$ is a smooth diffeomorphism. $\hat{=}$ smooth map with smooth inverse.



Each $(\mathcal{U}_i, \varphi_i)$ is called a chart for the manifold M.

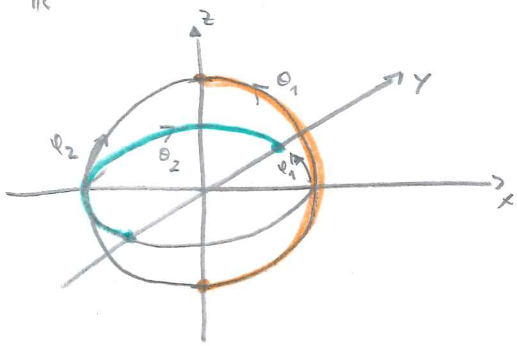
- Example: $S^2 := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$

$\varphi_1^{-1} : \underbrace{(0,\pi)}_{\theta_1} \times \underbrace{(0,2\pi)}_{\varphi_1} \to S^2 \setminus \{y=0, x \geq 0\} =: \mathcal{U}_1$, with $V_1 = (0,\pi)\times(0,2\pi) \subset \mathbb{R}^2$

$x = \cos\varphi_1 \sin\theta_1$
$y = \sin\varphi_1 \sin\theta_1$
$z = \cos\theta_1$

$\Rightarrow \theta_1 = \cos^{-1}(z)$

$\varphi_1 = \arg\left(\frac{x+iy}{\sqrt{x^2+y^2}}\right)$



$\varphi_2^{-1} : \underbrace{(0,\pi)}_{\theta_2} \times \underbrace{(0,2\pi)}_{\varphi_2} \to S^2 \setminus \{z=0, x \leq 0\} =: \mathcal{U}_2$, with $V_2 = (0,\pi)\times(0,2\pi)$

$x = -\cos\varphi_2 \sin\theta_2$
$y = \cos\theta_2$
$z = \sin\varphi_2 \sin\theta_2$

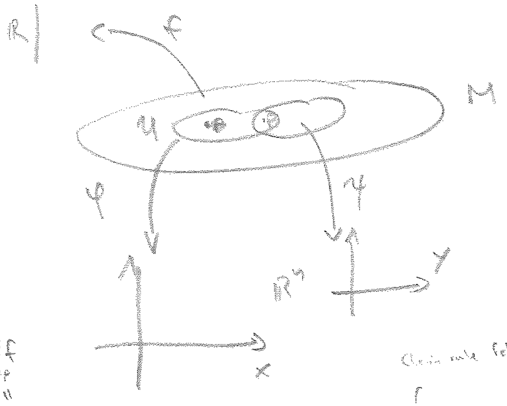
$$\varphi_1 \circ \varphi_2^{-1}(\theta_2, \varphi_2) = \left(\cos^{-1}(\sin\varphi_2 \sin\theta_2),\ \arg\left(\frac{-\cos\varphi_2 \sin\theta_2 + i\cos\theta_2}{\sqrt{\cos^2\varphi_2 \sin^2\theta_2 + \cos^2\theta_2}}\right)\right)$$

- A function $f: M \to \mathbb{R}$ is called smooth iff for all charts $(\mathcal{U}_i, \varphi_i)$ we have that $f \circ \varphi_i^{-1} : \mathcal{U}_i \to \mathbb{R}$ is smooth. $C^\infty(M)$ denotes the space of all smooth functions on M.

• Tangent space T_pM = space of (directional) derivations X at p

$$\downarrow$$

$$\begin{cases} X : C^\infty(M) \longrightarrow \mathbb{R} \quad \text{linear map} \\ \text{Leibniz rule} \quad X(fg) = X(f)\cdot g(p) + f(p)\, X(g) \end{cases}$$



Coordinate basis $\left.\frac{\partial}{\partial x^j}\right|_p (f) := \frac{\partial}{\partial x^j}\left(f \circ \varphi^{-1}\right)(\varphi(p))$

$X \in T_pM$, can write $X = X^\mu \frac{\partial}{\partial x^\mu}$

Chain rule for $\mathbb{R}^n \to \mathbb{R}^n$

$$\left.\frac{\partial}{\partial x^j}\right|_p f = \frac{\partial}{\partial x^j}\left(f\circ\psi^{-1}\circ(\psi\circ\varphi^{-1})\right)(\varphi(p)) = \frac{\partial}{\partial y^i}\left(f\circ\psi^{-1}\right)\psi(p)\cdot\frac{\partial(\psi\circ\varphi^{-1})^i}{\partial x^j}(\varphi(p)) = \left.\frac{\partial}{\partial y^i}\right|_p f \cdot \frac{\partial y^i}{\partial x^j}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x^j} = \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i}} \quad (1)$$

• Cotangent space T_p^*M is dual space of T_pM with coord basis dx^μ, $dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu$

$$\Rightarrow \alpha \in T_p^*M, \quad \alpha = \alpha_\mu dx^\mu \quad \overset{(1)}{\Rightarrow} \quad \boxed{dx^j = \frac{\partial x^j}{\partial y^i} dy^i} \quad (2)$$

Use $\frac{\partial}{\partial y^i} = \frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^k}$ $\Rightarrow$ $dx^j\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial x^j}{\partial y^i}$

• (k,l)-tensor space $T_p^{(k,l)}M$ consists of multilinear maps $T : \underbrace{T_p^*M \times \dots \times T_p^*M}_{k\text{-times}} \times \underbrace{T_pM \times \dots \times T_pM}_{l\text{-times}} \longrightarrow \mathbb{R}$

Note: $(0,1)$-tensor is just a covector and $(1,0)$-tensor is a vector (use $(T_p^*M)^* = T_pM$).

Local basis $\frac{\partial}{\partial x^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_l}$, $1 \le i_1,\dots,i_k,j_1,\dots,j_l \le n$

with $\frac{\partial}{\partial x^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_l}\left(dx^{a_1},\dots,dx^{a_k},\frac{\partial}{\partial x^{b_1}},\dots,\frac{\partial}{\partial x^{b_l}}\right)$

$$= \delta^{a_1}_{i_1}\dots\delta^{a_k}_{i_k}\,\delta^{j_1}_{b_1}\dots\delta^{j_l}_{b_l}$$

So T can be written as

$$T = T^{i_1\dots i_k}{}_{j_1\dots j_l}\frac{\partial}{\partial x^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_l} = \tilde{T}^{a_1\dots a_k}{}_{b_1\dots b_l}\frac{\partial}{\partial y^{a_1}}\otimes\dots\otimes\frac{\partial}{\partial y^{a_k}}\otimes dy^{b_1}\otimes\dots\otimes dy^{b_l}$$

Transformation of components of tensor fields.

with $\tilde{T}^{a_1\dots a_k}{}_{b_1\dots b_l} = T\left(\underbrace{dy^{a_1}}_{=\frac{\partial y^{a_1}}{\partial x^{i_1}}dx^{i_1}},\dots,dy^{a_k},\frac{\partial}{\partial y^{b_1}},\dots,\frac{\partial}{\partial y^{b_l}}\right)$

$$\Rightarrow \tilde{T}^{a_1\dots a_k}{}_{b_1\dots b_l} \overset{(1,2)}{=} T^{i_1\dots i_k}{}_{j_1\dots j_l}\frac{\partial y^{a_1}}{\partial x^{i_1}}\dots\frac{\partial y^{a_k}}{\partial x^{i_k}}\frac{\partial x^{j_1}}{\partial y^{b_1}}\dots\frac{\partial x^{j_l}}{\partial y^{b_l}}$$

- A smooth (k,ℓ)-tensor field T is a map $M \ni p \longmapsto T(p) \in T_p^{(k,\ell)}M \quad \forall p \in M$

s.t. in local coords $\varphi: \underset{M}{\underset{\cap}{\mathcal{U}}} \to \underset{\mathbb{R}^n}{\underset{\cap}{V}}$ the components $T^{i_1 \dots i_k}{}_{j_1 \dots j_\ell} : V \to \mathbb{R}$ are smooth functions.

- $\mathfrak{X}^\infty(M)$ space of smooth vector fields on M $\quad$ $\Omega^1(M)$ space of smooth 1-covector fields (1-forms)

Def: Given $f \in C^\infty(M)$, define $df \in \Omega^1(M)$ by $df(X) = X(f)$ for $X \in \mathfrak{X}^\infty(M)$. df is derivative of f. In coords $df = \partial_\mu f \, dx^\mu$. (easy exercise)

- A Lorentzian metric g on M is a smooth $(0,2)$-tensor field s.t. at every point $p \in M$

$g(p)$ is a non-degenerate, symmetric bilinear form $T_pM \times T_pM \to \mathbb{R}$ of signature $(-,+,\dots,+)$.

$$g_p(X,Y) = 0 \quad \forall Y \in T_pM \Rightarrow X = 0$$

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \quad \rightsquigarrow \quad g^{-1} = g^{\mu\nu} \partial_\mu \otimes \partial_\nu$$

- Tensor operations:
  - Contraction of contravariant & covariant index (trace)

    E.g. $T = T^i{}_{jk} \, \partial_i \otimes dx^j \otimes dx^k$ is $(1,2)$-tensor

    contract i & j $\rightsquigarrow$ $(\operatorname{tr} T) = T^i{}_{ik} \, dx^k$ is $(0,1)$-tensor.

  - Raising / Lowering index with metric

    E.g. $X = X^\mu \partial_\mu$ vector

    $$\rightsquigarrow X^\flat = \underbrace{X^\mu g_{\mu\nu}}_{= X^\flat{}_\nu \, = \, X_\nu} dx^\nu \quad \text{covector}$$

    α covector, $\alpha^\#$ vector.

