# II. Linearised General Relativity

1) Einstein eqs with matter
2) Linearised gravity (weak grav. field)
3) Newtonian limit
4) Far field of isolated gravitational body
5) Gravitational waves

## II.1. Einstein equations with matter

Continuum of matter described by stress-energy tensor $T_{ab}$ (symmetric 2-covariant tensor field). In flat space $\partial^a T_{ab} = 0$ expresses conservation laws (of energy, momentum, ...). (example sheet)

In curved spacetime $\Rightarrow$ $\nabla^a T_{ab} = 0$

Recall $G_{ab} := R_{ab} - \frac{1}{2} g_{ab} R$, Bianchi-eq $\Rightarrow$ $\nabla^a G_{ab} = 0$ (example sheet).

So Einstein tried

$$\underline{G_{ab} = \lambda \cdot T_{ab}}$$

, $\lambda$ constant, to be determined by comparison with Newtonian theory.

Find:
- $\lambda = 8\pi$ in geometrised units $G = c = 1$
- $\lambda = \frac{8\pi G}{c^4}$ non-geometrised units

Examples:

a) Perfect fluid: Described by
- 4-velocity $u$, unit timelike vector field
- $\rho$ mass-energy density in rest frame
- $p$ pressure in rest frame

  (scalars)
- equation of state $p = p(\rho)$

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab}$$

Choose orthonormal frame field $e_0, \dots, e_3$ with $e_0 = u$ ($\Leftrightarrow$ rest frame), then in this frame
$$T_{ab} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$$
, i.e. interpretation of $\rho$ & $p$ as given above.

$$\nabla^a T_{ab} = 0 \quad \Leftrightarrow \quad \begin{cases} u^a \nabla_a \rho + (\rho + p) \nabla^a u_a = 0 \\ (p + \rho) u^a \nabla_a u_b + (g_{ab} + u_a u_b) \nabla^a p = 0 \end{cases}$$

which, together with the equation of state, give the equations of motion for the fluid.

b) Dust : Perfect fluid with $p=0$

$$\rightsquigarrow \quad T_{ab} = \rho u_a u_b$$

c) Electromagnetic field : Faraday tensor $F_{ab}$ , 2-form

$$T_{ab} = \frac{1}{4\pi}\left( F_{ac}F_b{}^c - \frac{1}{4} g_{ab} F_{de}F^{de} \right)$$

$$\nabla^a T_{ab} = 0 \quad \Leftarrow \quad \left\{ \begin{array}{l} dF = 0 \\ \nabla^a F_{ab} = 0 \end{array} \right. \qquad \text{(example sheet)}$$

( Remark: $\nabla^a T_{ab} = 0$ does in general not imply the Maxwell equations. see MTW, p 483 )

- Rewriting the Einstein equations

$$R_{ab} - \frac{1}{2} g_{ab} R = \lambda T_{ab} \quad \xrightarrow{\text{trace}} \quad R - 2R = \lambda T \quad \Rightarrow \quad R = -\lambda T$$

EE are equivalent to $\underline{R_{ab} = \lambda \left( T_{ab} - \frac{1}{2} g_{ab} T \right)}$

In vacuum : $R_{ab} = 0$ .

( Remark: The reason for rewriting the EE in this form is that the principal part of $R_{ab}$ is nicer in harmonic gauge than that of $G_{ab}$. In linearised theory we thus obtain $R_{ab} \simeq \Box h_{ab}$ , while $G_{ab} \simeq \Box \bar{h}_{ab}$, where $\bar{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab} h$ )

## II.2 Linearising the Einstein equations around Minkowski space

Start with Minkowski spacetime $(\mathbb{R}^4, \eta)$ in inertial (Cartesian) coordinates $x^\mu$. In the following, the Greek indices $\mu, \nu, \kappa, \dots$ etc will not be abstract indices but always refer to this chosen coordinate system on $\mathbb{R}^4$.

Looking for an approx. solution $(\mathbb{R}^4, g = \eta + \varepsilon h, T = \varepsilon \overset{(1)}{T})$, $h$ symmetric 2-cov. tensor field on $\mathbb{R}^4$, $0 < \varepsilon \ll 1$ small parameter, require that $g$ satisfies the Einstein equations $R_{ab} = \lambda(T_{ab} - \frac{1}{2} g_{ab} T)$ to order $\varepsilon$. (ignore higher orders of $\varepsilon$) ; $\Rightarrow$ weak grav. field, small masses ; linearised equations around $\eta$ for $h$. (Theory for sym. 2-cov. tensor field $h$ on Minkowski spacetime)

Now : compute $R_{\mu\nu}(\eta + \varepsilon h)$ to order $\varepsilon$.

- Inverse of $g_{\mu\nu}$ : Ansatz $(g^{-1})^{\nu\kappa} = \eta^{\nu\kappa} - \varepsilon S^{\nu\kappa}$ , $S$ symmetric 2-contravariant tensor

Then $g_{\mu\nu}(g^{-1})^{\nu\kappa} = (\eta_{\mu\nu} + \varepsilon h_{\mu\nu})(\eta^{\nu\kappa} - \varepsilon S^{\nu\kappa}) = \delta_\mu{}^\kappa - \varepsilon \eta_{\mu\nu} S^{\nu\kappa} + \varepsilon h_{\mu\nu}\eta^{\nu\kappa} + \mathcal{O}(\varepsilon^2)$

$$\Rightarrow \quad \eta_{\mu\nu} S^{\nu\kappa} = h_{\mu\nu}\eta^{\nu\kappa} \qquad \Leftrightarrow \qquad S^{\rho\kappa} = h_{\mu\nu}\eta^{\nu\kappa}\eta^{\mu\rho}$$

(Confusing: two metrics $g$ & $\eta$, so not clear which used to raise indices)

Convention: Raise all indices with Minkowski metric $\eta_{\mu\nu}$, i.e. $h^{\varrho\kappa} = h_{\mu\nu}\eta^{\nu\kappa}\eta^{\mu\varrho}$

$$(g^{-1})^{\nu\kappa} = \eta^{\nu\kappa} - \varepsilon h^{\nu\kappa} + \mathcal{O}(\varepsilon^2)$$

• Christoffel symbols:

$$\Gamma^{\mu}_{\nu\kappa} = \frac{1}{2} g^{\mu\sigma}\left(\partial_\nu g_{\kappa\sigma} + \partial_\kappa g_{\nu\sigma} - \partial_\sigma g_{\nu\kappa}\right)$$

$$\overset{x^\mu \text{ are Cartesian coords.}}{=} \varepsilon \frac{1}{2}\eta^{\mu\sigma}\left(\partial_\nu h_{\kappa\sigma} + \partial_\kappa h_{\nu\sigma} - \partial_\sigma h_{\nu\kappa}\right) + \mathcal{O}(\varepsilon^2)$$

• Curvature:

$$R^{\mu}{}_{\kappa\varrho\nu} = \partial_\varrho \Gamma^{\mu}_{\nu\kappa} - \partial_\nu \Gamma^{\mu}_{\varrho\kappa} + \underbrace{\Gamma^{\mu}_{\varrho\sigma}\Gamma^{\sigma}_{\nu\kappa} - \Gamma^{\mu}_{\nu\sigma}\Gamma^{\sigma}_{\varrho\kappa}}_{=\mathcal{O}(\varepsilon^2)}$$

$$= \varepsilon\frac{1}{2}\eta^{\mu\sigma}\left(\partial_\varrho\partial_\nu h_{\kappa\sigma} + \partial_\varrho\partial_\kappa h_{\nu\sigma} - \partial_\varrho\partial_\sigma h_{\nu\kappa}\right)$$
$$- \varepsilon\frac{1}{2}\eta^{\mu\sigma}\left(\partial_\nu\partial_\varrho h_{\kappa\sigma} + \partial_\nu\partial_\kappa h_{\varrho\sigma} - \partial_\nu\partial_\sigma h_{\varrho\kappa}\right) + \mathcal{O}(\varepsilon^2)$$

$$= \varepsilon\frac{1}{2}\eta^{\mu\sigma}\left(\partial_\varrho\partial_\kappa h_{\nu\sigma} - \partial_\varrho\partial_\sigma h_{\nu\kappa} - \partial_\nu\partial_\kappa h_{\varrho\sigma} + \partial_\nu\partial_\sigma h_{\varrho\kappa}\right) + \mathcal{O}(\varepsilon^2)$$

• Ricci curvature:

$$R_{\kappa\nu} = R^{\mu}{}_{\kappa\mu\nu} = \varepsilon\frac{1}{2}\left(\partial^\mu\partial_\kappa h_{\nu\mu} - \Box h_{\nu\kappa} - \partial_\nu\partial_\kappa h + \partial_\nu\partial^\mu h_{\mu\kappa}\right) + \mathcal{O}(\varepsilon^2)$$

where $h = h_{\varrho\sigma}\eta^{\varrho\sigma}$ (trace) and $\Box = \eta^{\mu\sigma}\partial_\mu\partial_\sigma$ (wave operator in Minkowski)

We now introduce $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$

Compute

$$\underbrace{\partial_\kappa(\partial^\mu \bar{h}_{\mu\nu})}_{\overset{(1)}{\Gamma}_\nu \,\leftarrow\, \Gamma_\nu = \varepsilon\overset{(1)}{\Gamma}_\nu + \mathcal{O}(\varepsilon^2)} + \partial_\nu(\underbrace{\partial^\mu \bar{h}_{\mu\kappa}}_{\overset{(1)}{\Gamma}_\kappa}) = \partial_\kappa\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\kappa\partial_\nu h + \partial_\nu\partial^\mu h_{\mu\kappa} - \frac{1}{2}\partial_\nu\partial_\kappa h$$
$$= \partial_\kappa\partial^\mu h_{\mu\nu} + \partial_\nu\partial^\mu h_{\mu\kappa} - \partial_\kappa\partial_\nu h$$

Thus

$$R_{\kappa\nu}(\eta + \varepsilon h) = \frac{1}{2}\varepsilon\left(-\Box h_{\nu\kappa} + \partial_\kappa(\partial^\mu\bar{h}_{\mu\nu}) + \partial_\nu(\partial^\mu\bar{h}_{\mu\kappa})\right) + \mathcal{O}(\varepsilon^2)$$

Field equations $R_{ab} = \lambda(T_{ab} - \frac{1}{2}g_{ab}T)$ to order $\varepsilon$ are

$$\underline{\frac{1}{2}\left(-\Box h_{\nu\kappa} + \partial_\kappa\partial^\mu\bar{h}_{\mu\nu} + \partial_\nu\partial^\mu\bar{h}_{\mu\kappa}\right) = \lambda\left(\overset{(1)}{T}_{\nu\kappa} - \frac{1}{2}\eta_{\nu\kappa}\overset{(1)}{T}\right)} \qquad (*)$$

One can simplify these equations by choosing a suitable gauge.

Recall: $\Phi$ diffeomorphism, $g$ solution of $R_{\mu\nu}(g) = \lambda(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)$

then $\Phi^* g$ solution of $R_{\mu\nu}(\Phi^* g) = \Phi^* R_{\mu\nu}(g) = \lambda(\Phi^* T_{\mu\nu} - \frac{1}{2}\Phi^* g_{\mu\nu} \Phi^* T)$

$\rightsquigarrow$ solutions $(g, T)$ and $(\Phi^* g, \Phi^* T)$ are physically equivalent.

