III. Causality & Penrose diagrams

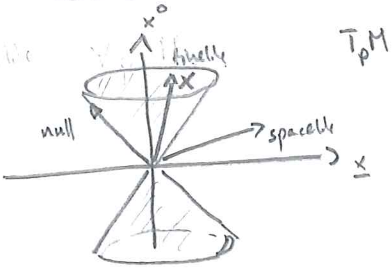
III.1 Lorentzian causality:

Def: (M,g) Lorentzian manifold, $p \in M$, $X \in T_pM$ tangent vector

$$g(X,X) = \begin{cases} < 0 & :\Leftrightarrow X \text{ timelike} \\ = 0 & :\Leftrightarrow X \text{ null (or light like)} \\ > 0 & :\Leftrightarrow X \text{ spacelike} \end{cases}$$

$\Big\}$ $:\Leftrightarrow X$ causal (timelike, null)

At $p \in M$ can choose coords x^μ s.t. $g_{\mu\nu}|_p = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots\ 1 \end{pmatrix}$, Minkowski metric.



Set of timelike vectors in T_pM forms disconnected double cone $C_p = \{ X = X^0\partial_0 + X^1\partial_1 + \ldots + X^n\partial_n \mid X^0 > \sqrt{(X^1)^2 + \ldots + (X^n)^2} \}$

$\cup \{ X \mid X^0 < -\sqrt{(X^1)^2 + \ldots + (X^n)^2} \}$.

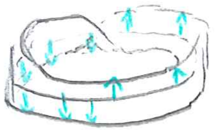
If we can single out one of those components throughout M in a continuous way, then we say that (M,g) is <u>time-orientable</u>. This is equivalent to the existence of a continuous timelike vector field on (M,g). Making such a continuous choice determines a <u>time-orientation</u>. Timelike vectors in this component are called <u>future directed</u>, timelike vectors in the other component are called <u>past directed</u> (extends by continuity to (non-vanishing) null vectors).

A time-oriented Lorentzian manifold (M,g) is also called a <u>spacetime</u>.

<u>Ex:</u>

- $M = \mathbb{R}^4$, $g = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots\ 1 \end{pmatrix}$ $\rightsquigarrow$ ∂_t provides a time orientation
- $M = \mathbb{R} \times (2m, \infty) \times S^2$, $g = -(1-\frac{2M}{r})dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$, $\rightsquigarrow$ ∂_t provides a time orientation
- $\underset{t}{(-1,1)} \times \underset{x}{[-10,10]}$, $g = -dt^2 + dx^2$, identify $(t,-10) \sim (-t,10)$

$\Rightarrow$ Möbius strip



$\rightsquigarrow$ not time-orientable.

Def: A smooth curve $\gamma : \underset{(-\infty,+\infty)}{\underset{\cap}{I}} \to M$ in a Lorentzian manifold (M,g) is called timelike / null / causal / spacelike iff its tangent vector $\dot{\gamma}(s)$ is timelike /... for all $s \in I$.

If (M,g) is time-oriented, then γ is called future directed timelike / null / causal iff $\dot{\gamma}(s)$ is future directed timelike /... for all $s \in I$.

Similar for past-directed.

As in special relativity, massive particles can only move along timelike curves, light rays follow null curves (in fact geodesics), nothing moves along spacelike curves.

Def: Let (M,g) be a spacetime. We write $\underset{M}{\underset{\uparrow}{p}} \ll \underset{M}{\underset{\uparrow}{q}}$ iff there exists a future directed timelike curve from p to q (i.e. if a massive object can travel from p to q).

We write $p \leq q$ iff there exist a future directed causal curve from p to q (i.e. if we can send a signal from p to q).

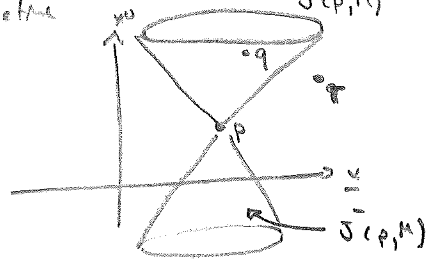
$I^+(p,M) := \{ q \in M \mid q \gg p \}$ timelike future of $p \in M$

$I^-(p,M) := \{ q \in M \mid q \ll p \}$ timelike past of $p \in M$

$J^+(p,M) := \{ q \in M \mid q \geq p \}$ causal future of $p \in M$ (set of points which can be causally influenced from p)

$J^-(p,M)$ - - - causal past - - - (set of points which can influence p)

Example: Minkowski spacetime



The sets $J^\pm(p,M)$ are of fundamental importance since they determine causal relations.