  - Tensor product.

    E.g. $\alpha = \alpha_\mu dx^\mu$, $\beta = \beta_\nu dx^\nu$ covectors

    $$\rightsquigarrow \alpha \otimes \beta = \underbrace{\alpha_\mu \beta_\nu}_{= (\alpha\otimes\beta)_{\mu\nu}} dx^\mu \otimes dx^\nu \quad (0,2)\text{-tensor}$$

Lie bracket

- Given two vector fields X, Y we define their Lie bracket $[X,Y]f := X(Yf) - Y(Xf)$, $f \in C^\infty(M)$

Clearly $[X,Y]: C^\infty(M) \to C^\infty(M)$ is linear. We show that $[X,Y]$ also satisfies the Leibniz rule, from which it then follows that $[X,Y]$ is a vector field.

could skip this

$$\begin{aligned}
\cdot \; [X,Y](fg) &= XY(fg) - YX(fg) = X\big((Yf)\cdot g + f(Yg)\big) - Y\big((Xf)g + f(Xg)\big) \\
&= (XYf)\cdot g + (Yf)(Xg) + (Xf)(Yg) + f(XYg) - (YXf)\cdot g - (Xf)(Yg) - (Yf)(Xg) - f(YXg) \\
&= ([X,Y]f)\cdot g + f([X,Y]g)
\end{aligned}$$

- In coords, $X = X^\mu \partial_\mu$, $Y = Y^\nu \partial_\nu$ $\rightsquigarrow$

$$\begin{aligned}
[X,Y]f &= X^\mu \partial_\mu (Y^\nu \partial_\nu f) - Y^\nu \partial_\nu (X^\mu \partial_\mu f) \\
&= X^\mu (\partial_\mu Y^\nu) \cdot \partial_\nu f - Y^\nu (\partial_\nu X^\mu)\, \partial_\mu f
\end{aligned}$$

clear from coord exp. that ∂ is a derivation!

$$\rightsquigarrow [X,Y] = [X,Y]^\nu \partial_\nu$$

with $[X,Y]^\nu = X(Y^\nu) - Y(X^\nu)$

~~Example sheet~~ ~~(Jacobi identity)~~

Properties:

- $[X,Y] = -[Y,X]$
- Bilinear: $[X, aY+bZ] = a[X,Y] + b[X,Z]$, $a,b \in \mathbb{R}$
- Jacobi identity: $[V,[W,X]] + [W,[X,V]] + [X,[V,W]] = 0$
- $f \in C^\infty(M)$ $\quad [X, fY] = f[X,Y] + (Xf)\cdot Y$

– end of 1st lecture

Connection: $\nabla : \mathfrak{X}^\infty(M) \times \mathfrak{X}^\infty(M) \to \mathfrak{X}^\infty(M)$ s.t.

- $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ $\quad$ ($f,g \in C^\infty(M)$, $f\in C^\infty(M)$) - linear in first entry)
- $\nabla_X(aY+bZ) = a\nabla_X Y + b\nabla_X Z$, $a,b\in\mathbb{R}$
- $\nabla_X(fY) = f\nabla_X Y + X(f)\cdot Y$ $\quad$ Leibniz rule

$dx^\kappa(\nabla_{\partial_\mu}\partial_\nu) =: \Gamma^\kappa_{\mu\nu}$ $\quad$ Christoffel symbols of the connection

$\Rightarrow \nabla_X Y = (X^\mu \partial_\mu Y^\nu + \Gamma^\nu_{\mu\kappa} X^\mu Y^\kappa)\partial_\nu$, $\nabla_\mu Y^\nu = dx^\nu(\nabla_{\partial_\mu} Y) = \partial_\mu(Y^\nu) + \Gamma^\nu_{\mu\kappa} Y^\kappa$

- Torsion tensor T ((1,2) - tensor field) of connection ∇: $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$.

(Ex: is indeed a tensor)

∇ is called symmetric $:\Leftrightarrow T = 0 \overset{[\partial_\mu,\partial_\nu]=0}{\Leftrightarrow} \Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu}$

Extends to all tensor fields by requiring:

1) $\nabla_X f := Xf$, $f\in C^\infty(M)$

2) $\nabla_X(\alpha\otimes\beta) = (\nabla_X\alpha)\otimes\beta + \alpha\otimes\nabla_X\beta$ $\quad$ Leibniz - rule.

3) ∇_X commutes with all contractions

$$\mathrm{tr}(\nabla_X\alpha) = \nabla_X(\mathrm{tr}\,\alpha).$$

Example: $\alpha\in\Omega^1(M)$, $X,Y\in\mathfrak{X}^\infty(M)$

$$(\nabla_X\alpha)(Y) = \mathrm{tr}(\nabla_X\alpha\otimes Y) \overset{2)}{=} \mathrm{tr}(\nabla_X(\alpha\otimes Y) - \alpha\otimes\nabla_X Y)$$

$$\overset{3,1)}{=} X(\alpha(Y)) - \alpha(\nabla_X Y)$$

$$\Rightarrow \nabla_\mu\alpha_\nu = (\nabla_{\partial_\mu}\alpha)(\partial_\nu) = \partial_\mu(\alpha_\nu) - \alpha_\kappa\Gamma^\kappa_{\mu\nu}$$

$\rightsquigarrow$ General tensor (GR1): E.g. $\nabla_a T^b{}_c = \partial_a(T^b{}_c) + \Gamma^b_{ad}T^d{}_c - \Gamma^d_{ac}T^b{}_d$

drop

$$\nabla_a T^{b_1\dots b_n}{}_{c_1\dots c_m} = \partial_a(T^{b_1\dots b_n}{}_{c_1\dots c_m}) + \Gamma^{b_1}_{ad}T^{d b_2\dots b_n}{}_{c_1\dots c_m} + \dots + \Gamma^{b_n}_{ad}T^{b_1\dots b_{n-1}d}{}_{c_1\dots c_m}$$
$$- \Gamma^d_{ac_1}T^{b_1\dots b_n}{}_{d c_2\dots c_m} - \dots - \Gamma^d_{ac_m}T^{b_1\dots b_n}{}_{c_1\dots c_{m-1}d}$$

<u>Thm</u>: (M,g) Lorentzian manifold, there is exactly one connection ∇ which is

1) metric : $\nabla g \equiv 0$

2) symmetric : Torsion $T \equiv 0$

∇ is called the Levi-Civita connection, in coords $\Gamma^{\mu}{}_{\nu\kappa} = \frac{1}{2} g^{\mu\sigma}(g_{\nu\sigma,\kappa} + g_{\sigma\kappa,\nu} - g_{\nu\kappa,\sigma})$.

$\longrightarrow$ Notions of parallel transport of X along curve γ $\Leftrightarrow$ $\nabla_{\dot\gamma} X = 0$).

Curvature : $X, Y, Z \in \mathfrak{X}^\infty(M)$

Define $R(\cdot,\cdot)\cdot : \mathfrak{X}^\infty(M) \times \mathfrak{X}^\infty(M) \times \mathfrak{X}^\infty(M) \longrightarrow \mathfrak{X}^\infty(M)$

by $R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$

Perhaps better like this
$R(X,Y)Z = (\nabla\nabla Z)(Y,Y)$
$-(\nabla\nabla Z)(Y,X)$

(GR1 $\leadsto$ R is a $(1,3)$-tensor field $R(\partial_\mu, \partial_\nu)\partial_\kappa = R^\sigma{}_{\kappa\mu\nu}\partial_\sigma$

$\Rightarrow$ $R^\sigma{}_{\kappa\mu\nu} = \partial_\mu \Gamma^\sigma_{\nu\kappa} - \partial_\nu \Gamma^\sigma_{\mu\kappa} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\nu\kappa} - \Gamma^\sigma_{\nu\rho}\Gamma^\rho_{\mu\kappa}$

Symmetries
- $R_{\sigma\kappa\mu\nu} = -R_{\kappa\sigma\mu\nu}$
- $R_{\sigma\kappa\mu\nu} = -R_{\sigma\kappa\nu\mu}$
- $R_{\sigma\kappa\mu\nu} = R_{\mu\nu\sigma\kappa}$
- $R_{\sigma\kappa\mu\nu} + R_{\sigma\mu\nu\kappa} + R_{\sigma\nu\kappa\mu} = 0$
- $\nabla_\rho R_{\sigma\kappa\mu\nu} + \nabla_\sigma R_{\kappa\rho\mu\nu} + \nabla_\kappa R_{\rho\sigma\mu\nu} = 0$

- One interpretation of curvature as geodesic deviation (GR1)
- Smaller : Example sheet

$R_{\kappa\nu} = R^\sigma{}_{\kappa\sigma\nu}$ Ricci tensor

$R = g^{\kappa\nu} R_{\kappa\nu}$ scalar curvature

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ Einstein tensor

<u>Einstein equations</u>: $G_{\mu\nu} = 8\pi T_{\mu\nu}$ $(G = c = 1)$

stress-energy tensor of matter

$\boxed{\text{End of recap of GRI}}$

Smooth maps between manifolds.

Let M, N be smooth manifolds. A map $F: M \to N$ is smooth :⟺ it is smooth in coordinates, i.e.