Now, let $\xi \in \mathfrak{X}^1(M)$ vector field and let $\phi_t$ denote the 1-parameter group of diffeomorphisms it generates.

Then • $(\phi_\varepsilon^* g)_{\mu\nu} = (\phi_\varepsilon^* \eta)_{\mu\nu} + \varepsilon(\phi_\varepsilon^* h)_{\mu\nu} = \eta_{\mu\nu} + \varepsilon \frac{d}{dt}\Big|_{t=0} (\phi_t^* \eta)_{\mu\nu} + \mathcal{O}(\varepsilon^2) + \varepsilon h_{\mu\nu} + \mathcal{O}(\varepsilon^2)$

$$= \eta_{\mu\nu} + \varepsilon\left(h_{\mu\nu} + (\mathcal{L}_\xi \eta)_{\mu\nu}\right) + \mathcal{O}(\varepsilon^2)$$

• $(\phi_\varepsilon^* T)_{\mu\nu} = (\phi_\varepsilon^* \varepsilon \overset{(1)}{T}_{\mu\nu}) = \varepsilon \overset{(1)}{T}_{\mu\nu} + \mathcal{O}(\varepsilon^2)$

↳ gauge invariant at linear level.

Thus if $(\underbrace{\eta_{\mu\nu} + \varepsilon h_{\mu\nu}}_{= g_{\mu\nu}}, \varepsilon \overset{(1)}{T}_{\mu\nu})$ satisfy the Einstein equations $R_{\mu\nu} = \lambda(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)$

to order $\varepsilon$, then so does $(\eta_{\mu\nu} + \varepsilon(h_{\mu\nu} + (\mathcal{L}_\xi \eta)_{\mu\nu}), \varepsilon \overset{(1)}{T}_{\mu\nu})$ for any $\xi \in \mathfrak{X}^1(M)$

and one considers them as physically equivalent.

often called "infinitesimal diffeomorphism"

$\Rightarrow$ if $h_{\mu\nu}$ satisfies $(*)$ then so does $\tilde{h}_{\mu\nu} := h_{\mu\nu} + (\mathcal{L}_\xi \eta)_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$.

Choose gauge $\xi$ by solving $\Box \xi_\mu = -\partial^\rho \bar{h}_{\mu\rho}$ (called wave gauge)

Just wave eq. with RHS in Minkowski, easily solvable, freedom of prescribing I.D. at $t=0$.

This determines $\xi$ up to addition of solutions $\eta$ of homogeneous wave equation $\Box \eta_\mu = 0$ $\rightsquigarrow$ residual gauge freedom.

Then $\partial^\mu \bar{\tilde{h}}_{\mu\nu} = \partial^\mu(\tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{h}) = \partial^\mu(h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu) - \frac{1}{2}\partial_\nu(h + 2\partial^\mu \xi_\mu)$

$$= \partial^\mu h_{\mu\nu} + \Box \xi_\nu + \partial_\nu \partial^\mu \xi_\mu - \frac{1}{2}\partial_\nu h - \partial_\nu \partial^\mu \xi_\mu$$

$$= \partial^\mu \bar{h}_{\mu\nu} + \Box \xi_\nu$$

$$= 0$$

Thus $\tilde{h}_{\mu\nu}$ satisfies

$$\Box \tilde{h}_{\mu\nu} = -2\lambda(\overset{(1)}{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overset{(1)}{T})$$

$$\partial^\mu \bar{\tilde{h}}_{\mu\nu} = 0$$

Linearised Einstein equations in wave gauge. (LEWG)

Remark (alternative formulation): (for quadrupole formula)

Sometimes convenient to rephrase first eq. also in terms of $\bar{\tilde{h}}_{\mu\nu} \doteq \tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{h}$

$$\Rightarrow \left\{ \begin{array}{l} \Box \bar{\tilde{h}}_{\mu\nu} = -2\lambda \overset{(1)}{\tilde{T}}_{\mu\nu} \\ \partial^\mu \bar{\tilde{h}}_{\mu\nu} = 0 \end{array} \right\} \quad \text{LEWh for } \bar{\tilde{h}}_{\mu\nu}$$

Note that $\bar{\bar{\tilde{h}}}_{\mu\nu} = \tilde{h}_{\mu\nu}$.

Remark (why is it called wave gauge?):

$(M,g)$ Lorentzian mfd , $\Box_g \psi := g^{\rho\lambda}\nabla_\rho \nabla_\lambda \psi$ wave operator on $(M,g)$

$\rightsquigarrow \quad \Box = \Box_\eta$ on $(\mathbb{R}^4, \eta)$

Let $x^\mu$ be coord. function on $M$, then $\Box_g x^\mu = -g^{\rho\lambda}\Gamma^\mu_{\rho\lambda} =: -\Gamma^\mu$

Also $\Gamma_\nu := g_{\nu\mu}\Gamma^\mu$.

Set of coords $x^\mu$ satisfy wave eq. $\iff$ $\Gamma_\nu = 0 \quad \forall \nu$.

In linearised gravity, Cartesian coords $x^\mu$ satisfy $\Box_g x^\mu = 0 + \mathcal{O}(\varepsilon^2)$ (wave eq. to first order)

iff $\mathcal{O}(\varepsilon)\overset{!}{=}\Gamma_\nu = g_{\nu\mu} g^{\rho\lambda}\Gamma^\mu_{\rho\lambda} = \eta_{\nu\mu}\eta^{\rho\lambda}\varepsilon\frac{1}{2}\eta^{\mu\sigma}\left(\partial_\rho \tilde h_{\sigma\lambda} + \partial_\lambda \tilde h_{\sigma\rho} - \partial_\sigma \tilde h_{\rho\lambda}\right) + \mathcal{O}(\varepsilon^2)$

$$= \varepsilon\frac{1}{2}\left(\partial^\lambda \tilde h_{\nu\lambda} + \partial^\lambda \tilde h_{\nu\lambda} - \partial_\nu \tilde h\right) + \mathcal{O}(\varepsilon^2)$$

$$= \varepsilon\,\partial^\lambda\left(\tilde h_{\nu\lambda} - \frac{1}{2}\eta_{\lambda\nu}\tilde h\right) + \mathcal{O}(\varepsilon^2)$$

$$= \varepsilon\,\partial^\lambda \tilde{\tilde h}_{\nu\lambda} + \mathcal{O}(\varepsilon^2)$$

i.e. iff $\partial^\lambda \tilde{\tilde h}_{\nu\lambda} = 0$.

[ $\sim$ in $\tilde h_{\mu\nu}$ stands for wave gauge ... :) ]

Remark (analogy with gauge freedom in Maxwell's equations)

Maxwell's equation in Minkowski : $dF = 0$ & $\partial^\mu F_{\mu\nu} = 4\pi J_\nu$ (MTW convention)

$\uparrow$ source

$\downarrow$

$F = dA$ $\quad$ $A$ one-form, the electromagnetic potential.

Then, $dF = ddA = 0$ is trivially satisfied and $\partial^\mu F_{\mu\nu} = \partial^\mu\partial_\mu A_\nu - \partial^\mu\partial_\nu A_\mu = J_\nu \rightsquigarrow \Box A_\nu - \partial_\nu\partial^\mu A_\mu = 4\pi J_\nu$ is second eq.

If $A$ solves $\partial^\mu F_{\mu\nu} = 4\pi J_\nu$ with $F = dA$, and $\chi \in C^\infty(\mathbb{R}^4)$, then so does $\tilde A = A + d\chi$

(since $\tilde F = d\tilde A = dA + \underbrace{dd\chi}_{=0} = F$). Here, addition of $d\chi$ to $A$ represents gauge freedom.

Fixing gauge: E.g. choose Lorentz gauge $\partial^\mu \tilde A_\mu = 0$. This can be arranged by

solving $\Box\chi = -\partial^\mu A_\mu$.

Since then $\partial^\mu \tilde A_\mu = \partial^\mu(A_\mu + \partial_\mu\chi) = \partial^\mu A_\mu + \Box\chi = 0$.

In this gauge, second equation simplifies to $\Box \tilde A_\nu = 4\pi J_\nu$ (inhomogeneous wave eq.)

$\rightsquigarrow$ residual gauge freedom $\hat A_\mu = \tilde A_\mu + d\eta$ with $\Box\eta = 0$.

Remark (spin 2 particle)

Equations (LEWG) with $\overset{(1)}{T} \equiv 0$ describe a massless spin-2 field on Minkowski spacetime.

$\rightsquigarrow$ "graviton"

Meaning only in linear approx. around Minkowski spacetime.

## II.3 Newtonian limit.

Newtonian theory well verified if

i) gravity is weak $\rightsquigarrow$ Linearised theory

Consider perfect fluid $\overset{(1)}{T}_{\mu\nu} = (\rho + p) u_\mu u_\nu + p\,\eta_{\mu\nu}$

ii) relative motion of sources much slower than the speed of light $c = 1$

$$\Longrightarrow \quad u \simeq \partial_t$$

iii) material stresses much smaller than mass-energy density

$$\Longrightarrow \quad p \simeq 0$$

$$\Longrightarrow \quad \overset{(1)}{T}_{\mu\nu} \simeq \rho\, dt \otimes dt \qquad (\sim \text{pressureless dust at rest})$$

Since sources are slowly varying, also expect spacetime geometry to vary slowly

$$\Longrightarrow \quad \partial_t \tilde{h}_{\mu\nu},\ \partial_t^2 \tilde{h}_{\mu\nu} \simeq 0 .$$

$$\text{(LEWG)} \Longrightarrow \begin{cases} \Delta \tilde{h}_{00} = -2\lambda\left(\rho - \tfrac{1}{2}\rho\right) = -\lambda\rho & (**) \quad \Big| \text{ use } \overset{(1)}{T} = -\rho \\ \Delta \tilde{h}_{0i} = 0 & \text{for } i = 1,2,3 \\ \Delta \tilde{h}_{ij} = -2\lambda\left(0 + \tfrac{1}{2}\eta_{ij}\rho\right) = -\eta_{ij}\lambda\rho & \text{for } i,j = 1,2,3 \end{cases}$$

Unique solution of $\Delta \tilde{h}_{\mu\nu} = 0$ s.t. $\tilde{h}_{\mu\nu} \longrightarrow 0$ for $r \longrightarrow \infty$ is $\tilde{h}_{\mu\nu} \equiv 0$.

ex. $\Longrightarrow \partial^\mu \bar{\tilde{h}}_{\mu\nu} = 0$ satisfied. $\left( \begin{array}{l} \Delta \bar{\tilde{h}}_{\mu\nu} = 0 \quad \text{for } (\mu\nu) \neq (0,0) \quad \rightsquigarrow \bar{\tilde{h}}_{\mu\nu} = 0 \\ \Delta \bar{\tilde{h}}_{00} = -2\lambda\rho \end{array} \right)$

$\rightsquigarrow$ whole content of gravity encoded in one scalar function $\tilde{h}_{00} = \tilde{h}_{ii}$, $i = 1,2,3$, which satisfies $\Delta \tilde{h}_{00} = -\lambda\rho$

Action on test body $\rightsquigarrow$ geodesic equation $\dfrac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\rho\sigma} \dfrac{dx^\rho}{d\tau}\dfrac{dx^\sigma}{d\tau} = 0$.