What are $J^\pm(p,M)$ for e.g. Schwarzschild? Penrose diagrams easy way to visualise causal structure of (spherically symmetric) spacetimes.

III.2 Penrose diagrams

Def: Let (M,g) be a Lorentzian manifold. Another Lorentzian metric $\tilde{g}$ on M is called conformal to g iff there exists a smooth function Ω on M s.t. $\tilde{g} = \Omega^2 \cdot g$.

Note that for $X \in T_pM$, $\tilde{g}(X,X) = \Omega^2 g(X,X)$, thus X is $\tilde{g}$-timelike / null / causal / spacelike if X is g-timelike / ...

$\Rightarrow$ $J^{\pm}_{\tilde{g}}(p,M) = J^{\pm}_{g}(p,M)$ & same for $I^{\pm}$

$\rightsquigarrow$ Conformal metrics have the same causal structure

Idea of Penrose diagram:

1) Want to understand global structure of spacetime (M,g), i.e. by suitable coordinate transformation bring in the infinities of (M,g) to a finite coordinate range.

$\rightsquigarrow$ As a consequence, $g_{\mu\nu}$ blows up in these coordinates.

illustrate orally by $(\mathbb{R}, dx^2)$ $\downarrow$ bring to $(-1,1)$

2) Choose a conformal factor Ω to make $\tilde{g}_{\mu\nu} = \Omega^2 \cdot g_{\mu\nu}$ regular at the infinities.

3) Can add the infinities as boundary to the spacetime to create a "conformal compactification"

4) If needed, drop some (spherically symmetric) dimensions and draw 2-dimensional diagram with causality as in Minkowski spacetime.

Examples:

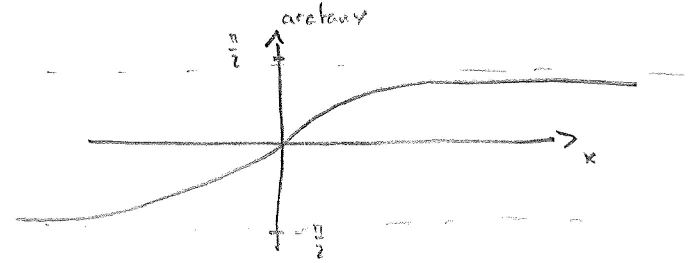
i) 1+1-dimensional Minkowski spacetime, $M = \mathbb{R}^2$, $g = -dt^2 + dx^2$

1) Null coordinates: $v = t+x$, $u = t-x$ $\quad \left(dt = \frac{1}{2}(dv+du),\ dx = \frac{1}{2}(dv-du) \right)$

$$\rightsquigarrow \quad g = -\frac{1}{2}(dv \otimes du + du \otimes dv), \quad u,v \in \mathbb{R}$$

Bring infinities to finite coord. range: $\tilde{u} = \arctan u$, $\tilde{v} = \arctan v$

Recall:



$u = \tan \tilde u = \frac{\sin \tilde u}{\cos \tilde u} \quad \rightsquigarrow \quad du = \frac{1}{\cos^2 \tilde u} d\tilde u$

$$\Rightarrow \quad g = - \frac{1}{2 \cos^2 \tilde u \cos^2 \tilde v} (d\tilde v \otimes d\tilde u + d\tilde u \otimes d\tilde v) \quad , \quad \tilde u, \tilde v \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

(Note metric diverges at infinities $\tilde v, \tilde v \to \pm \frac{\pi}{2}$

2) Choose conformal factor $\Omega^2 = \cos^2 \tilde u \cos^2 \tilde v$, then

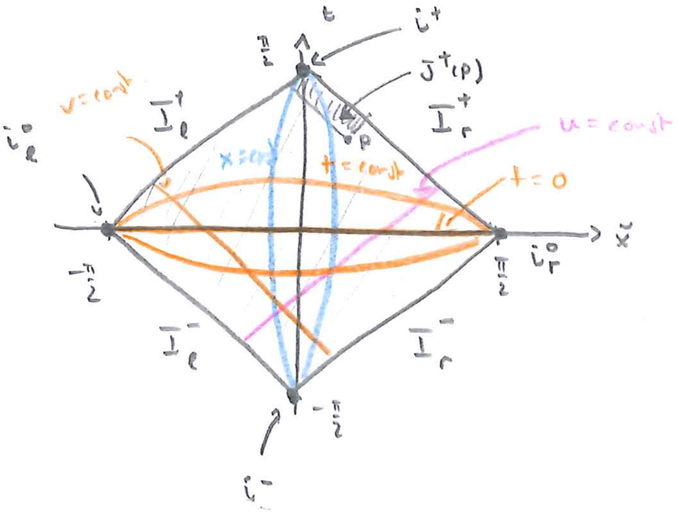
$$\tilde g = \Omega^2 g = -\tfrac{1}{2} (d\tilde v \otimes d\tilde u + d\tilde u \otimes d\tilde v)$$ is regular for $\tilde u, \tilde v \to \pm \frac{\pi}{2}$