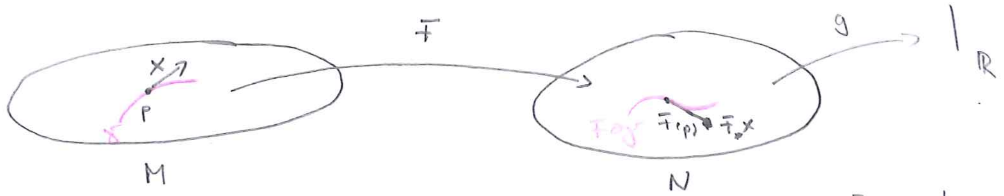
$\varphi: U \to V$ chart for M, $\psi: \tilde{U} \to \tilde{V}$ chart for N with $F(U) \subseteq \tilde{U}$, then (shrink, replace by preimage)

$U \subseteq M$, $V \subseteq \mathbb{R}^n$, $\tilde{U} \subseteq N$, $\tilde{V} \subseteq \mathbb{R}^m$

$$\psi \circ F \circ \varphi^{-1} : V \to \tilde{V} \quad \text{is smooth.}$$

Pushforward: Let $X \in T_pM$. Define pushforward $F_*X \in T_{F(p)}N$ of X via F by

$$(F_*X)(g) := X(\underbrace{g \circ F}_{\in C^\infty(M)}), \quad g \in C^\infty(N)$$



Easy to check that F_*X is indeed a derivation, i.e. $F_*X \in T_{F(p)}N$.

Thus $F: M \to N$ induces map $F_*: T_pM \to T_{F(p)}N$ $\forall p \in M$. (check: F_* is a linear map.)

Let x be local coords around $p \in M$, and y be local coords around $F(p) \in N$.

$$X = \frac{\partial}{\partial x^\mu}, \qquad F_*X = (F_*X)^\nu \frac{\partial}{\partial y^\nu}$$

$$\Rightarrow (F_*X)^\nu\Big|_{F(p)} = (F_*X)(y^\nu) = X(y^\nu \circ F)\Big|_p = \frac{\partial F^\nu}{\partial x^\mu}\Big|_p \qquad \left(\Rightarrow F_*\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial F^\nu}{\partial x^\mu}\Big|_p \frac{\partial}{\partial y^\nu}\right)$$

Thus if $X = X^\mu \frac{\partial}{\partial x^\mu}$ (using linearity), then $F_*X = X^\mu \frac{\partial F^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu}$. ⇝ Thus, F_* also often denoted by DF or dF, the derivative of F. It generalises the derivative of a map $F: \mathbb{R}^n \to \mathbb{R}^m$.

(Examples: 1) $\gamma: \mathbb{R} \to M$ curve. $\frac{\partial}{\partial s}$ tangent vector in $\mathbb{R}$, then $\gamma_*\left(\frac{\partial}{\partial s}\right) = \frac{\partial \gamma^\mu}{\partial s} \frac{\partial}{\partial x^\mu}$ ⟷ tangent vector of curve γ in M.)
2) $f: M \to \mathbb{R}$.

Pullback: Let $\alpha \in T^*_{F(p)}N$. Then define $F^*\alpha \in T^*_pM$, the pullback of α via F, by

$$(F^*\alpha)\Big|_p(X) := \alpha\Big|_{F(p)}(F_*X) \qquad X \in T_pM$$

$$\rightsquigarrow (F^*\alpha)_\mu = (F^*\alpha)\left(\frac{\partial}{\partial x^\mu}\right) = \alpha\Big|_{F(p)}\left(F_*\frac{\partial}{\partial x^\mu}\right) = \alpha\Big|_{F(p)}\left(\frac{\partial F^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right) = \frac{\partial F^\nu}{\partial x^\mu}\Big|_p \alpha_\nu(F(p))$$

Diffeomorphisms and Einstein's hole argument

A smooth map $F: M \to N$ is a diffeomorphism iff F is bijective and $F^{-1}: N \to M$ is also smooth.

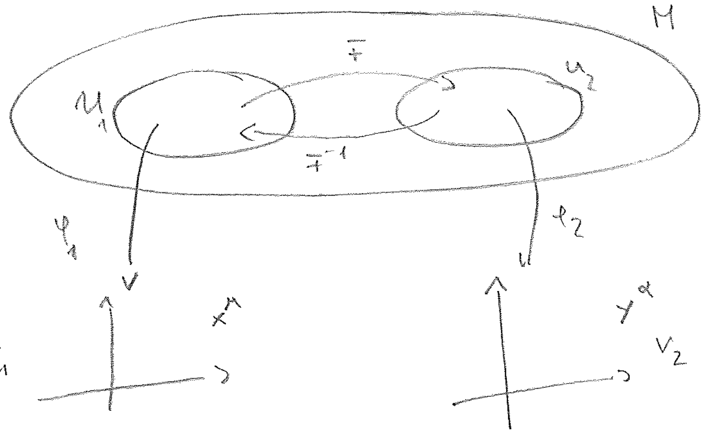
Ex. $\Rightarrow \dim M = \dim N$

(In general, if $F: M \to N$ is smooth, can only pushforward vectors & pullback covectors. If F is diffeo can pullback vector $Y \in T_{F(p)}N$ by pushing forward via F^{-1}. Similarly, push forward forms via pulling back with F^{-1}.)

Let now (M,g) be Lorentzian manifold and $F: M \to M$ diffeomorphism

Can define another Lorentzian metric on M by (F^*g), i.e. $(F^*g)(X,Y) = g(F_*X, F_*Y)$
$X, Y \in T_pM$.

Then $(F^*g)_{\mu\nu}(p) = g_{\alpha\beta}\Big|_{F(p)} \frac{\partial F^\alpha}{\partial x^\mu}\Big|_p \frac{\partial F^\beta}{\partial x^\nu}\Big|_p$. <~> Looks like a coord transformation, since F diffeo.
(thus indeed F^*g is Lorentzian metric (doesn't degenerate))



Two viewpoints on a diffeomorphism

1) Active viewpoint: F^{-1} maps point $F(p)$ to p, F^* maps tensors from $F(p)$ to tensors at p

$g \to$ get new metric F^*g on M

2) Passive viewpoint: Diffeomorphism induces change of coordinates.

x^μ coords on U_1, $F(U_1) = U_2$, y^α coords on U_2

Introduce new chart $\psi = \varphi_1 \circ F^{-1}: U_2 \to V_1$ on U_2

Then transition fct is $\varphi_2 \circ \psi^{-1}: V_1 \to V_2$

$$\varphi_2 \circ \psi^{-1} = \varphi_2 \circ F \circ \varphi_1^{-1}$$

and $$\frac{\partial(\varphi_2 \circ F \circ \varphi_1^{-1})^\alpha}{\partial x^\mu} = \frac{\partial F^\alpha}{\partial x^\mu}$$

→ In this new coord. system on U_2 g has components $g_{\mu\nu} = g_{\alpha\beta}\frac{\partial F^\alpha}{\partial x^\mu}\frac{\partial F^\beta}{\partial x^\nu}$

→ same as above.

→ Although two viewpoints are philosophically very different, computationally they are equivalent.

Assume now $R_{\mu\nu}(g) = 0$. By the above $(F^*g)_{\mu\nu}$ are the same coord. components as of $g_{\mu\nu} = g_{\alpha\beta}\frac{\partial F^\alpha}{\partial x^\mu}\frac{\partial F^\beta}{\partial x^\nu}$. $R_{\mu\nu}$ is tensor, so does not depend on which coord. system you compute it in $\Rightarrow R_{\mu\nu}(F^*g) = 0$.

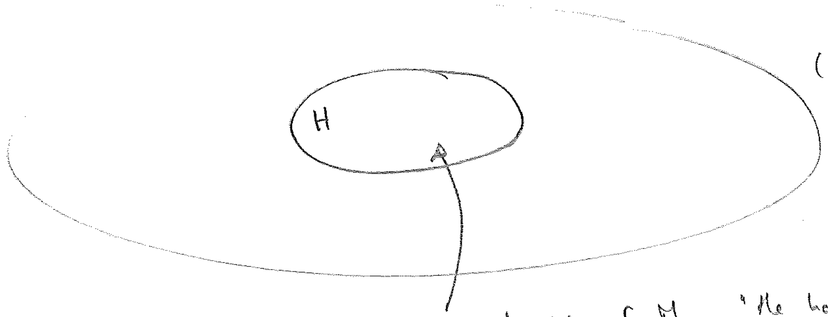
Thus: (M,g) solution of vacuum EE, $Ric(g)=0$. $F: M \to M$ diffeo.
(M, F^*g) also solution of vacuum EE.

(General, if (M, g, ϕ) solves $G_{\mu\nu} = 8\pi T_{\mu\nu}$, then so does $(M, F^*g, F^*\phi)$ for any diffeo $F: M \to M$
matter fields
require $T(F^*\phi) = F^*(T(\phi))$.)

$\Rightarrow$ Einstein equations are <u>diffeomorphism invariant</u>.

Following implications (<u>Einstein's hole argument</u>): <u>If</u> one gives reality to spacetime points $p \in M$, then following problem

(M,g) solution of $R_{\mu\nu}(g)=0$



compact region of M, "the hole".

Choose $F: M \to M$ diffeomorphism s.t. $F|_{M \setminus H} = id_{M \setminus H}$. Get new solution F^*g, which agrees with g on $M \setminus H$ but is different in H.

$\Rightarrow$ Physics in $M \setminus H$ do not determine physics in H. ↯
Einstein equations are not deterministic

Einstein's resolution: Points in M have no physical reality that is independent of g. Only in conjunction with the metric do spacetime points have physical reality.

$\Rightarrow$ (M,g) & (M, F^*g) are physically the same

$\Rightarrow$ Group of diffeomorphism form gauge group for GR.