Non-relativistic motion $\left|\dfrac{dx^i}{d\tau}\right| \ll \dfrac{dt}{d\tau} \simeq 1$, $\rightsquigarrow t \simeq \tau$. $\quad i = 1,2,3$

Hence leading order contribution $\dfrac{d^2x^i}{dt^2} \simeq \dfrac{d^2x^i}{d\tau} \simeq -\Gamma^i{}_{00}(x^\mu(t))$.

Also $\Gamma^i{}_{00} = \dfrac{1}{2}\eta^{i\sigma}\left(2\partial_0 \tilde{h}_{0\sigma} - \partial_\sigma \tilde{h}_{00}\right) = -\dfrac{1}{2}\partial_i \tilde{h}_{00}$ $\rightsquigarrow$ only $\tilde{h}_{00}$ enters.

if we hadn't assumed that $\overset{(1)}{T}_{0i}$ negligible, would get contribution here. see Wald

$$\Longrightarrow \quad \frac{d^2x^i}{dt^2} \simeq \frac{1}{2}\partial_i \tilde{h}_{00} \qquad (*)$$

Compare with Newtonian theory:

$$\begin{cases} \Delta\phi = 4\pi\rho & \text{Poisson's equation } (G\equiv 1) \\ \vec{F} = -\vec{\nabla}\phi & \rightsquigarrow \quad \dfrac{d^2x^i}{dt^2} = -\partial_i\phi \end{cases}$$

Compare with (*) $\Rightarrow$ $\phi = -\frac{1}{2}\tilde{h}_{00}$

Compare with (**) $\quad -\lambda\rho = \Delta\tilde{h}_{00} = -2\Delta\phi \quad \Longrightarrow \quad \underline{\underline{\lambda = 8\pi}}$

[Newtonian theory: space & time a priori given by coords. Einstein: have to determine length and time geometrically. Can identify Minkowski $\leftrightarrow \mathbb{R}\times\mathbb{R}^3$ (Newton without mass). Now introduce small mass, in harmonic coords. computation of orbits etc (grav. exp.) reduce to Newtonian computations. Moreover, the "need" coords are approx. meters/sec. what $g^{ab}$ from geom. scale...]

Thus Einstein field equations $\quad \underline{\underline{G_{ab} = 8\pi T_{ab}}}$

(Perhaps remark on geodesic hypothesis coming from 2nd order approximation ($= \mathcal{O}(\varepsilon^2)$), so we use ... also aspects of higher orders as well $\rightsquigarrow$ Wald, p. 78)

## II.4 Far field of isolated gravitational body in linear approximation

Recall: Linearized Einstein equations in wave gauge:

$$(1) \qquad \Box\tilde{h}_{\mu\nu} = -16\pi\left(\overset{(1)}{T}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\overset{(1)}{T}\right) =: -16\pi\,\overset{(1)}{\hat{T}}_{\mu\nu}$$

$$\partial^\mu\bar{\tilde{h}}_{\mu\nu} = 0 \qquad , \qquad \bar{\tilde{h}}_{\mu\nu} = \tilde{h}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\tilde{h}$$

Here, looking for solution with
- $\partial^\mu\overset{(1)}{T}_{\mu\nu} = 0$ (of course!)
- $\overset{(1)}{T}_{\mu\nu}(t,\cdot)$ compact support in $\mathbb{R}^3$ for all $t\in\mathbb{R}$ (isolated gravitational body)
- $\partial_t\overset{(1)}{T}_{\mu\nu} = 0 = \partial_t\tilde{h}_{\mu\nu}$ (time independent (stationary $\leftrightarrow \partial_t$ KVF) solution)

$\Rightarrow$ (1) becomes $\Delta\tilde{h}_{\mu\nu} = -16\pi\,\overset{(1)}{\hat{T}}_{\mu\nu}$ where $\Delta = \eta^{ij}\partial_i\partial_j$, $i,j,k,\dots$ are space indices running from 1 to 3.

Interested in the asymptotics $\tilde{h}_{\mu\nu}(\underline{x})$ for $|\underline{x}|\to\infty$, $\underline{x} = (x^1,x^2,x^3)\in\mathbb{R}^3$

Recall: Poisson's equation $\Delta\varphi = f$ in $\mathbb{R}^3$, then $\varphi(\underline{x}) = -\displaystyle\int_{\mathbb{R}^3}\frac{f(\underline{x}')}{4\pi|\underline{x}-\underline{x}'|}\,d\underline{x}'$

Thus

$$\tilde{h}_{\mu\nu}(\underline{x}) = 4\int_{\mathbb{R}^3}\frac{\overset{(1)}{\hat{T}}_{\mu\nu}(\underline{x}')}{|\underline{x}-\underline{x}'|}\,d\underline{x}'$$

Check wave gauge is satisfied: $\bar{\tilde{h}}_{\mu\nu}(\underline{x}) = 4\displaystyle\int_{\mathbb{R}^3}\frac{1}{|\underline{x}-\underline{x}'|}\left(\overset{(1)}{\hat{T}}_{\mu\nu}(\underline{x}') - \tfrac{1}{2}\eta_{\mu\nu}\overset{(1)}{\hat{T}}(\underline{x}')\right)d\underline{x}' = 4\int_{\mathbb{R}^3}\frac{1}{|\underline{x}-\underline{x}'|}\overset{(1)}{T}_{\mu\nu}(\underline{x}')\,d\underline{x}'$

(Formally!) Thus $\partial^\mu\bar{\tilde{h}}_{\mu\nu}(\underline{x}) = 4\displaystyle\int_{\mathbb{R}^3}\partial^\mu_{\underline{x}}\left(\frac{1}{|\underline{x}-\underline{x}'|}\overset{(1)}{T}_{\mu\nu}(\underline{x}')\,d\underline{x}'\right) = -4\int_{\mathbb{R}^3}\left(\partial^\mu_{\underline{x}'}\frac{1}{|\underline{x}-\underline{x}'|}\right)\overset{(1)}{T}_{\mu\nu}(\underline{x}')\,d\underline{x}' \underset{\substack{\uparrow \\ T \text{ cpct. support}}}{=} 4\int_{\mathbb{R}^3}\frac{1}{|\underline{x}-\underline{x}'|}\underbrace{\partial^\mu_{\underline{x}'}\overset{(1)}{T}_{\mu\nu}(\underline{x}')}_{=0 \text{ (conservation eq. for } T_{\mu\nu})}\,d\underline{x}'$

Now: Asymptotics $\rightsquigarrow$ expand $\tilde{h}_{\mu\nu}(\underline{x})$ in powers of $\frac{1}{r}$.

Use following identities (2nd problem sheet, problem 7)

1) $$\frac{1}{|\underline{x}-\underline{x}'|} = \frac{1}{r} + \frac{1}{r^3}\,\underline{x}\cdot\underline{x}' + \mathcal{O}\left(\frac{1}{r^3}\right)$$ as a function of $\underline{x}$, uniformly for bounded $\underline{x}''$.

2) $$\int_{\mathbb{R}^3} T^{ij}(1,\underline{x})\,d\underline{x} = 0$$

3) $$\int_{\mathbb{R}^3} T^{0j}(1,\underline{x})\,d\underline{x} = 0$$

4) $$\int_{\mathbb{R}^3} T^i{}_i(1,\underline{x})\,d\underline{x} = 0$$

5) $$\int_{\mathbb{R}^3} (T^{0j}x^k + T^{0k}x^j)(1,\underline{x})\,d\underline{x} = 0$$

6) $$\int_{\mathbb{R}^3} T^i{}_i(1,\underline{x})\,x^j\,d\underline{x} = 0$$

• $\tilde{h}_{00}(\underline{x})$ :
$$\overset{(1)}{\hat{T}}_{00} = \overset{(1)}{T}_{00} + \frac{1}{2}\overset{(1)}{T} = \overset{(1)}{T}_{00} + \frac{1}{2}\left(-\overset{(1)}{T}_{00} + \overset{(1)}{T}{}^i{}_i\right) = \frac{1}{2}\left(\overset{(1)}{T}_{00} + \overset{(1)}{T}{}^i{}_i\right)$$

Thus
$$\tilde{h}_{00}(\underline{x}) = 4\int_{\mathbb{R}^3}\frac{1}{r}\left(1 + \frac{\underline{x}\cdot\underline{x}'}{r^2} + \mathcal{O}\left(\frac{1}{r^2}\right)\right)\frac{1}{2}\left(\overset{(1)}{T}_{00} + \overset{(1)}{T}{}^i{}_i\right)(\underline{x}')\,d\underline{x}'$$
$$= \frac{2}{r}\int_{\mathbb{R}^3}\Big(\overset{(1)}{T}_{00} + \underset{\text{use 4)}}{\overset{(1)}{T}{}^i{}_i} + \frac{x_j(x')^j}{r^2}\big(\underset{(*)}{\overset{(1)}{T}_{00}(\underline{x}')} + \underset{\text{use 6)}}{\overset{(1)}{T}{}^i{}_i(\underline{x}')}\big) + \mathcal{O}\left(\frac{1}{r^2}\right)\Big)\,d\underline{x}'$$
$$= \frac{2M}{r} + \mathcal{O}\left(\frac{1}{r^3}\right)$$

where $M := \int_{\mathbb{R}^3}\overset{(1)}{T}_{00}(\underline{x}')\,d\underline{x}'$ total mass

and we have used (*) $\int_{\mathbb{R}^3}(x')^j\,\overset{(1)}{T}_{00}(\underline{x}')\,d\underline{x}' = 0$ by choosing the Cartesian coords. s.t. the origin is the centre of mass, i.e. exactly the above is true.