3) Can add the boundaries to create conformal compactification $\tilde g = -\frac{1}{2} (d\tilde v \otimes d\tilde u + d\tilde u \otimes d\tilde v)$
$(\tilde u, \tilde v) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2$

4) Let $\tilde t = \frac{1}{2}(\tilde v + \tilde u)$, $\tilde x = \frac{1}{2}(\tilde v - \tilde u)$, then $\tilde g = -d\tilde t^2 + d\tilde x^2$

defined on $\{(\tilde t, \tilde x) \in \mathbb{R}^2 \mid -\frac{\pi}{2} \leq \overset{\tilde v}{\overset{\shortparallel}{\tilde t + \tilde x}} \leq \frac{\pi}{2},$
$-\frac{\pi}{2} \leq \underset{\tilde u}{\underset{\shortparallel}{\tilde t - \tilde x}} \leq \frac{\pi}{2}\}$

<u>Penrose diagram:</u>



$t = \frac{1}{2}(v+u) = \frac{1}{2}(\tan \tilde v + \tan \tilde u) = \frac{1}{2}(\tan(\tilde t + \tilde x)$
$+ \tan(\tilde t - \tilde x))$

$x = \frac{1}{2}(v-u) = \frac{1}{2}(\tan(\tilde t + \tilde x) - \tan(\tilde t - \tilde x))$

Since $\tilde g$ and g are conformal, $J^+_{\tilde g}(p) = J^+_g(p)$.

<u>Infinities:</u> • $\mathcal{I}^+_r / \mathcal{I}^+_l$ are called right & left <u>future null infinities</u>, they form the asymptotic endpoints of all future-directed right / left -going null geodesics.
(null geodesics are exactly the null lines)

• $\mathcal{I}^+_r / \mathcal{I}^+_l$ — " — <u>past null infinities</u>, — " —
— " — past-directed — " — .

• i^+ is called <u>future timelike infinity</u>, it's the endpoint of all future directed timelike geodesics. (<u>Exercise</u>: timelike geodesic has to intersect all null lines.)

• i^- — " — <u>past timelike infinity</u>, — " — past directed — " — .

- i^0_e / i^0_r are called left/right spacelike infinities, they form the endpoints of all left/right going spacelike geodesics.

(Remark: Note that there are timelike / spacelike curves going to $I^{\pm}_{r/e}$ - but they are not geodesics)

ii) 3+1 - dimensional Minkowski spacetime

$$M = \mathbb{R}^4 \quad , \quad g = -dt^2 + dr^2 + r^2 d\sigma^2 \quad , \quad \text{where } d\sigma^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

metric on S^2.

1) Spherically symmetric null coordinates: $v = t+r$, $u = t-r$

$$\rightsquigarrow \quad g = -\frac{1}{2}\left(dv\otimes du + du\otimes dv\right) + \frac{1}{4}(v-u)^2 d\sigma^2 \quad , \quad r \geq 0 \iff v-u \geq 0$$

$\infty > v \underline{\geq} u > -\infty$

(difference to before)

$\tilde u = \arctan u$, $\tilde v = \arctan v$ $\left(\text{and use } v-u = \tan\tilde v - \tan\tilde u = \frac{\sin\tilde v}{\cos\tilde v} - \frac{\sin\tilde u}{\cos\tilde u} = \frac{\sin\tilde v\cos\tilde u - \sin\tilde u\cos\tilde v}{\cos\tilde v\cos\tilde u} \right.$

$$\left. = \frac{\sin(\tilde v - \tilde u)}{\cos\tilde v\cos\tilde u} \right)$$

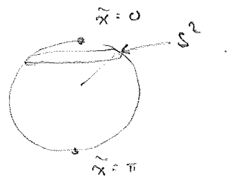
$$\Rightarrow \quad g = \frac{1}{\cos^2\tilde u\cos^2\tilde v}\left(-\frac{1}{2}\left(d\tilde v\otimes d\tilde u + d\tilde u\otimes d\tilde v\right) + \frac{1}{4}\sin^2(\tilde v - \tilde u)\, d\sigma^2 \right) \quad , \quad \frac{\pi}{2} > \tilde v \geq \tilde u > -\frac{\pi}{2}$$