– end of 1st lecture.

<u>One-parameter groups of diffeomorphism</u>

A one-parameter group of diffeomorphisms on a smooth manifold M is a smooth map

$$F: \mathbb{R} \times M \to M$$
$$(t, x) \longmapsto F_t(x)$$

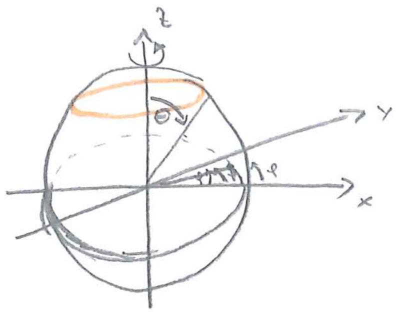
s.t. • $\forall t \in \mathbb{R}$ $F_t : M \to M$ is a diffeomorphism

• $F_0 = id_M$

• $F_s \circ F_t = F_{s+t}$ $\forall s,t \in \mathbb{R}$ (group action $\mathbb{R}$)

Also called: <u>global flow</u>

Example: S^2 with coordinates (θ, φ), i.e. $x = \cos\varphi \sin\theta$
$y = \sin\varphi \sin\theta$
$z = \cos\theta$



$F: \mathbb{R}\times S^2 \to S^2$, $F_t(\theta,\varphi) = (\theta, \varphi + t)$ rotation by angle $\Delta\varphi = t$ around z-axis

For fixed $(\theta,\varphi)\in S^2$, $t \longmapsto F_t(\theta,\varphi) = (\theta, \varphi+t)$ is a curve with tangent

$$\frac{d}{dt}\Big|_{t_0} F_t(\theta,\varphi) = \frac{\partial}{\partial\varphi} \quad \text{at} \quad (\theta,\varphi).$$

Let V be a smooth vector field on M. An <u>integral curve</u> of V is a curve $\gamma: \mathbb{R} \to M$ s.t. $\dot\gamma(s) = V(\gamma(s))$, i.e. s.t. V is tangent to the curve.

<u>Example:</u> a) In the above example if $V = \frac{\partial}{\partial\varphi}$, then $t \mapsto (\theta, \varphi+t)$ are integral curves of V

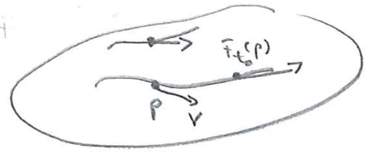
b) $M = (-1,1)^2$, $V = \frac{\partial}{\partial x}$

Need curve $s \mapsto \begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$
with $\dot x(s) = 1$
$\dot y(s) = 0$

Then the integral curves are $s \mapsto (x+s, y)$.
They are not defined for all $t \in \mathbb{R}$.
For simplicity we will however always assume that they are.
(following discussion is slightly more complicated in the general case ...)

Given a 1-parameter group of diffeomorphisms $F: \mathbb{R}\times M \to M$, obtain smooth curves
$\gamma(t) = F_t(p)$ for $p \in M$.
Define a smooth vector field V on M by

$$V(p) := \frac{d}{dt}\Big|_{t=0} F_t(p) = \dot\gamma(0)$$



V is also called the <u>infinitesimal generator</u> of F for the following reason:

$t \longmapsto F_t(p)$ are integral curves of V

<u>Pf:</u>
$$\frac{d}{dt}\Big|_{t=t_0} F_t(p) = \frac{d}{dt}\Big|_{t=t_0} F_{t-t_0}(F_{t_0}(p)) = \frac{d}{ds}\Big|_{s=0} F_s(F_{t_0}(p)) = V(F_{t_0}(p))$$

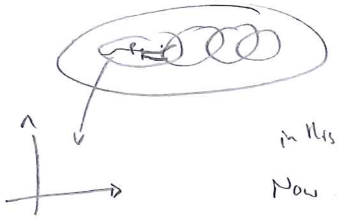
Thm (Relation between 1-param. groups of diffeos & vector fields)

i) Given a 1-param. group of diffeomorphisms $F : \mathbb{R} \times M \to M$, we associate the smooth vector field
$V(x) := \frac{d}{dt}\Big|_{t=0} F_t(x)$ and its integral curves are given by $t \mapsto F_t(x)$.

ii) Given a smooth vector field $V \in \mathfrak{X}^\infty(M)$ (whose integral curves are defined on all of $\mathbb{R}$)
there exists a unique 1-param. group of diffeomorphisms $F : \mathbb{R} \times M \to M$
with $V(x) := \frac{d}{dt}\Big|_{t=0} F_t(x)$.

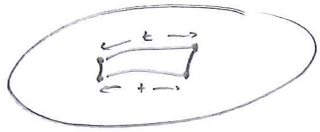
Sketch of Pf: i) already proven

ii) Choose local coords x^μ (around $p \in M$). We then consider the ODE $\dot{\gamma}^\mu(s) = V^\mu(\gamma(s))$ with initial condition $\gamma(0) = p$.

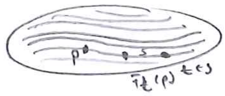


By fundamental theorem on ODEs, there exists a unique solution in this chart which depends smoothly on the initial data.

Now cover M with charts and repeat.



$\Rightarrow$ Obtain a foliation of M by integral curves of V, i.e. a family of integral curves of V s.t. through every $p \in M$ there passes exactly one such integral curve.



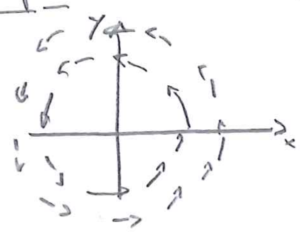
For $t \in \mathbb{R}$, define $F_t : M \to M$ by flowing points for time t along the integral curves. Clearly $F_{t+s}(p) = F_t(F_s(p))$. Inverse of F_t is given by F_{-t}.

By smooth dependence of integral curves on initial data the map

$$F : \mathbb{R} \times M \to M$$

is smooth and thus is the wanted 1-param. group of diffeomorphisms. $\square$

Examples: $M = \mathbb{R}^2 \setminus \{0\}$, $V = x\partial_y - y\partial_x$. Polar coords: $\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}$ $\Rightarrow \frac{\partial}{\partial\varphi} = -y\partial_x + x\partial_y$



$\Rightarrow V = \partial_\varphi$

Integral curves: solve $\begin{array}{l} \dot\varphi(s) = 1 \\ \dot r(s) = 0 \end{array}$ $\Rightarrow s \mapsto (\varphi + s, r)$

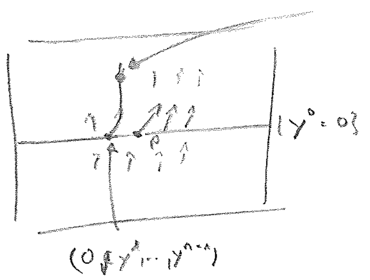
$\Rightarrow F : \mathbb{R} \times M \to M$
$(s, (\varphi, r)) \longmapsto (\varphi + s, r)$. corresponding 1-parameter group.

$\Rightarrow \partial_\varphi$ is infinitesimal generator of rotations around origin.

Proposition (Coordinates adapted to a non-vanishing vector field)

M smooth manifold, $X \in \mathfrak{X}^\infty(M)$, $X(p) \neq 0$. Then there exist smooth coordinates x^μ on a nbhd of p s.t. $X = \frac{\partial}{\partial x^0}$.

Pf: (Take coord. chart $\varphi: \mathcal{U} \to \mathbb{R}^n$ s.t. the coords.) y^ν centred at p ($y(p)=0$) and wlog assume that the hypersurface $\{y^0 = 0\}$ is not tangent to X, i.e. X has non-vanishing ∂_{y_0} component.



Define a map
$$(x^0, x^1, \dots, x^{n-1}) \xmapsto{\psi} F_{x_0}(0, x^1, \dots, x^{n-1})$$
i.e. flow the point $(0, x^1, \dots, x^{n-1})$ on $\{y^0=0\}$ for time x_0 along integral curve of V.

Then by previous result $\frac{\partial}{\partial x^0} = X$. Just need to show that ψ is locally a diffeomorphism.

Easy: $D\psi(0) = \begin{pmatrix} X^0 & 0 & 0 & \dots & 0 \\ \vdots & 1 & & & 0 \\ & & 1 & & \\ X^n & 0 & & \ddots & 1 \end{pmatrix}$.

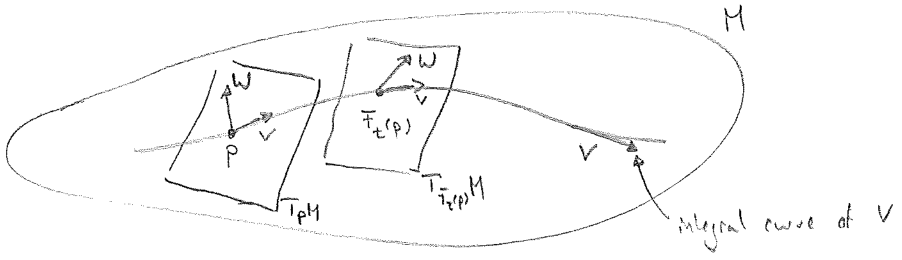
By inverse fct. theorem, in a small enough neighbourhood, ψ^{-1} exists and is smooth.

Then $\psi^{-1} \circ \varphi$ is the wanted coord. chart.

Lie Derivative

V smooth vector field on M, $F: \mathbb{R} \times M \to M$ associated 1-parameter group of diffeomorphisms.

W another smooth vector field on M. Want to take derivative of W along V.