• $\tilde{h}_{0i}(\underline{x})$ :
$$\overset{(1)}{\hat{T}}_{0i} = \overset{(1)}{T}_{0i}$$

Thus
$$\tilde{h}_{0i}(\underline{x}) = 4\int_{\mathbb{R}^3}\frac{1}{r}\left(1 + \frac{\underline{x}\cdot\underline{x}'}{r^2} + \mathcal{O}\left(\frac{1}{r^2}\right)\right)\overset{(1)}{T}_{0i}(\underline{x}')\,d\underline{x}'$$

use (3)
$$= \frac{4}{r^3}x^j\int_{\mathbb{R}^3}x'_j\,\overset{(1)}{T}_{0i}(\underline{x}')\,d\underline{x}' + \mathcal{O}\left(\frac{1}{r^3}\right)$$

$$\overset{(1)}{T}_{0i}x'_j = \underbrace{\overset{(1)}{T}_{0(i}x'_{j)}}_{\substack{=0\\ \text{by 5)}}} + \overset{(1)}{T}_{0[i}x'_{j]}$$

$$= \frac{4}{r^3}x^j\int_{\mathbb{R}^3}\overset{(1)}{T}_{0[i}x'_{j]}\,d\underline{x}' + \mathcal{O}\left(\frac{1}{r^3}\right)$$

Now use $\quad J_k = \int_{\mathbb{R}^3} \varepsilon_{\ell m k} (x')^\ell \, \overset{(1)}{T}{}^{0m}(\underline{x}')\, d\underline{x}'$ $\quad$ total angular momentum around $x^k$ axis.

$$\leadsto \quad \varepsilon_{ijk} J^k = \int_{\mathbb{R}^3} \left[ (x')^i \, \overset{(1)}{T}{}^{0j}(\underline{x}') - (x')^j \, \overset{(1)}{T}{}^{0i}(\underline{x}') \right] d\underline{x}' = 2 \int_{\mathbb{R}^3} \overset{(1)}{T}_{0[i}\, x'_{j]} \, d\underline{x}'$$

(minus sign from lowering index)

$$\leadsto \quad \tilde{h}_{0i}(\underline{x}) = \frac{2}{r^3} \varepsilon_{ijk} x^j J^k + \mathcal{O}\left(\tfrac{1}{r^3}\right) \left( = \frac{2}{r^3} \underbrace{(\vec{x} \times \vec{J})_i}_{\text{cross-product in } \mathbb{R}^3} + \mathcal{O}\left(\tfrac{1}{r^3}\right) \right)$$

- $\underline{\tilde{h}_{ij}(\underline{x})}$: $\quad \hat{\overset{(1)}{T}}_{ij} = \overset{(1)}{T}_{ij} - \frac{1}{2} \eta_{ij} \left( -\overset{(1)}{T}_{00} + \overset{(1)}{T}{}^k{}_k \right)$

Thus

$$\tilde{h}_{ij}(\underline{x}) = 4 \int_{\mathbb{R}^3} \frac{1}{r} \left( 1 + \mathcal{O}\left(\tfrac{1}{r}\right) \right) \left( \overset{(1)}{T}_{ij} - \frac{1}{2} \eta_{ij} \left( -\overset{(1)}{T}_{00} + \overset{(1)}{T}{}^k{}_k \right) \right) (\underline{x}')\, d\underline{x}'$$

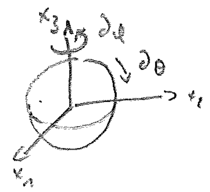
$$= \frac{4}{r} \int_{\mathbb{R}^3} \Big[ \underbrace{\overset{(1)}{T}_{ij}(\underline{x}')}_{\text{use (2)}} + \frac{1}{2} \eta_{ij} \underbrace{\left( \overset{(1)}{T}_{00} - \overset{(1)}{T}{}^k{}_k \right)(\underline{x}')}_{\text{use (4)}} + \mathcal{O}\left(\tfrac{1}{r}\right) \Big] \, d\underline{x}'$$

$$= \frac{2}{r} M \eta_{ij} + \mathcal{O}\left(\tfrac{1}{r^2}\right)$$

Thus obtain the asymptotic form of metric

$$\boxed{\begin{aligned} g &\simeq (\eta_{\mu\nu} + \varepsilon \tilde{h}_{\mu\nu})\, dx^\mu \otimes dx^\nu \\ &= -\left(1 - \frac{2\varepsilon M}{r} + \mathcal{O}\left(\tfrac{1}{r^2}\right)\right) dt^2 + \left( \frac{2}{r^3} \varepsilon_{ijk} x^j \varepsilon J^k + \mathcal{O}\left(\tfrac{1}{r^3}\right) \right) \left( dt \otimes dx^i + dx^i \otimes dt \right) \\ &\quad + \left( \left(1 + \frac{2\varepsilon M}{r}\right) \eta_{ij} + \mathcal{O}\left(\tfrac{1}{r^2}\right) \right) dx^i \otimes dx^j \end{aligned}}$$

Wlog can assume that $\vec{J} = J \partial_{x^3}$ (rotate coordinate system) and introduce spherical polar coords $(r, \theta, \varphi)$




Then $\quad \varepsilon_{ijk} x^j J^k dx^i = J(x^2 dx^1 - x^1 dx^2) = -J r^2 \sin^2\theta \, d\varphi$

$x^1 = r\cos\varphi \sin\theta$
$x^2 = r\sin\varphi \sin\theta$
$x^3 = r\cos\theta$

$\Rightarrow$ $x^2 dx^1 - x^1 dx^2 = r\sin\varphi\sin\theta \cdot r\sin\theta(-\sin\varphi)\, d\varphi$
$\qquad - r\cos\varphi\sin\theta \, r\cos\varphi\sin\theta \, d\varphi$
$\qquad = -r^2 \sin^2\theta \, d\varphi$

and thus

$$g \simeq -\left(1-\frac{2\varepsilon M}{r}+\mathcal{O}\left(\frac{1}{r^2}\right)\right)dt^2 - \frac{2}{r}\varepsilon J\sin^2\theta\,(dt\otimes d\varphi + d\varphi\otimes dt) + \mathcal{O}\left(\frac{1}{r^3}\right)(dt\otimes d\underline{x} + d\underline{x}\otimes dt)$$
$$+\left(1+\frac{2\varepsilon M}{r}\right)\left(dr^2 + r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\right) + \mathcal{O}\left(\frac{1}{r^2}\right)(dx^i\otimes dx^j + dx^j\otimes dx^i)$$

$\varepsilon M$ total mass of body, $\varepsilon J$ total angular momentum of body.

Remark:
- Compare with last section on Newtonian limit, $\phi = -\frac{1}{2}\tilde{h}_{00} = -\frac{1}{2}\tilde{h}_{ii} = -\frac{\varepsilon M}{r}$ $\rightsquigarrow$ Newtonian grav. potential of body with mass $\varepsilon M$.
- In contrast to last section we allowed relative motion of sources to be comparable to the speed of light ($\overset{(1)}{T}_{0i} \neq 0$), which gives us the $\tilde{h}_{0i}$ terms (problem: effect on geodesic motion?)
(allows rotations, compatible with stationarity, distribution of mass-energy, of stresses, of momentum does not change) – is it static? when is it static?)
- We allowed material stresses to be comparable to mass energy density ($\overset{(1)}{T}_{ij} \neq 0$).
- We restricted to the far field.
- The same asymptotic form of the metric can be derived for a stationary strongly gravitating isolated body in the fully non-linear theory.
  - ↳ Now, it doesn't in general hold anymore that $\int_{\mathbb{R}^3} T_{00}\, d^3\underline{x} = \varepsilon M$ and $\varepsilon J_k = \int_{\mathbb{R}^3} \varepsilon_{\ell m k}\,(x')^\ell\, T^{0m}(\underline{x}')\, d\underline{x}'$ but still define the parameters $\varepsilon M$ & $\varepsilon J$ appearing in the asymptotic form of the metric above to be mass & angular momentum of the spacetime. (another approach of defining M & J by Arnowitt, Deser, Misner using Hamiltonian formulation of GR)
  - ↳ M can be measured by looking at trajectory of test particle:
    As in Newtonian limit find $\frac{d^2x^i}{dt^2} \simeq \frac{1}{2}\partial_i\tilde{h}_{00} = \partial_i\left(\frac{\varepsilon M}{r}\right) + \mathcal{O}\left(\frac{1}{r^3}\right)$
    For large r Newton's laws are thus valid and can measure M e.g. from Kepler's third law $M \approx \omega^2 a^3$ ($\omega$: orbital period; $a$: semi-major axis of elliptic orbit).

    $\rightsquigarrow$ good 'definition' of M.
  - ↳ Can measure J from precession of gyroscope, see MTW p. 457

Example:

Example: Schwarzschild $g = -(1-\frac{2M}{r})dt^2 + \frac{1}{1-\frac{2M}{r}} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$

To bring into above form, find $\varrho(r)$ new coordinate s.t.

$$g = -A(\varrho)^2 dt^2 + B(\varrho)^2 \underbrace{[d\varrho^2 + \varrho^2(d\theta^2 + \sin^2\theta d\varphi^2)]}_{dx^2+dy^2+dz^2}$$

with $x = \varrho \sin\theta \cos\varphi$
⋮

$$\Rightarrow B(\varrho)^2\varrho^2 = r^2 \quad \& \quad B(\varrho)^2 d\varrho^2 = \frac{1}{1-\frac{2M}{r}} dr^2 \quad \& \quad A(\varrho)^2 = 1 - \frac{2M}{r}$$

$$\left(\frac{d\varrho}{dr}\right)^2 = \frac{1}{(1-\frac{2M}{r})B(\varrho)^2} = \frac{\varrho^2}{(1-\frac{2M}{r})r^2}$$

$$\Rightarrow \frac{d\varrho}{dr} = \pm \frac{\varrho}{(1-\frac{2M}{r})^{1/2} r} \quad \Rightarrow \log\varrho = \pm\int \frac{1}{r(1-\frac{2M}{r})^{1/2}} dr$$

We use $\int \frac{dy}{\sqrt{y^2-a^2}} = \ln\left|y + \sqrt{y^2-a^2}\right| + \text{const}$ (check or derive with trigonometric substitution $y = a\sec\theta = \frac{a}{\cos\theta}$ + $\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta|$)

$$\rightsquigarrow \log\varrho = \pm\int \frac{1}{\sqrt{r^2-2Mr}} dr = \pm\int \frac{1}{\sqrt{(r-M)^2 - M^2}} dr$$

$$= \pm\log\left(r - M + \sqrt{(r-M)^2 - M^2}\right) + \text{const}$$

$$= \pm\log\left(r - M + r\left(1-\frac{2M}{r}\right)^{1/2}\right) + \text{const.}$$

Use '+' sign s.t. $\varrho \to +\infty$ for $r \to +\infty$

Then $\underline{\varrho = C\left(r - M + r\left(1-\frac{2M}{r}\right)^{1/2}\right)}$ $C > 0$ constant

Solve for $r(\varrho)$ (to get $B(\varrho)^2 = \frac{r^2}{\varrho^2}$): $\varrho - C(r-M) = rC\left(1-\frac{2M}{r}\right)^{1/2}$

$$\rightsquigarrow (\varrho + CM - Cr)^2 = r^2C^2\left(1-\frac{2M}{r}\right)$$

$$\Leftrightarrow (\varrho+CM)^2 - 2(\varrho+CM)Cr + C^2r^2 = r^2C^2 - 2MrC^2$$

$$\Leftrightarrow (\varrho+CM)^2 - 2\varrho Cr = 0$$

$$\Leftrightarrow r = \frac{(\varrho+CM)^2}{2\varrho C} = \underline{\frac{\varrho}{2C}\left(1+\frac{CM}{\varrho}\right)^2}$$

This gives $B(\varrho)^2 = \frac{r^2}{\varrho^2} = \frac{1}{4C^2}\left(1+\frac{CM}{\varrho}\right)^4$ Choose $C = \frac{1}{2}$ s.t. $B^2(\varrho) \to 1$ for $\varrho \to +\infty$

32 additional i.e. $g$ is of the previous form to leading order.

$$\Rightarrow \quad \underline{r = \rho\left(1+\frac{M}{2\rho}\right)^2} \qquad \& \qquad \underline{B(\rho)^2 = \left(1+\frac{M}{2\rho}\right)^4}$$

$$\& \quad \underline{A(\rho)^2} = 1 - \frac{2M}{r} = 1 - \frac{2M}{\rho\left(1+\frac{M}{2\rho}\right)^2} = \frac{\rho\left(1+\frac{M}{2\rho}\right)^2 - 2M}{\rho\left(1+\frac{M}{2\rho}\right)^2}$$

$$= \frac{\rho + M + \frac{M^2}{4\rho} - 2M}{\rho\left(1+\frac{M}{2\rho}\right)^2} = \frac{\rho - M + \frac{M^2}{4\rho}}{\rho\left(1+\frac{M}{2\rho}\right)^2} = \underline{\frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2}}$$

$$\Rightarrow \quad g = -\frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2} dt^2 + \left(1+\frac{M}{2\rho}\right)^4 \underbrace{\left(d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right)}_{= dx^2 + dy^2 + dz^2}$$

$(t, \rho, \theta, \varphi)$ called <u>isotropic coordinates</u> for Schwarzschild.

Taylor expanding in $\frac{1}{\rho}$ gives
$$\left(1-\frac{M}{2\rho}\right)^2 = 1 - \frac{M}{\rho} + \dots$$
$$\frac{1}{\left(1+\frac{M}{2\rho}\right)^2} = 1 - \frac{M}{\rho} + \dots$$

$$\left(\left(\frac{1}{\left(1+\frac{Mx}{2}\right)^2}\right)'\right)\Big|_{x=0} = -\frac{2\cdot\frac{M}{2}}{1} = -M$$

$$\Rightarrow \quad g = -\left(1 - \frac{2M}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right)\right) dt^2 + \left(1 + \frac{2M}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right)\right)\left[dx^2 + dy^2 + dz^2\right]$$

Comparison $\Rightarrow$ Thus $M$ indeed the total mass and $\vec{J} = 0$.

## II.5 Gravitational waves

Now vacuum $T_{\mu\nu} \equiv 0$.
Recall linearised Einstein equations in wave gauge in vacuum

$$\begin{cases} \Box \tilde{h}_{\mu\nu} = 0 \\ \partial^\mu \bar{\tilde{h}}_{\mu\nu} = 0 \end{cases}$$

Recall we have residual gauge freedom of choosing linearised diffeomorphism $\eta_\mu$ that satisfies

$$\Box \eta_\mu = 0 \, ,$$

then can go over to $\hat{h}_{\mu\nu} := \tilde{h}_{\mu\nu} + \partial_\mu \eta_\nu + \partial_\nu \eta_\mu$ which is physically equivalent to $\tilde{h}_{\mu\nu}$ and still in wave gauge $\partial^\mu \bar{\hat{h}}_{\mu\nu} = 0$.

<u>Show:</u> Can choose $\eta$ (depending on $\tilde{h}_{\mu\nu}$) s.t. $\hat{h}_{0\mu} = 0 = \hat{h}$, $\mu = 0,1,2,3$.
[This requires $T_{\mu\nu} \equiv 0$ !]

<u>Idea illustrated by Maxwell's equations</u>:

Maxwell's eq. in Lorentz gauge
$$\begin{cases} \Box \tilde{A}_\mu = 0 \\ \partial^\mu \tilde{A}_\mu = 0 \end{cases}$$

and recall residual gauge freedom $\hat{A}_\mu = \tilde{A}_\mu + \partial_\mu \eta$ with $\Box \eta = 0$
(since then $\partial^\mu \hat{A}_\mu = \partial^\mu \tilde{A}_\mu + \Box \eta = 0$.)

Can use this freedom to set $\hat{A}_0 = 0$ (Coulomb or radiation gauge):

On $\{t = 0\}$ solve Poisson's eq. $\Delta \eta_0|_{t=0} = -\partial^i \tilde{A}_i|_{t=0}$ $\left(= -\operatorname{div}(\tilde{A}^1\partial_1 + \tilde{A}^2\partial_2 + \tilde{A}^3\partial_3)\right)$

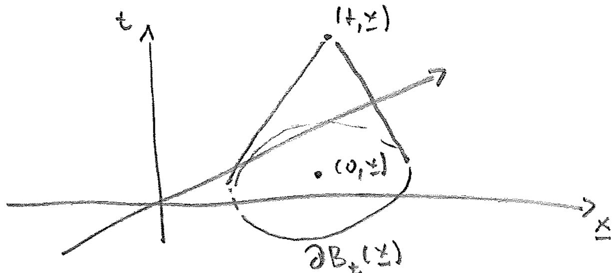
$\rightsquigarrow$ get solution $\eta_0$

Then solve wave equation $\Box \eta = 0$ with initial data
$$\begin{cases} \eta|_{t=0} = \eta_0 \\ \partial_t \eta|_{t=0} = \eta_1 = -\tilde{A}_0 \end{cases}$$

There exists a unique such solution. Indeed, it is given by Kirchhoff's formula

$$\eta(t, \underline{x}) = \frac{1}{4\pi t^2} \int_{\partial B_t(\underline{x})} \left( \eta_0(\underline{y}) + \nabla \eta_0(\underline{y}) \cdot (\underline{y} - \underline{x}) + t\, \eta_1(\underline{y}) \right) d\sigma(\underline{y})$$

(Non-examinable)



Then $f := \tilde{A}_0 + \partial_t \eta$ satisfies $\Box f = \Box \tilde{A}_0 + \partial_t \Box \eta = 0$

and we have $f|_{t=0} = 0$ & $\partial_t f|_{t=0} = (\partial_0 \tilde{A}_0 + \partial_t^2 \eta)|_{t=0}$

$$\underset{\substack{\partial^\mu \tilde{A}_\mu = 0 \\ \Box = -\partial_t^2 + \Delta}}{=} (\partial^i \tilde{A}_i + \Delta \eta)|_{t=0}$$

$$= 0$$

Thus $f \equiv 0$ by Kirchhoff's formula and thus $\hat{A}_0 = \tilde{A}_0 + \partial_0 \eta = 0$.

---

Now for linearised gravity:

Solve on $\{t=0\}$

$$\left.\begin{aligned} 2(-\partial_t \eta_0 + \partial^i \eta_i) &= -\tilde{h} && (1) \\ 2[-\Delta \eta_0 + \partial^i \partial_t \eta_i] &= -\partial_t \tilde{h} && (2) \\ \partial_t \eta_i + \partial_i \eta_0 &= -\tilde{h}_{0i} && (3) \\ \Delta \eta_i + \partial_i \partial_t \eta_0 &= -\partial_t \tilde{h}_{0i} && (4) \end{aligned}\right\}$$

to obtain $\eta_\mu|_{t=0}$, $\partial_t \eta_\mu|_{t=0}$.

Exercise: Show that one can indeed solve the above system to obtain $\eta_\mu|_{t=0}$, $\partial_t \eta_\mu|_{t=0}$ in terms of $\tilde{h}_{\mu\nu}$.

Solution: Put (3) into (2) to obtain $2[-\Delta \eta_0 + \partial^i(-\partial_i \eta_0 - \tilde{h}_{0i})] = -\partial_t \tilde{h}$

$$\Leftrightarrow 2[-2\Delta \eta_0 - \partial^i \tilde{h}_{0i}] = -\partial_t \tilde{h}$$

$\rightsquigarrow$ gives $\eta_0|_{t=0}$.

Use $\eta_0|_{t=0}$ in (3) to obtain $\partial_t \eta_i|_{t=0}$. Then $\partial_t \eta_i|_{t=0}$ & $\eta_0|_{t=0}$ satisfy (2) & (3).

Similarly use (1) to eliminate $\partial_t \eta_0$ from (4) to get Poisson eq for $\eta_i$. Then use (1) to get $\partial_t \eta_0|_{t=0}$. ∎

Now solve $\Box \eta_\mu = 0$ with determined initial data and it follows as before that

$$\hat{h}_{\mu\nu} = \tilde{h}_{\mu\nu} + \partial_\mu \eta_\nu + \partial_\nu \eta_\mu$$

satisfies

$$\Box \hat{h}_{\mu\nu} = 0 \qquad \text{and} \qquad \hat{h}|_{t=0} = 0 = \partial_t \hat{h}|_{t=0} \qquad (\text{from } ① \;\&\; ②)$$

$$\text{and} \qquad \hat{h}_{0i}|_{t=0} = 0 = \partial_t \hat{h}_{0i}|_{t=0} \qquad (\text{from } ③ \;\&\; ④)$$

$$\Rightarrow \quad \hat{h} \equiv 0 \equiv \hat{h}_{0i}$$

In order to see that also $\hat{h}_{00} = 0$, we observe that since $\hat{h} = 0$, we have $\hat{h}_{\mu\nu} = \bar{\hat{h}}_{\mu\nu}$.

$$\Rightarrow \quad \underset{\substack{\uparrow \\ \text{wave} \\ \text{gauge}}}{0} = \partial^\mu \bar{\hat{h}}_{\mu 0} = \partial^\mu \hat{h}_{\mu 0} \underset{\substack{\uparrow \\ \hat{h}_{0i}=0}}{=} \partial^0 \hat{h}_{00} \quad \leadsto \quad \partial_t \hat{h}_{00} = 0$$

Then use linearized EE in wave gauge $\quad 0 = \Box \hat{h}_{00} = \Delta \hat{h}_{00}$.

$$\leadsto \quad \hat{h}_{00} \equiv 0$$

( $\hat{h}_{00}$ = const. is ruled out since we from the very beginning assumed $\partial^{(k)} h_{\mu\nu} \to 0$ for $r \to \infty$ and all our gauge choices respect this condition)

**Example**: Consider plane wave solutions $\quad \hat{h}_{\mu\nu}(x) = \hat{H}_{\mu\nu}(k)\, e^{ik_\alpha x^\alpha}$, $\quad k = (k_0, \dots, k_3) \in \mathbb{R}^4$

( Note that $\hat{h}_{\mu\nu} \not\to 0$ for $r \to \infty$, but the actual solution will be a superposition of real part of plane wave solutions which will satisfy it as usual story ...)

- $\Box \hat{h}_{\mu\nu} = 0 \quad \Leftrightarrow \quad \eta^{\alpha\beta} k_\alpha k_\beta = 0 \quad \leadsto \quad k$ null vector
- $\partial^\mu \bar{\hat{h}}_{\mu\nu} = \partial^\mu \hat{h}_{\mu\nu} = 0 \quad \Leftrightarrow \quad \underline{k^\mu \hat{H}_{\mu\nu}(k) = 0} \qquad ( \underset{3 \text{ cond.}}{k^\mu \hat{H}_{\mu i}(k) = 0}$, $\quad k^\mu \hat{H}_{\mu 0}(k) = 0$ implied by $)$
- $\hat{h}_{0\mu} = 0 \quad \Leftrightarrow \quad \underline{\hat{H}_{0\mu}(k) = 0} \qquad (4 \text{ cond})$
- $\hat{h} = 0 \quad \Leftrightarrow \quad \underline{\underline{\hat{H}^\mu{}_\mu(k) = 0}} \qquad (1 \text{ cond.})$

$\hat{H}_{\mu\nu}(k)$ symmetric $\leadsto$ 10 dof $\quad + \quad$ 8 cond. $\quad = \quad$ 2 degrees of freedom.