Problem: $W(p)$ & $W(F_t(p))$ lie in different tangent spaces, we can't compare them.

Solution:

Recall F_t is diffeomorphism with inverse $(F_t)^{-1} = F_{-t}$. Have $F_{-t}(F_t(p)) = p$, thus $(F_{-t})_*: T_{F_t(p)}M \to T_pM$ gives identification of tangent spaces depending on the flow lines of V.

Now define the Lie derivative $\mathcal{L}_V W$ of W with respect to V by

$$\mathcal{L}_V W\,(p) := \frac{d}{dt}\Big|_{t=0} (F_{-t})_*\left(W_{F_t(p)}\right) = \frac{d}{dt}\Big|_{t=0} \left(F_t^* W\right)(p) \qquad (*)$$
$$= \lim_{t \to 0} \frac{(F_{-t})_* W_{F_t(p)} - W_p}{t} \in T_pM$$

In coords $(F_{-t})_* W_{F_t(p)} = \underbrace{\frac{\partial F^\mu_{-t}}{\partial x^\nu}(F_t(p)) \cdot W^\nu(F_t(p))}_{\text{smooth expr. in } t, x} \frac{\partial}{\partial x^\mu}$ (recalling from p.6)

2 observations

① Thus $(\mathcal{L}_V W)^\mu = \frac{d}{dt}\Big|_{t=0}\left(\frac{\partial F^\mu_{-t}}{\partial x^\nu}(F_t(p))\, W^\nu(F_t(p))\right)$ depends smoothly on x

$\Rightarrow \mathcal{L}_V W$ is smooth vector field

② Let $R \subseteq M$ be the open set of all $p \in M$ s.t. $V(p) \neq 0$.

By Proposition can find local coords x^μ s.t. $V = \frac{\partial}{\partial x^0}$.

$$\Rightarrow F_t(x^0, \dots, x^{n-1}) = (x^0 + t, x^1, \dots, x^{n-1})$$

$$\Rightarrow \frac{\partial F^\mu_{-t}}{\partial x^\nu} = \delta^\mu{}_\nu$$

$$\boxed{\begin{aligned}\Rightarrow (\mathcal{L}_V W)^\mu_{(x^0,\dots,x^{n-1})} &= \frac{d}{dt}\Big|_{t=0}\left(W^\mu(x^0+t, x^1, \dots, x^{n-1})\right) \\ &= (\partial_{x^0} W^\mu)(x^0, \dots, x^{n-1})\end{aligned}}$$

Or as in Hawking & Ellis, p.32

Also note that $[V, W]^\mu = [\partial_0, W]^\mu = (\partial_0 W)^\mu$.

$\Rightarrow \mathcal{L}_V W = [V, W]$ (first only showed on R, by continuity also on $\bar{R}$. On $M \setminus \bar{R}$, where $V \equiv 0$, easy to show that both expression vanish.)

$\rightarrow$ gives interpretation of Lie-bracket!!

The Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ gives

$$\mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z = \mathcal{L}_{[X,Y]} Z$$

$\rightarrow$ interpretation: $[X,Y]$ measure of non-commutativity of 1-param. groups.

The definition (*) can easily be extended to general tensor fields T by

$$\mathcal{L}_V T(p) := \frac{d}{dt}\Big|_{t=0} (F_t^* T)(p)$$

where we recall that if T is a (k,l)-tensor field and $\alpha_1, \dots, \alpha_k \in T_p^* M$, $X_1, \dots, X_l \in T_p M$, then

$$\begin{aligned}(F_t^* T)(p)\,(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l) &= T(F_t(p))\,((F_t)_* \alpha_1, \dots, (F_t)_* \alpha_k, (F_t)_* X_1, \dots, (F_t)_* X_l) \\ &= T(F_t(p))\,((F_{-t})^* \alpha_1, \dots, (F_{-t})^* \alpha_k, (F_t)_* X_1, \dots, (F_t)_* X_l)\end{aligned}$$

If $f \in C^\infty(M)$, we define $\mathcal{L}_V f(p) := \frac{d}{dt}\Big|_{t=0} F_t^* f(p) = \frac{d}{dt}\Big|_{t=0} (f \circ F_t)(p) = V|_p(f)$

$t \mapsto F_t(p)$ curve with tangent V

Proposition (Properties of Lie derivative)

i) $\mathcal{L}_V f = Vf$ for $f \in C^\infty(M)$

ii) $\mathcal{L}_V (aT + bS) = a\mathcal{L}_V T + b\mathcal{L}_V S$ for $a,b \in \mathbb{R}$, T,S tensor fields (linearity)

iii) $\mathcal{L}_V (T \otimes S) = (\mathcal{L}_V T) \otimes S + T \otimes (\mathcal{L}_V S)$ Leibniz rule

iv) $\mathcal{L}_V (\mathrm{tr}\, T) = \mathrm{tr}\,(\mathcal{L}_V T)$ commutes with contractions

add ix) $\mathcal{L}_X \mathcal{L}_Y T - \mathcal{L}_Y \mathcal{L}_X T = \mathcal{L}_{[X,Y]} T$
Pf: exercise

v) $\mathcal{L}_V W = [V,W]$ for W vector field

vi) $\mathcal{L}_V (df) = d(\mathcal{L}_V f) = d(Vf)$ for $f \in C^\infty(M)$

vii) T (k,l)-tensor field, then in coords

$$(\mathcal{L}_V T)^{a_1 \dots a_k}{}_{b_1 \dots b_l} = V^c \partial_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} - T^{c a_2 \dots a_k}{}_{b_1 \dots b_l} \partial_c V^{a_1} - \dots + T^{a_1 \dots a_k}{}_{c b_2 \dots b_l} \partial_{b_1} V^c + \dots$$

viii) In adapted coords st $V = \partial_0$, then $\mathcal{L}_V T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = \partial_0 (T^{a_1 \dots a_k}{}_{b_1 \dots b_l})$

Remark: Lie derivative depends only on smooth manifold structure, it does not depend on a metric.

(Pf: i) ✓ , ii) easy

iii)

$$\mathcal{L}_V (T \otimes S)|_p = \lim_{t \to 0} \frac{F_t^* (T \otimes S)|_p - T \otimes S|_p}{t} = \lim_{t \to 0} \frac{(F_t^* T) \otimes (F_t^* S)|_p - T \otimes S|_p}{t}$$

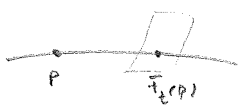
$$= \lim_{t \to 0} \frac{F_t^* T \otimes F_t^* S|_p - F_t^* T \otimes S|_p}{t} + \lim_{t \to 0} \frac{F_t^* T \otimes S|_p - T \otimes S|_p}{t}$$

$$= T \otimes \mathcal{L}_V S|_p + \mathcal{L}_V T \otimes S|_p$$

iv) α 1-form, X vector field

$$\mathrm{tr}\,(\mathcal{L}_V (\alpha \otimes X))|_p = \mathrm{tr}\left(\frac{d}{dt}\Big|_{t=0} F_t^* (\alpha \otimes X)\right)\Big|_p = \mathrm{tr}\left(\frac{d}{dt}\Big|_{t=0} (F_t^* \alpha)|_p \otimes (F_{-t})_* X|_p \right)$$

$$= \delta^\mu{}_\kappa \frac{d}{dt}\Big|_{t=0} \left(\frac{\partial F_t^\nu}{\partial x^\mu}(p)\, \alpha_\nu (F_t(p)) \cdot \frac{\partial F_{-t}^\kappa}{\partial x^\sigma}(F_t(p))\, X^\sigma (F_t(p)) \right)$$



$$= \frac{d}{dt}\Big|_{t=0} \left(\alpha_\nu (F_t(p)) X^\nu (F_t(p)) \right)$$

$$= \frac{d}{dt}\Big|_{t=0} F_t^* (\alpha(X)) = \mathcal{L}_V (\mathrm{tr}\,(\alpha \otimes X))$$

$F_{-t} = (F_t)^{-1}$

$F_{-t} \circ F_t = \mathrm{id}$

$\Rightarrow \frac{\partial F_{-t}^\kappa}{\partial x^\sigma}(F_t(p)) \frac{\partial F_t^\sigma}{\partial x^\mu}(p) = \delta^\kappa{}_\mu$

boils down to essentially

$\frac{d}{dt}(\delta^\mu{}_\nu \alpha^\nu X_\mu)$
$=$
$\delta^\mu{}_\nu \frac{d}{dt}(\alpha^\nu X_\mu)$

General case follows from this and iii)

v) ✓

vi) $\mathcal{L}_V (df)(X) = \mathrm{tr}(\mathcal{L}_V df \otimes X) = \mathcal{L}_V (df(X)) - df(\mathcal{L}_V X) = V(X(f)) - [V,X]f = X(Vf) = d(Vf)(X)$

$\Rightarrow \mathcal{L}_V dx^\mu = dV^\mu = \partial_\kappa V^\mu dx^\kappa$

vii) Now in general:

Using iii) & iv) we compute

$$(\mathcal{L}_V T)^{a_1 \dots a_k}{}_{b_1 \dots b_l} = (\mathcal{L}_V T)(dx^{a_1}, \dots, dx^{a_k}, \partial_{b_1}, \dots, \partial_{b_l}) = \mathrm{tr}(\mathcal{L}_V T \otimes dx^{a_1} \otimes \dots \otimes \partial_{b_l})$$

$$= \mathcal{L}_V (T^{a_1 \dots a_k}{}_{b_1 \dots b_l}) - T(\mathcal{L}_V dx^{a_1}, \dots, dx^{a_k}, \partial_{b_1}, \dots, \partial_{b_l}) - \dots$$

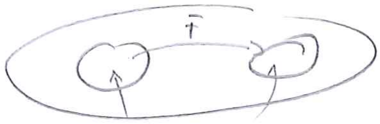
$$- T(dx^{a_1}, \dots, dx^{a_k}, \mathcal{L}_V \partial_{b_1}, \dots, \partial_{b_l}) - \dots$$

$$= V^c \partial_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} - T^{c a_2 \dots a_k}{}_{b_1 \dots b_l} \partial_c V^{a_1} - \dots + T^{a_1 \dots a_k}{}_{c b_2 \dots b_l} \partial_{b_1} V^c + \dots$$

viii) From vii).