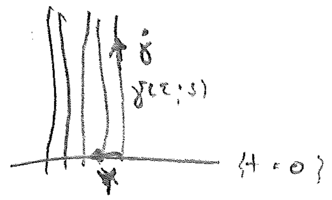
e.g. if $\quad k = \omega(\partial_t + \partial_3) \quad$ (grav. wave propagating in z-direction)

then $$\hat{H}_{\mu\nu}(k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A(k) & B(k) & 0 \\ 0 & B(k) & -A(k) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\leadsto$ distortion of spacetime geometry is transverse to the direction of propagation.

Detection of gravitational waves via gravitational tidal forces.

If $\gamma(\tau;s)$ is a one parameter family of timelike geodesics $\tau \longmapsto \gamma(\tau;s)$ and let $Y^\mu(\gamma(\tau,s)) = \frac{\partial}{\partial s}\gamma^\mu(\tau;s)$ ) be the deviation vector.




Then
$$D_\tau^2 Y) = R(\dot\gamma, Y)\dot\gamma \qquad (*)$$
$$\left( \left(\nabla_{\dot\gamma}(\nabla_{\dot\gamma} Y)\right)^\mu = R^\mu{}_{\nu\rho\sigma}\dot\gamma^\nu \dot\gamma^\rho Y^\sigma \right)$$

Set up s.t. $s \mapsto \gamma(0;s) \subseteq \{t=0\}$ and

Assume that test particles are nearly at rest wrt Minkowskian coords $x^\mu$ $\rightsquigarrow$ $\dot\gamma \simeq \partial_t$ + lower order terms.

and thus $\tau \simeq t$. Also to highest order $D_\tau^2 \simeq \left(\frac{d}{d\tau}\right)^2 \simeq \left(\frac{d}{dt}\right)^2$ and thus

$(*)$ becomes
$$\frac{d^2}{dt^2}Y^\mu \simeq R^\mu{}_{00\sigma}Y^\sigma$$

Moreover $Y^0(0;s) = \partial_s\gamma^0(0;s) = 0$ & $\frac{d}{dt}\Big|_{t=0} Y^0(t;s) \simeq \frac{\partial}{\partial s}\frac{\partial}{\partial \tau}\Big|_{\tau=0}\gamma^0(\tau;s) \simeq \frac{\partial}{\partial s}1 = 0$. Since $R^0{}_{00\sigma} = 0$, $\Rightarrow Y^0 \equiv 0$

Recall $R^\mu{}_{\kappa\rho\nu} \simeq \varepsilon\frac{1}{2}\eta^{\mu\sigma}(\partial_\rho\partial_\kappa\hat h_{\nu\sigma} - \partial_\rho\partial_\sigma\hat h_{\nu\kappa} - \partial_\nu\partial_\kappa\hat h_{\rho\sigma} + \partial_\nu\partial_\sigma\hat h_{\rho\kappa})$

and thus
$$R^\mu{}_{00\nu} \simeq \left(\varepsilon\frac{1}{2}\eta^{\mu\sigma}\left(\partial_0\partial_0\hat h_{\nu\sigma} - \partial_0\partial_\sigma\hat h_{\nu 0} - \partial_\nu\partial_0\hat h_{0\sigma} + \partial_\nu\partial_\sigma\hat h_{00}\right)\right)$$
$$\simeq \varepsilon\frac{1}{2}\eta^{\mu\sigma}\partial_0\partial_0\hat h_{\nu\sigma}$$
using radiation gauge $\hat h_{\mu 0} = 0$.

$$\Rightarrow \boxed{\frac{d^2}{dt^2}Y^i \simeq \frac{1}{2}\varepsilon(\partial_t^2\hat h_{ij})\cdot Y^j}$$

Now $\frac{d}{dt}Y^i(0) = 0$ (test particles at rest initially) and since $\varepsilon$ very small get

$Y^i(t) \simeq Y^i(0)$ + small

To get leading order of correction $\frac{d^2}{dt^2}Y^i(t) \simeq \frac{1}{2}\varepsilon\,\partial_t^2\hat h_{ij}\cdot Y^j(0)$

$\frac{d}{dt}Y^i(0) = 0 \rightsquigarrow$
$$\boxed{Y^i(t) \simeq Y^i(0) + \frac{\varepsilon}{2}\hat h_{ij}(t)\,Y^j(0)}$$

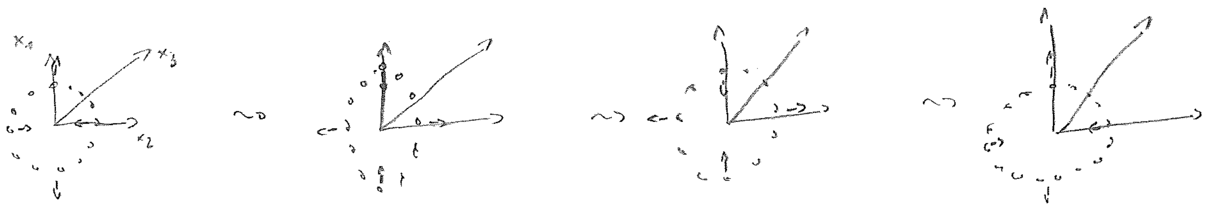
Example: Evaluate for plane wave solution $\hat h_{\mu\nu}(x) = \begin{pmatrix} 0&0&0&0\\ 0&A&B&0\\ 0&B&-A&0\\ 0&0&0&0\end{pmatrix} \mathrm{Re}\left(e^{-i\omega(t-x_3)}\right)$

(But of course in general won't have this harmonic dependence!) $= \cos(\omega(t-x_3))$

Linear polarisations:

i) $B = 0$ , "+" polarisation

$$Y^1(t) \simeq Y^1(0) + \frac{\varepsilon}{2} A \cos(\omega(t - x_3))\, Y^1(0)$$
$$Y^2(t) \simeq Y^2(0) - \frac{\varepsilon}{2} A \cos(\omega(t - x_3))\, Y^2(0)$$
$$Y^3(t) = Y^3(0)$$




Distances in x & y directions oscillate ⇝ can be measured with lasers in interferometer (LIGO)

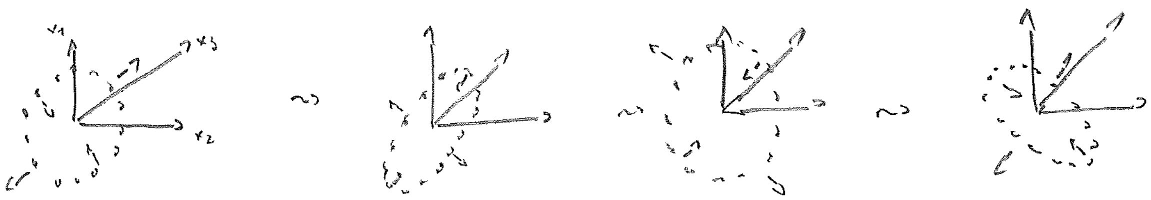
(Don't think position in absolute space...)

ii) $A = 0$ , "x" polarisation

$$Y^1(t) \simeq Y^1(0) + \frac{\varepsilon}{2} B \cos(\omega(t - x_3))\, Y^2(0)$$
$$Y^2(t) \simeq Y^2(0) + \frac{\varepsilon}{2} B \cos(\omega(t - x_3))\, Y^1(0)$$
$$Y^3(t) = Y^3(0)$$

⇝ $(Y^1 + Y^2)(t) = Y^1(0) + Y^2(0) + \frac{\varepsilon}{2} B \cos(\ldots)\,(Y^1(0) + Y^2(0))$

$(Y^1 - Y^2)(t) = Y^1(0) - Y^2(0) - \frac{\varepsilon}{2} B \cos(\ldots)\,(Y^1(0) - Y^2(0))$




⇝ circular polarisations (MTW, p. 952)

Remark: very weak effect : LIGO : $\frac{1}{1000}$ diameter of proton over 4 km

$\underbrace{\qquad}_{\approx 10^{-15}\,m}$

$\approx 10^{-18}\,m$

## Generation of gravitational waves

- Gravitational waves produced by collision of two BH's ⇝ outside validity of linear approximation
- Here, derive formula for generation of grav. waves by (weak) non-self-gravitating sources in the linear approximation (spinning rods, rotating inhomogeneous stars) in the far field

Moreover Assume:
- $\overset{(1)}{T}_{\mu\nu}(t, \underline{x})$ compactly supported $\forall t$, coords s.t. $T_{\mu\nu}(t, \underline{x})$ centred around $\underline{x} = 0$.
- Charact. time over which source varies small compared to spatial extent of source. (e.g. rotation slow) (same as: typical wavelength of radiation much larger than extent of source)

Recall
$$\begin{cases} \Box \tilde{\bar{h}}_{\mu\nu} = -16\pi \overset{(1)}{T}_{\mu\nu} \\ \partial^\mu \tilde{\bar{h}}_{\mu\nu} = 0 \end{cases} \qquad \tilde{\bar{h}}_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{h}$$

Retarded solution (no incoming radiation)

$$\tilde{\bar{h}}_{\mu\nu}(t,\underline{x}) = 4\int_{\mathbb{R}^3} \frac{\overset{(1)}{T}_{\mu\nu}(t-|\underline{x}'-\underline{x}|, \underline{x}')}{|\underline{x}'-\underline{x}|} d\underline{x}'$$

**Ex**: show that gauge condition $\partial^\mu \tilde{\bar{h}}_{\mu\nu} = 0$ is satisfied by virtue of $\partial^\mu \overset{(1)}{T}_{\mu\nu} = 0$.

Pf

$$\partial^0 \tilde{\bar{h}}_{0\nu}(t,\underline{x}) = 4\int_{\mathbb{R}^3} \frac{\partial^0 \overset{(1)}{T}_{0\nu}(t-|\underline{x}'-\underline{x}|, \underline{x}')}{|\underline{x}'-\underline{x}|} d\underline{x}'$$

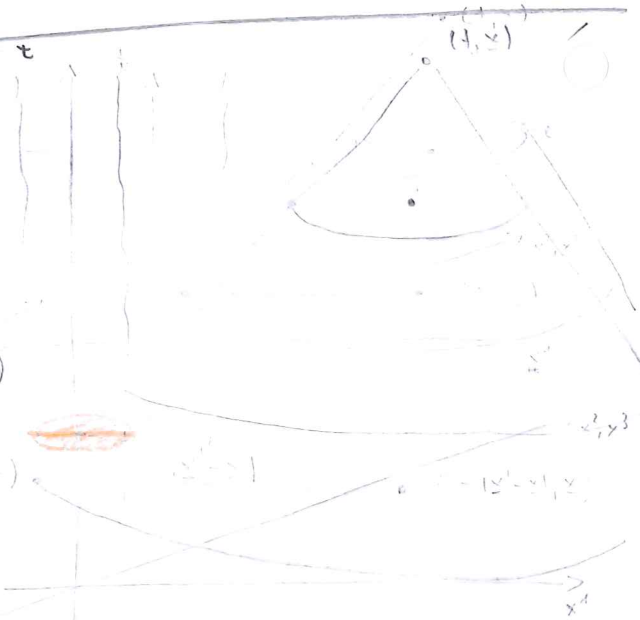
$$\partial^i \tilde{\bar{h}}_{i\nu}(t,\underline{x}) = 4\int_{\mathbb{R}^3} \left[ \frac{\partial_i\left[\overset{(1)}{T}_{i\nu}(t-|\underline{x}'-\underline{x}|, \underline{x}')\right]}{|\underline{x}'-\underline{x}|} + \frac{\overset{(1)}{T}_{i\nu}(t-|\underline{x}'-\underline{x}|, \underline{x}')}{1} \underbrace{\partial_i\left(\frac{1}{|\underline{x}'-\underline{x}|}\right)}_{= -\partial_i'\left(\frac{1}{|\underline{x}'-\underline{x}|}\right)} \right] d\underline{x}'$$

I.P. since $T_{\mu\nu}$ is compactly supported on light cone

$$\overset{?}{=} 4\int_{\mathbb{R}^3} \frac{(\partial_i + \partial_i')\left[\overset{(1)}{T}_{i\nu}(t-|\underline{x}'-\underline{x}|, \underline{x}')\right]}{|\underline{x}'-\underline{x}|} d\underline{x}'$$

$\partial_i|\underline{x}'-\underline{x}| = -\partial_i'|\underline{x}'-\underline{x}|$
$$\overset{?}{=} 4\int_{\mathbb{R}^3} \frac{(\partial^i \overset{(1)}{T}_{i\nu})(t-|\underline{x}'-\underline{x}|, \underline{x}')}{|\underline{x}'-\underline{x}|} d\underline{x}'$$

$$\simeq \frac{4}{r}\int_{\mathbb{R}^3} \overset{(1)}{T}_{\mu\nu}(t-|\underline{x}'-\underline{x}|, \underline{x}')\, d\underline{x}' + \mathcal{O}\left(\frac{1}{r^2}\right) \qquad \left(\text{for field approx}, \frac{1}{|\underline{x}'-\underline{x}|} = \frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right)$$



We next expand the integrand using the slow motion approximation, more precisely assuming that if $\omega$ is a characteristic frequency of change of $\overset{(1)}{T}_{\mu\nu}$ and $R_0$ is the radius of its support, then $R_0 \omega \ll 1$.

Let $f(\underline{x}') = t - |\underline{x}' - \underline{x}|$. Then

- $\partial_i f(\underline{x}') = -\dfrac{x_i' - x_i}{\sqrt{\sum_j (x_j' - x_j)^2}}$, $\quad \partial_i f(0) = \dfrac{x_i}{r} = \mathcal{O}(1)$

- $\partial_k \partial_i f(\underline{x}') = -\dfrac{\delta_{ik}}{|\underline{x}' - \underline{x}|} + \dfrac{(x_i' - x_i)(x_k' - x_k)}{|\underline{x}' - \underline{x}|^3}$, $\quad \partial_k \partial_i f(0) = -\dfrac{\delta_{ik}}{r} + \dfrac{x_i x_k}{r^3} = \mathcal{O}\left(\dfrac{1}{r}\right)$

- $\partial_\ell \partial_k \partial_i f(\underline{x}') = \mathcal{O}\left(\dfrac{1}{r^2}\right)$ for $|\underline{x}'| \le R_0$.

Also $\partial_i \overset{(1)}{T}_{\mu\nu}(f(\underline{x}'), \underline{x}') = \partial_0 \overset{(1)}{T}_{\mu\nu}(f(\underline{x}'), \underline{x}') \cdot \partial_i f(\underline{x}')$, $\quad \partial_k \partial_i \overset{(1)}{T}_{\mu\nu}(f(\underline{x}'), \underline{x}') = \partial_0^2 \overset{(1)}{T}(f(\underline{x}'), \underline{x}')\, \partial_i f(\underline{x}')\, \partial_k f(\underline{x}') + \partial_0 \overset{(1)}{T}(f(\underline{x}), \underline{x}')\, \partial_k \partial_i f(\underline{x}')$

Then
$$\overset{(1)}{T}_{\mu\nu}(f(\underline{x}'), \underline{x}') = \overset{(1)}{T}_{\mu\nu}(f(0), \underline{x}') + \partial_0 \overset{(1)}{T}_{\mu\nu}(f(0), \underline{x}')\, \partial_i f(0)\, x_i'$$
$$+ \frac{1}{2}\left[\partial_0^2 \overset{(1)}{T}_{\mu\nu}(f(0), \underline{x}')\, \partial_k f(0)\, \partial_i f(0) + \partial_0 \overset{(1)}{T}_{\mu\nu}(f(0), \underline{x}')\, \partial_k \partial_i f(0)\right] x_i' x_k'$$
$$+ \frac{1}{6}\left[\partial_0^3 \overset{(1)}{T}(f(\underline{\xi}'), \underline{x}')\, \partial_i f(\underline{\xi}')\, \partial_k f(\underline{\xi}')\, \partial_\ell f(\underline{\xi}') + \mathcal{O}\left(\frac{1}{r^2}\right)\right] x_i' x_k' x_\ell'$$

Assume now $\overset{(1)}{T}_{\mu\nu}(t', \underline{x}') = \tau_{\mu\nu}(\underline{x}') \cos(\omega t')$. Then $\left|\partial_0^k \overset{(1)}{T}_{\mu\nu}(t', \underline{x}')\right| \lesssim \omega^k$

Thus we obtain

$$\boxed{\begin{aligned} \overset{(1)}{T}_{\mu\nu}(t - |\underline{x}' - \underline{x}|, \underline{x}') &= \overset{(1)}{T}_{\mu\nu}(t - r, \underline{x}') + \underbrace{\partial_0 \overset{(1)}{T}_{\mu\nu}(t - r, \underline{x}')\, \frac{x_i}{r} x_i'}_{\mathcal{O}(\omega R_0)} \\ &+ \underbrace{\frac{1}{2} \partial_0^2 \overset{(1)}{T}_{\mu\nu}(t - r, \underline{x}')\, \frac{x_i}{r} \frac{x_k}{r} x_i' x_k'}_{\mathcal{O}(\omega^2 R_0^2)} + \mathcal{O}\left(\frac{1}{r^2}\right) + \mathcal{O}(\omega^3 R_0^3 |\tau_{\mu\nu}|) \end{aligned}}$$

Now compute metric to leading order in $\frac{1}{r}$ and $\omega R_0$:

- $\tilde{\bar{h}}_{ij}(t, \underline{x}) \simeq \dfrac{4}{r} \displaystyle\int_{\mathbb{R}^3} \overset{(1)}{T}_{ij}(t - r, \underline{x}')\, d\underline{x}'$ + higher order terms.

$$= \underbrace{\frac{2}{r} \frac{d^2}{dt^2} Q_{ij}(t - r)}_{\mathcal{O}(\omega^2 R_0^2)}, \quad \text{where } Q_{ij}(t) = \int_{\mathbb{R}^3} T_{00}(t, \underline{x}')\, x_i' x_j'\, d\underline{x}' \sim R_0^2$$

radial propagation $\qquad$ quadrupole moment

Problem sheet 2

• $\tilde{\bar{h}}_{0i}(t,\underline{x}) = \underbrace{\frac{4}{r}\int_{\mathbb{R}^3} \overset{(1)}{T}_{0i}(t-r,\underline{x}')\,d\underline{x}'}_{= -\frac{4}{r}P^i(t-r) \overset{\uparrow}{=} 0} + \frac{4}{r}\frac{x_j}{r}\int_{\mathbb{R}^3} \partial_0 \overset{(1)}{T}_{0i}(t-r,\underline{x}')\,x_j'\,d\underline{x}'$ + higher order terms

By Problem sheet 2 $P^i = \text{const}$, can choose s.t. $P^i \equiv 0$

$\partial_\mu \overset{(1)}{T}^{\mu\nu} = 0$

$$\overset{\downarrow}{=} \frac{4}{r}\frac{x_j}{r}\int_{\mathbb{R}^3} \partial_k \overset{(1)}{T}{}^{ki}(t-r,\underline{x}')\,x_j'\,d\underline{x}' \quad + \text{ higher order terms}$$

$$= -\frac{4}{r}\frac{x_j}{r}\int_{\mathbb{R}^3} \overset{(1)}{T}{}^{ji}(t-r,\underline{x}')\,d\underline{x}' + \ldots$$

$$= -\frac{2}{r}\frac{x_j}{r}\underbrace{\frac{d^2}{dt^2}Q_{ij}(t-r)}_{O(\omega^2 R_0^2)} + \text{ higher order terms.}$$

• $\tilde{\bar{h}}_{00}(t,\underline{x}) = \underbrace{\frac{4}{r}\int_{\mathbb{R}^3} \overset{(1)}{T}_{00}(t-r,\underline{x}')\,d\underline{x}'}_{=\frac{4M}{r}} + \underbrace{\frac{4}{r}\frac{x_i}{r}\int_{\mathbb{R}^3}\partial_0 \overset{(1)}{T}_{00}(t-r,\underline{x}')\,x_i'\,d\underline{x}'}_{\sim \frac{d}{dt}D^i = P^i = 0}$

$= \frac{4M}{r}$ ← time indep, non-radiating contribution.

$$+ \frac{4}{r}\cdot\frac{1}{2}\frac{x_i}{r}\frac{x_k}{r}\int_{\mathbb{R}^3}\partial_0^2 \overset{(1)}{T}_{00}(t-r,\underline{x}')\,x_i'x_k'\,d\underline{x}' \quad + \text{h.o.t}$$

$$= \frac{4M}{r} + \underbrace{\frac{2}{r}\frac{x_i}{r}\frac{x_k}{r}\frac{d^2}{dt^2}Q_{ik}(t-r)}_{O(\omega^2 R_0^2)} + \text{h.o.t.}$$

$$\Rightarrow \left.\begin{aligned} \tilde{\bar{h}}_{00}(t,\underline{x}) &\simeq \frac{4M}{r} + \frac{2}{r}\frac{x_i}{r}\frac{x_k}{r}\frac{d^2}{dt^2}Q_{ik}(t-r) \\ \tilde{\bar{h}}_{0i}(t,\underline{x}) &\simeq -\frac{2}{r}\frac{x_j}{r}\frac{d^2}{dt^2}Q_{ij}(t-r) \\ \tilde{\bar{h}}_{ij}(t,\underline{x}) &\simeq \frac{2}{r}\frac{d^2}{dt^2}Q_{ij}(t-r) \end{aligned}\right\} \quad (*)$$

Often called the **Quadrupole formula**

Check: Wave gauge is satisfied to highest order: Use $\partial_i r = \frac{x_i}{r}$, then

$$\partial^i \tilde{\bar{h}}_{ij} \simeq -\frac{2}{r}\frac{x_i}{r}\frac{d^3}{dt^3}Q_{ij}(t-r)$$
$$\partial^0 \tilde{\bar{h}}_{0j} \simeq \frac{2}{r}\frac{x_i}{r}\frac{d^3}{dt^3}Q_{ij}(t-r)$$
$$\partial^i \tilde{\bar{h}}_{0i} \simeq \frac{2}{r}\frac{x_i}{r}\frac{x_j}{r}\frac{d^3}{dt^3}Q_{ij}(t-r)$$
$$\partial^0 \tilde{\bar{h}}_{00} \simeq -\frac{2}{r}\frac{x_i}{r}\frac{x_j}{r}\frac{d^2}{dt^2}Q_{ij}(t-r)$$

✓

Remarks:

• Monopole moment $\int_{\mathbb{R}^3} \overset{(1)}{T}_{00}(t,\underline{x}')\,d\underline{x}' = M$ indep. of time

• Dipole moment $\int_{\mathbb{R}^3} \overset{(1)}{T}_{00}(t,\underline{x}')\,x_i'\,d\underline{x}' = D^i(t)$ ~ centre of mass ( = 0 wlog, since momentum $\frac{d}{dt}D^i(t) = P^i(t)$ is conserved and ...