Killing vector fields & isometries

Def: Let (M,g) Lorentzian (Riemannian) manifold. A diffeomorphism $F: M \to M$ is an <u>isometry</u> if $F^* g = g$



metrics we are the same.

Now let $F: \mathbb{R} \times M \to M$ be a 1-parameter group of isometries, i.e. $F_t: M \to M$ is an isometry for every $t \in \mathbb{R}$. Let $V \in \mathfrak{X}^\infty(M)$ denote the infinitesimal generator. We then have

$$\mathcal{L}_V g = \frac{d}{dt}\Big|_{t=0} F_t^* g = \frac{d}{dt}\Big|_{t=0} g = 0$$

Def: A vector field V on (M,g) satisfying $\mathcal{L}_V g = 0$ is called a <u>Killing vector field (KVF)</u>.

Vice versa, let V be a KVF (again assume that it has integral curves defined for $t \in \mathbb{R}$) and let $F: \mathbb{R} \times M \to M$ be the associated 1-parameter group of diffeomorphisms.

Thus we have $\frac{d}{dt}\Big|_{t=0} F_t^* g = \mathcal{L}_V g = 0$ . . . Fix $p \in M$ and consider the curve $t \xmapsto{\psi} F_t^* g|_p$. We have

$$\psi'(t_0) = \frac{d}{dt}\Big|_{t=t_0} F_t^* g\Big|_p = \frac{d}{dt}\Big|_{t=t_0} F_{t_0}^* F_{t-t_0}^* g\Big|_p = F_{t_0}^*\left(\frac{d}{dt}\Big|_{t=0} F_t^* g\Big|_{F_{t_0}(p)}\right)$$

$$= F_{t_0}^*\big(\underbrace{\mathcal{L}_V g|_{F_{t_0}(p)}}_{=0}\big) = 0$$

$\Rightarrow$ ψ is constant curve and $\psi(0) = g|_p$ $\Rightarrow$ $F_t^* g = g$ $\forall t \in \mathbb{R}$.

We have thus shown

Proposition: Let (M,g) be Lorentzian (Riemannian) manifold and $F: \mathbb{R} \times M \to M$ a 1-parameter group of diffeomorphism. Then F is a 1-param. group of isometries if, and only if, the infinitesimal generator V is a Killing vector field.

Remark: Let V be a vector field and x^μ a coord. chart s.t. $V = \frac{\partial}{\partial x^0}$. Then V is a KVF iff $\partial_0 g_{\mu\nu} = 0$ (i.e. if metric components are independent of x^0)

Proposition (Properties of KVFs). (M,g) Lorentzian manifold (Riemannian)

i) Killing vector fields form a Lie algebra: if V, K are KVFs on (M,g), so is $[V,K]$.

ii) V is a KVF iff $\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$

iii) If V is a KVF, then $\nabla_a \nabla_b V_c = -R_{adbc} V^d$

iv) Let V be a KVF and $\gamma : I \to M$ an affinely parametrised geodesic $(\nabla_{\dot\gamma}\dot\gamma = 0)$. Then $g(V,\dot\gamma)$ is constant along γ. (KVFs give rise to first integrals / conserved quantities for geodesics)

Pf: i) Follows from $\mathcal{L}_{[V,K]}g = \mathcal{L}_V\mathcal{L}_K g - \mathcal{L}_K\mathcal{L}_V g = 0$

ii)
$$(\mathcal{L}_V g)(\partial_\mu, \partial_\nu) \overset{\text{Leibniz}}{=} V(g_{\mu\nu}) - g([V,\partial_\mu],\partial_\nu) - g(\partial_\mu,[V,\partial_\nu])$$
$$\overset{\nabla \text{ symmetric}}{=} V(g_{\mu\nu}) - g(\nabla_V\partial_\mu,\partial_\nu) - g(\partial_\mu,\nabla_V\partial_\nu) + g(\nabla_{\partial_\mu}V,\partial_\nu) + g(\partial_\mu,\nabla_{\partial_\nu}V)$$
$$= \underbrace{(\nabla_V g)}_{\substack{=0\\ \nabla \text{ metric}}}(\partial_\mu,\partial_\nu) + g(\nabla_{\partial_\mu}V,\partial_\nu) + g(\partial_\mu,\nabla_{\partial_\nu}V)$$
$$= \nabla_\mu V_\nu + \nabla_\nu V_\mu$$

(Same curvature convention as Wald)

iii) Recall $\nabla_a\nabla_b V_c - \nabla_b\nabla_a V_c = R_{cdab}V^d = -R_{dcab}V^d$

Also recall 1st Bianchi identity $R_{dcab} + R_{dabc} + R_{dbca} = 0$

$$\Rightarrow 0 = \nabla_a\nabla_b V_c - \nabla_b\nabla_a V_c + \nabla_b\nabla_c V_a - \nabla_c\nabla_b V_a + \nabla_c\nabla_a V_b - \nabla_a\nabla_c V_b$$
$$\overset{\text{ii)}}{=} \nabla_a\nabla_b V_c + \nabla_b\nabla_c V_a + \nabla_b\nabla_c V_a - \nabla_c\nabla_b V_a - \nabla_c\nabla_b V_a + \nabla_a\nabla_b V_c$$
$$= 2\left(\nabla_a\nabla_b V_c + \nabla_b\nabla_c V_a - \nabla_c\nabla_b V_a\right)$$
$$\Rightarrow \nabla_a\nabla_b V_c = -R_{adbc}V^d$$

iv)
$$\dot\gamma(g(V,\dot\gamma)) = \nabla_{\dot\gamma}(g(V,\dot\gamma)) = \underbrace{g(\nabla_{\dot\gamma}V,\dot\gamma)}_{=0} + g(V,\underbrace{\nabla_{\dot\gamma}\dot\gamma}_{=0})$$

$=0$ by ii) which says $\nabla_\mu V_\nu$ is antisymmetric; $=0$ by γ being geodesic

∎

Example: Consider Minkowski space $\mathbb{R}^4$, $g = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$

- Then $V = \partial_t$ is a KVF by last Remark & g being independent of t.
- It generates 1-param. group of isometries $F: \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^4$, $F_{t_0}(t, \underline{x}) = (t + t_0, \underline{x})$, the time translations.
- Consider unit timelike vector $U \in \mathbb{R}^4$, $g(U, U) = -1$. Then $s \xmapsto{\gamma} s \cdot U \in \mathbb{R}^4$ is timelike geodesic with $\dot\gamma = U$. $-g(\partial_t, U) = U^0$ is the conserved energy of the particle.

Submanifolds

- M n-dim smooth manifold. A subset $S \subseteq M$ is called a k-dimensional (embedded) submanifold of M ($k \leq n$), iff $\forall p \in S$ $\exists$ coordinate chart $\varphi: M \supseteq U \to V \subseteq \mathbb{R}^n$, $U \ni p$, s.t.

$$S \cap U = \{(x^1, \dots, x^k, x^{k+1}, \dots, x^n) \mid x^{k+1} = \dots = x^n = 0\}$$

- A hypersurface $S \subseteq M$ is an $(n-1)$-dimensional (embedded) submanifold
- Proposition:
  Let $f: M \to \mathbb{R}$ be a smooth function s.t. $df|_p \neq 0$ $\forall p \in f^{-1}(0)$.
  Then $f^{-1}(0) =: S \subseteq M$ is a hypersurface in M.

Pf: Let $p \in S$, $(x^1, \dots, x^n)$ a coord. system centred at p and let $\frac{\partial f}{\partial x^n}(0) \neq 0$.

Consider the map $(x^1, \dots, x^n) \xmapsto{\psi} (x^1, \dots, x^{n-1}, f(x_1, \dots, x_n)) = (y^1, \dots, y^n)$. Then

$$D\psi = \begin{pmatrix} 1 & & & 0 & 0 \\ & \ddots & & & \vdots \\ 0 & & & 1 & 0 \\ \frac{\partial f}{\partial x^1} & \cdots & & \cdots & \frac{\partial f}{\partial x^n} \end{pmatrix}$$

is invertible at $x = 0$ and thus ψ is a diffeomorphism in a nbhd of 0 and thus gives rise to a new coord. system $(y^1, \dots, y^n)$ in which S is locally the level set $y^n = 0$.

$\boxtimes$

Examples:

- $M = \mathbb{R}^n$, $S = \{x^{k+1} = \dots = x^n = 0\} \simeq \mathbb{R}^k$ is k-dim submfd.

$$f(x) = x_1^2 + \dots + x_n^2 - 1, \qquad f^{-1}(0) = S^{n-1}$$

$$df(x) = 2(x_1 dx_1 + \dots + x_n dx_n) \neq 0 \text{ for } x \neq 0 \quad \Rightarrow S^{n-1} \text{ is hypersurface.}$$

- M n-dim mfd, V vector field, $V(p) \neq 0$ $\forall p \in M$. Let $\{\gamma\}_I$ be the family of integral curves on V. Then each γ is a 1-dim submanifold, since can choose locally adapted coords $\{x^1, \dots, x^n\}$ s.t. $V = \frac{\partial}{\partial x^1}$ and thus $\gamma = \{x^2 = \dots = x^n = 0\}$.