• Lowest moment that radiates in GR is quadrupole moment (in contrast to electromagnetism, where dipole moment gives leading order (40) of radiation (see problem sheet 2)

Using $\tilde{h}_{\mu\nu} = \bar{\tilde{h}}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{\tilde{h}}$ we can now compute $\tilde{h}_{\mu\nu}$ from (*) and then $R^i{}_{00j}$ of the metric $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \varepsilon \tilde{h}_{\mu\nu}$ to leading order in $\varepsilon$ and $\left(\frac{1}{r}\right)$ we find (problem sheet 3)

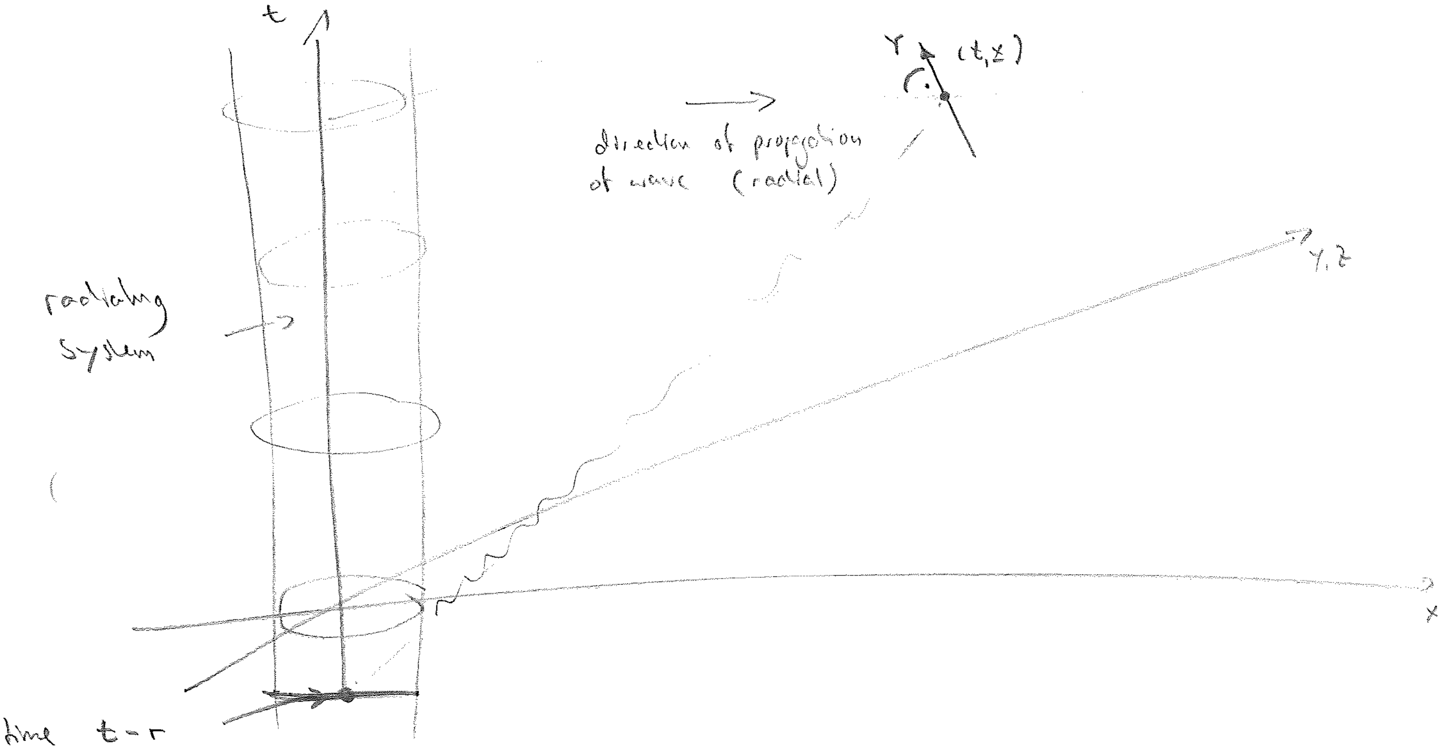
(**) $$R^i{}_{00j} \simeq \frac{\varepsilon}{r}\left[\pi_i{}^m \pi_j{}^n - \frac{1}{2}\pi^{mn}\pi_{ij}\right]\frac{d^4}{dt^4}Q_{mn}(t-r)$$ where $\pi^{mn}(x) = \delta^{mn} - \frac{x_m}{r}\frac{x_n}{r}$ is projection on 2-space orthogonal to $x \in \mathbb{R}^3$

(tangent space of sphere of radius $|x|$ at point $x$).

For detection of gravitational waves set up one-parameter family of timelike geodesics as before, with Jacobi field $Y$

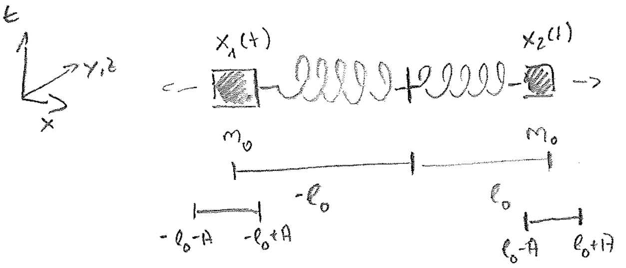
$$\Rightarrow \quad \frac{d^2}{dt^2}Y^i \simeq R^i{}_{00j}Y^j \simeq \frac{\varepsilon}{r}\left[\pi_i{}^m \pi_j{}^n - \frac{1}{2}\pi^{mn}\pi_{ij}\right]\frac{d^4}{dt^4}Q_{mn}(t-r)\, Y^j$$

( as before $\Rightarrow$ $$\boxed{Y^i(t) \simeq Y^i(0) + \frac{\varepsilon}{r_i}\left[\pi_i{}^m \pi_j{}^n - \frac{1}{2}\pi^{mn}\pi_{ij}\right]\frac{d^2}{dt^2}Q_{mn}(t-r)\, Y^j(0)}$$




Remarks. Oscillation of test masses only in directions in $\mathbb{R}^3$ orthogonal to propagation direction of wave ( $\pi_{ij}x^j = 0$. )

Example: Laboratory gravitational wave generator




Two masses with mass $m_0$ oscillating with angular frequency $\omega$ and amplitude $A$ around positions $\pm l_0$ on $x$-axis.

$$\rightsquigarrow \quad x_1(t) = -l_0 - A\cos(\omega t)$$
$$x_2(t) = l_0 + A\cos(\omega t)$$

$$\overset{(1)}{T}_{00}(t,x,y,z) \underset{\uparrow}{\approx} m_0\,\delta^3(x - x_1(t)) + m_0\,\delta^3(x - x_2(t))$$

movement slow compared to speed of light.

$$\Rightarrow \quad Q_{xx}(t) = \int_{\mathbb{R}^3} \overset{(1)}{T}(t,x',y',z')\, x'x'\, dx'dy'dz' = m_0\left[x_1^2(t) + x_2^2(t)\right]$$

$$= 2m_0\left[l_0 + A\cos(\omega t)\right]^2$$

$$= 2m_0\left[l_0^2 + 2 l_0 A\cos(\omega t) + A^2\cos^2(\omega t)\right]$$

All other components $Q_{..} = 0$. It follows from $\overline{T}^{ix}(x,0,0) = 0$ that there is no radiation in the $x$-direction. (But radiation in $y$ & $z$-direction)

$$\rightsquigarrow \quad \frac{d^2}{dt^2} Q_{xx}(t) \sim m_0 l_0 A\omega^2 + m_0 A^2\omega^2$$

Take $m_0 = 10^3\,\text{kg} = 10^6\,\text{g}$, $l_0 = 1\,\text{m} = 100\,\text{cm}$, $A = 10^{-1}\,\text{cm}$, $\omega = 10^4\,\text{s}^{-1}$

Use $c = 3\cdot 10^{10}\,\frac{\text{cm}}{\text{s}}$, $G = 6.67\cdot 10^{-8}\,\frac{\text{cm}^3}{\text{g}\,\text{s}^2}$

$$\rightsquigarrow \quad m_0 = 10^6\,\text{g}\cdot 6.67\cdot 10^{-8}\,\frac{\text{cm}^3}{\text{g}\,\text{s}^2}\cdot\frac{1}{(3\cdot 10^{10}\,\frac{\text{cm}}{\text{s}})^2} \simeq 0.75\cdot 10^{-22}\,\text{cm}$$

$$l_0 = 10^2\,\text{cm}$$
$$A = 10^{-1}\,\text{cm}$$
$$\omega \approx 0.3\; 10^{-6}\,\text{cm}^{-1}$$

$\rightarrow$ First term leading order ($l_0 \gg A$) $\rightsquigarrow$ $\frac{d^2}{dt^2}Q_{xx}(t) \sim 0.0675\cdot 10^{-22}\, 10^2\, 10^{-1}\, \underset{10^{-33}}{10^{-12}}\,\text{cm}$

$$= 6.75\; 10^{-35}\,\text{cm}$$

$$\rightsquigarrow \quad \Delta Y^i \simeq \frac{1}{r}\, 6.75\cdot 10^{-35}\cdot Y^i(0)\,\text{cm}$$

If test masses are $1\,\text{km} = 10^3\,\text{cm}$ apart, this gives change in amplitude of order $\frac{1}{r}\cdot 6.75\; 10^{-32}\,\text{cm}$ with $r$ in cm.

$\rightarrow$ too small to be detectable.

**Remark:** Derivation of Quadrupole formula only valid for non-self-gravitating systems because we used $\partial_\mu T^{\mu\nu} = 0$. However, it is expected that it still is a good approximation for example for orbiting binaries. Indeed Hulse & Taylor observed in 1975 the increase of orbiting frequency of a binary system which contains a pulsar at a rate compatible with the loss of energy due to emission of gravitational waves predicted by the quadrupole formula.

First indirect observation of gravitational waves, Nobel prize for physics 1993.

**Remark:** Already Einstein noted that on small scales (quantum scales) one has to modify GR because otherwise the H-atom would be unstable due to emission of gravitational radiation.