Think: "Prescribe 1-dim tangent spaces at each point by giving vector field V", then we showed that one can "integrate" them, i.e. find 1-dim submanifolds which have the prescribed tangent spaces.

<u>Integral manifolds</u>: Given now two smooth vector fields V, W on M s.t. at every $p \in M$ $V(p), W(p)$ are linearly independent. Then $\mathrm{span}\{V,W\} \subseteq TM$ is called a 2-dim distribution of TM.
It is called <u>integrable</u> if there exists locally a family of 2-dim submanifolds which have $\mathrm{span}\{V,W\}$ as their tangent spaces.

Assume there exists such a submanifold S. Then V, W restrict to vector fields on S and thus $[V,W]$ is also tangent to S, i.e. $[V,W] \subseteq \mathrm{span}\{V,W\}$.

$\Rightarrow$ $[V,W] \subseteq \mathrm{span}\{V,W\}$ is necessary cond.

(We show that it is also sufficient.

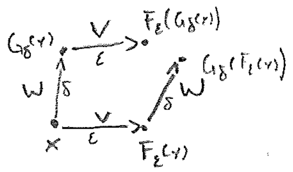
First: <u>Geometric interpretation of $[V,W]$</u>

<u>Heuristics:</u> $V, W \in \mathcal{X}^\infty(M)$, $F_t : \mathbb{R}\times M \to M$ 1-param. group of diff generated by V
$G_t : \mathbb{R}\times M \to M$ — " — W

Let $p \in M$ and $\{x^1, \dots, x^n\}$ local coords centred at p.

Then
$$F^\mu_\varepsilon(G_\delta(x)) = F^\mu_\varepsilon(\underbrace{x^\nu + \delta W^\nu(x)}) = x^\mu + \delta W^\mu(x) + \varepsilon V^\mu(x^\nu + \delta W^\nu(x))$$
$$\frac{d}{d\delta}\Big|_{\delta=0} G^\nu_\delta(x) = W^\nu(x)$$
$$= x^\mu + \delta W^\mu(x) + \varepsilon V^\mu(x) + \varepsilon\delta W^\nu(x)\partial_\nu V^\mu(x)$$

(



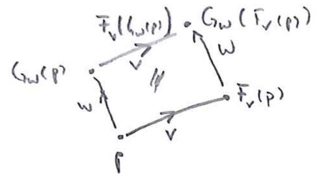
Similarly
$$G^\mu_\delta(F_\varepsilon(x)) = x^\mu + \varepsilon V^\mu(x) + \delta W^\mu(x) + \varepsilon\delta V^\nu(x)\partial_\nu W^\mu(x)$$

$$\Rightarrow \quad F^\mu_\varepsilon(G_\delta(x)) - G^\mu_\delta(F_\varepsilon(x)) = \varepsilon\delta\left(W^\nu\partial_\nu V^\mu - V^\nu\partial_\nu W^\mu\right)(x) = \varepsilon\delta\,[W,V]^\mu(x).$$

$\Rightarrow$ $[W,V]$ measures the lack of commutation of the associated flows F_t & G_s.

One can show that $F_s \circ G_t = G_t \circ F_s$ iff $[V,W] \equiv 0$.

Back to integral manifolds: V, W smooth, linearly independent, $[V,W]=0$, $p \in M$.



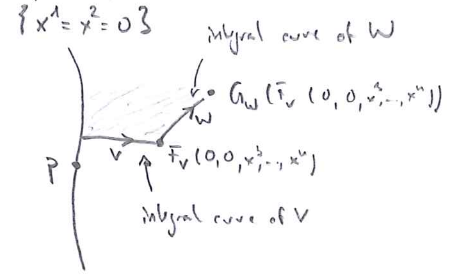
(w,v) form coord. system on 2-dim surface.

(can give proof along these lines, see Lee Thm 18.6, p. 471)

Thm: M n-dim smooth manifold, V, W smooth vector fields pointwise linearly independent, $[V,W]=0$. Then, locally there are coords $(v, w, x^3, \dots, x^n)$ for M s.t. $V = \frac{\partial}{\partial v}$, $W = \frac{\partial}{\partial w}$.

Pf: Let $p \in M$, can find local coords $(x^1, \dots, x^n)$ centred at p s.t. $\frac{\partial}{\partial x^1}\Big|_p = V|_p$ & $\frac{\partial}{\partial x^2}\Big|_p = W|_p$ (exercise! (hint: linear change of coords.)). Let F be flow of V, G flow of W.

Define $(v, w, x^3, \dots, x^n) \overset{\psi}{\longmapsto} G_w(F_v(0,0,x^3,\dots,x^n))$



Then $D\psi|_0 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, thus ψ is a local diffeomorphism and $(v, w, x^3, \dots, x^n)$ form new coords. on M.

Clearly $W = \frac{\partial}{\partial w}$, but in general only $V = a(v,w,x^i)\frac{\partial}{\partial v} + b(v,w,x^i)\frac{\partial}{\partial w} + \sum_{j=3}^{n} c^j(v,w,x^i)\frac{\partial}{\partial x^j}$

with $a(v,0,x^i) = 1$, $b(v,0,x^i) = 0 = c^j(v,w,x^i)$, $j = 3, \dots, n$

(for $w = 0$ we are on the integral curve of V).

Now $0 = [W,V] = \frac{\partial}{\partial w}\left(a\frac{\partial}{\partial v} + b\frac{\partial}{\partial w} + \sum c^j \frac{\partial}{\partial x^j}\right) - \left(a\frac{\partial}{\partial v} + b\frac{\partial}{\partial w} + \sum c^j\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial w}$

$= \partial_w a \cdot \frac{\partial}{\partial v} + \partial_w b \cdot \frac{\partial}{\partial w} + \sum \partial_w c^j \cdot \frac{\partial}{\partial x^j} + \cancel{VW - VW}$

partial derivatives commuting.

$\frac{\partial}{\partial v}, \frac{\partial}{\partial w}, \frac{\partial}{\partial x^j}$ $j = 3, \dots, n$ linearly independent $\Rightarrow \partial_w a = \partial_w b = \partial_w c^j = 0$ $j = 3, \dots, n$

$\Rightarrow a = 1$, $b = 0 = c^j$ $j = 3, \dots, n$

$\Rightarrow W = \frac{\partial}{\partial w}$ & $V = \frac{\partial}{\partial v}$

∎

In particular if $[V,W]=0$, then there exists 2-dim submanifold $S=\{x^3=c^3,\dots,x^n=c^n\}$ with $TS=\operatorname{span}\{V,W\}$.

Following lemma reduces general case to this one.

__Lemma.__ M smooth manifold, $V,W\in\mathfrak{X}^\infty(M)$ pointwise linearly independent with $[V,W]\in\operatorname{span}\{V,W\}$.
Then there exists $\hat V,\hat W\in\mathfrak{X}^\infty(M)$ with $\operatorname{span}\{\hat V,\hat W\}=\operatorname{span}\{V,W\}$ and $[\hat V,\hat W]=0$.

__Pf.__ Let $\hat V=\lambda V$, $\hat W=\mu W$, $\lambda,\mu\in C^\infty(M)$.

$$0 \overset{!}{=} [\hat V,\hat W]f = [\lambda V,\mu W]f = \lambda[V,\mu W]f-\mu W(\lambda)(Vf) = \lambda\mu[V,W]f+\lambda V(\mu)(Wf)-\mu W(\lambda)(Vf)$$

$$= \lambda\mu a V(f)+\lambda\mu b W(f)+\lambda V(\mu)(Wf)-\mu W(\lambda)(Vf)$$

$[V,W]\in\operatorname{span}\{V,W\}$
$\Rightarrow [V,W]=a\cdot V+b\cdot W$, $a,b\in C^\infty(M)$

If we now choose μ s.t. $\mu\cdot b+V(\mu)=0 \iff V(\ln\mu)=-b$
and λ s.t. $\lambda a-W(\lambda)=0 \iff W(\ln\lambda)=a$

solve for μ,λ by integrating along integral curves of V,W, respectively.

then $[\hat V,\hat W]=0$.

∎

Following theorem is a generalisation to higher dimensional case.

__Thm (Frobenius)__ Let M n-dim smooth manifold, $V_1,\dots,V_k\in\mathfrak{X}^\infty(M)$, $k<n$, pointwise linearly independent s.t. $[V_i,V_j]\in\operatorname{span}\{V_1,\dots,V_k\}$ $\forall\, 1\le i,j\le k$. Then ($\operatorname{span}\{V_1,\dots,V_k\}$ is integrable, i.e.) locally there exist coords. $\{v_1,\dots,v_k,x_{k+1},\dots,x_n\}$ s.t. $\operatorname{span}\{V_1,\dots,V_k\}$ are the tangent spaces of the family of submanifolds $\{x_{k+1}=c_{k+1},\dots,x_n=c_n\}$, c_i = consts.

__Proof:__ For $k=2$ by previous Lemma & Theorem. General case by induction.

not examinable

k-forms and dual version of Frobenius

Recall $(0,k)$ tensor field ω := at every $p\in M$ $\omega|_p: T_pM\times\dots\times T_pM\to\mathbb{R}$ multilinear map.

If $\omega|_p$ is totally antisymmetric then we say ω is a k-form.

If α is $(0,k)$-tensor field, then $\alpha_{[a_1\dots a_k]} := \frac{1}{k!}\sum_\sigma \operatorname{sgn}(\sigma)\,\alpha_{\sigma(a_1)\dots\sigma(a_k)}$ $\qquad \operatorname{sgn}(\sigma)=\begin{cases}+1 & \text{for even permutations}\\ -1 & \text{for odd permutations}\end{cases}$

is the total antisymmetrisation of α, a k-form.

Given k-form α & l-form β define $(k+l)$-form $\alpha\wedge\beta$ by $(\alpha\wedge\beta)_{a_1\dots a_{k+l}} := \frac{(k+l)!}{k!\,l!}\alpha_{[a_1\dots a_k}\beta_{a_{k+1}\dots a_{k+l}]}$ $\leftarrow \binom{k+l}{k}$ ⇝ see Lee p.293

⇝ antisymmetrisation of tensor product.

Example:
- α,β 1-forms $\Rightarrow (\alpha\wedge\beta)(X,Y)=\alpha(X)\beta(Y)-\beta(X)\alpha(Y)$.
- $\alpha\wedge\alpha=0$.

Given a k-form α, then $\nabla\alpha$ is a $(0,k+1)$-tensor and $(d\alpha)_{a_1\dots a_{k+1}} := (k+1)\nabla_{[a_1}\alpha_{a_2\dots a_{k+1}]}$ a $(k+1)$-form.

Note that
$$\nabla_{[a_1}\alpha_{a_2\dots a_{k+1}]} = \partial_{[a_1}\alpha_{a_2\dots a_{k+1}]} - \underbrace{T^b_{[a_1 a_2}}_{T^b_{a_1a_2} = T^b_{a_2a_1}}\alpha_{|b| a_3\dots a_{k+1}]} - \dots\dots$$
$$= \partial_{[a_1}\alpha_{a_2\dots a_{k+1}]}$$

Get operator d : k-forms $\to (k+1)$-forms which is independent of metric g (and ∇).

It only depends on manifold structure.

Exercise : $d\circ d = 0$. (Use $\partial_{a_1}\partial_{a_2} = \partial_{a_2}\partial_{a_1}$ ...)

Proposition: ω 1-form, X, Y smooth vector fields. Then

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

Pf. $d\omega(X,Y) = (d\omega)_{ab}X^aY^b = 2\partial_{[a}\omega_{b]}X^aY^b = X(\omega_b)\cdot Y^b - Y(\omega_b)X^b$

$$= X(\omega_bY^b) - \omega_b X(Y^b) - Y(\omega_bX^b) + \omega_b Y(X^b)$$

$[X,Y]^b = X(Y^b) - Y(X^b)$
$$\overset{?}{=} X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

$\square$

Let M be n-dim manifold and $\alpha \in \Omega^1(M)$ (non-vanishing). For $p\in M$ $\alpha: T_pM \to \mathbb{R}$ linear, so $\ker\alpha_p$ is $(n-1)$-dimensional. It follows that there are locally $(n-1)$-smooth pointwise linearly indep. vector fields $V_1,\dots,V_{n-1}$ s.t. $\ker\alpha = \operatorname{span}\{V_1,\dots,V_{n-1}\}$. Vice versa, given an $(n-1)$-dimensional distribution $\operatorname{span}\{V_1,\dots,V_{n-1}\}$ there is locally a 1-form α with $\ker\omega = \operatorname{span}\{V_1,\dots,V_{n-1}\}$.

$\to$ 1-forms easy way to specify $(n-1)$-dim distribution.

Proposition. M n-dim manifold, $\alpha\in\Omega^1(M)$ pointwise non-vanishing. Then the following are equivalent

i) For $V,W \in \ker\alpha$ smooth vector fields we have $[V,W]\in\ker\alpha$ ($\overset{\text{Frobenius}}{\iff}$ $\ker\omega$ is integrable).

ii) $d\alpha|_{\ker\alpha} \equiv 0$

iii) $\alpha\wedge d\alpha \equiv 0$

Pf: By Proposition for $V,W\in\ker\alpha$ we have $d\alpha(V,W) = V(\underbrace{\alpha(W)}_{=0}) - W(\underbrace{\alpha(V)}_{=0}) - \alpha([V,W])$

So $d\alpha(V,W) = -\alpha([V,W])$ and we obtain $d\alpha(V,W) = 0 \iff [V,W]\in\ker\alpha$.

This shows i) $\iff$ ii)

To see ii) $\Leftrightarrow$ iii). Let $p \in M$, and let $\alpha_1, \dots, \alpha_n$ be a basis of T_p^*M with $\alpha_1 = \alpha(p)$.

Then $\alpha_i \wedge \alpha_j$, $1 \le i < j \le n$ is a basis for all antisymmetric $(0,2)$-tensors on T_pM (exercise)

Thus $\displaystyle d\alpha|_p = \sum_{1 \le i < j \le n} f_{ij}\, \alpha_i \wedge \alpha_j$ $\quad f_{ij} \in \mathbb{R}$.

Now, if $d\alpha|_{\ker\alpha} = 0$, then $\displaystyle d\alpha|_p = \sum_{1 < j \le n} f_{1j}\, \alpha \wedge \alpha_j$. $\Rightarrow$ $\displaystyle \alpha \wedge d\alpha|_p = \sum_{1<j\le n} f_{1j} \underbrace{\alpha \wedge \alpha \wedge \alpha_j}_{=0} = 0$.

Vice versa if $\displaystyle 0 \overset{!}{=} \alpha \wedge d\alpha = \alpha \wedge \sum_{1 \le i < j \le n} f_{ij}\, \alpha_i \wedge \alpha_j$ $\Rightarrow$ $\displaystyle d\alpha = \sum_{1<j\le n} f_{1j}\, \alpha \wedge \alpha_j$

$\Rightarrow d\alpha|_{\ker\alpha} \equiv 0$. ▫

<u>Corollary:</u> Let $\alpha \in \Omega^1(M)$, $\alpha \neq 0$ pointwise, with $\alpha \wedge d\alpha = 0$. Then there exist locally coords $\{x^1, \dots, x^n\}$ and a fct f s.t. $\alpha = f \cdot dx^n$.

<u>Pf:</u> By Proposition and Frobenius $\ker\alpha$ is integrable $\rightsquigarrow$ exist locally coords $\{x^1, \dots, x^n\}$ s.t. $\{x^n = \text{const}\}$ are integral manifolds of $\ker\alpha$. Thus dx^n is proportional to α. ▫

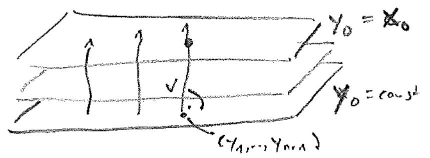
<u>Applications:</u> • Let (M,g) be a Lorentzian manifold and let V be a vector field that is nowhere vanishing. We say that V is <u>hypersurface orthogonal</u> iff the distribution orthogonal to it $V^\perp$ is integrable, i.e. $V^\perp = \{X \in \mathfrak{X}(M) \mid g(V,X) = 0\}$. Note that $V^\perp = \ker V^\flat$.

Proposition $\Rightarrow$ V is hypersurface orthogonal iff $V^\flat \wedge dV^\flat \equiv 0$ $\quad (\Leftrightarrow V_{[a}\partial_b V_{c]} \equiv 0)$.

<u>Corollary:</u> Let V be hypersurface orthogonal. Then there exist coords $x_0, \dots, x_{n-1}$ s.t.

$\partial_0 \sim V$ and $g = g_{00}\, dx_0^2 + g_{ij}\, dx_i dx_j$ $\quad$ (i.e. $g_{0i} = 0$)

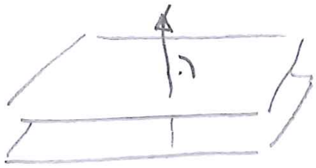
<u>Pf:</u> By corollary $\exists$ coords $\{y^0, \dots, y^{n-1}\}$ s.t. $\{y^0 = \text{const}\}$ are integral manifolds of $V^\perp$.



Pick $\{y_0 = 0\}$, on which we have coords $y_1, \dots, y_{n-1}$. Consider the family of integral curves of V (which are orthogonal to $\{y_0 = 0\}$) and let $(x_0, x_1, \dots, x_{n-1})$ refer to the point that is the intersection of the integral curve through $\{y_0 = 0, y_i = x_i\}$ with $\{y_0 = x_0\}$, in a small enough nbhd $(x_0, \dots, x_{n-1})$ form coordinates (exercise) and since x_i, $i = 1, \dots, n-1$, is constant along integral curves of V we have $V \sim \partial_0$. Moreover $\partial_i \in V^\perp$.

▫

Example: Cosmology, FLRW models $g = -d\tau^2 + a(\tau)^2 \bar{g}$. ∂_τ is hypersurface orthogonal



Def: A spacetime is called static iff there is a timelike & hypersurface orthogonal Killing vector field V.

Remark: In such a spacetime can locally introduce coords $x_0, \dots, x_{n-1}$ s.t. $V = \partial_0$ and

$$g = g_{00} dx_0^2 + g_{ij} dx^i dx^j$$ with $g_{\mu\nu}$ independent of x_0.

Pf: example sheet.

Example: Schwarzschild $g = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$ with $r > 2m$ is static

with timelike & hypersurface orthogonal KVF ∂_t.

Def: A spacetime is called stationary iff there exists a timelike KVF (not necessarily everywhere in the spacetime, tbd) (not necessarily hypersurface orthogonal)

$\rightarrow$ Kerr, later.