2) Conformal factor $\Omega^2 = 4\cos^2\tilde u\cos^2\tilde v$ and choose coord trafo $\tilde t = \tilde v + \tilde u$, $\tilde x = \tilde v - \tilde u$

$$\Rightarrow \quad \tilde g = \Omega^2 \cdot g = -d\tilde t^2 + d\tilde x^2 + \sin^2\tilde x\, d\sigma^2$$

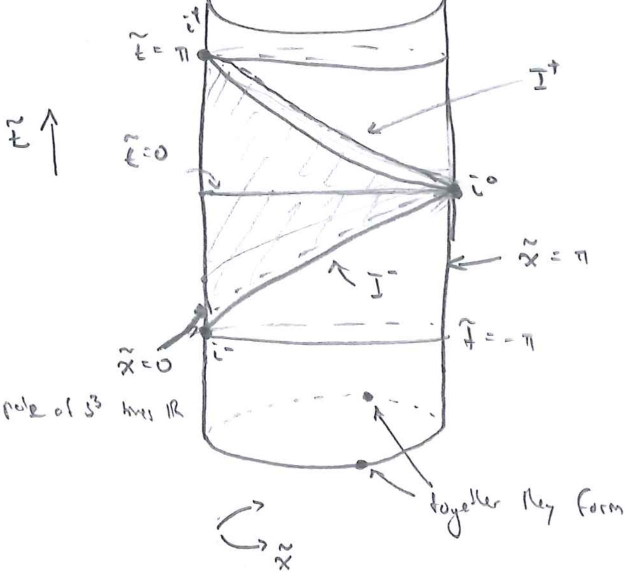
$$\text{with} \begin{cases} -\frac{\pi}{2} < \frac{1}{2}(\tilde t \pm \tilde x) < \frac{\pi}{2} & (\tilde u, \tilde v \in (-\frac{\pi}{2}, \frac{\pi}{2})) \\ \pi > \tilde x \geq 0 & (r \geq 0 / \tilde v \geq \tilde u) \end{cases}$$

3) Observe that $d\tilde x^2 + \sin^2\tilde x\, d\sigma^2$ is round metric on S^3, $\tilde x = 0, \pi$ are poles of S^3 and $\tilde x = \text{const.} \neq 0, \pi$ are 2-spheres of radius $\sin\tilde x$.



The spacetime $(\tilde M, \tilde g)$ with $\tilde M = \mathbb{R}\times S^3$, $\tilde g = -d\tilde t^2 + d\tilde x^2 + \sin^2\tilde x\, d\sigma^2$ is known as the Einstein static universe.

$\rightsquigarrow$ We have mapped 3+1-dim. Minkowski spacetime conformally into a portion of the Einstein static universe!



(south pole of S^3 times $\mathbb{R}$)

(north pole of S^3 times $\mathbb{R}$

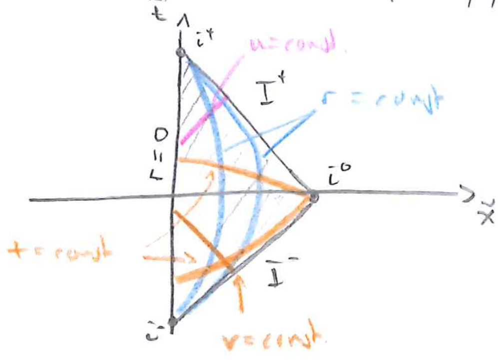
together they form an S^2 of radius $\sin \tilde{x}$

(diagram supposed to illustrate that the S^2's open up between the poles

3) Can add future/past timelike infinity i^+/i^- & spacelike infinity i^0 (all points in $\tilde{M}$)

and future/past null infinity $\mathcal{I}^+/\mathcal{I}^-$ (topology $(0,\pi) \times S^2$)

4) Quotient out the spheres of symmetry, draw quotient (Penrose diagram)



- Every point corresponds to an S^2 except $\{r = 0\}$, i^+, i^0, i^-.
- $t = \frac{1}{2}(v+u) = \frac{1}{2}(\tan(\tilde{t}+\tilde{x}) + \tan(\tilde{t}-\tilde{x}))$
- $r = \frac{1}{2}(\tan(\tilde{t}+\tilde{x}) - \tan(\tilde{t}-\tilde{x}))$

- i^+ future endpoint of future directed timelike geodesics
- i^- past –"– past –"–
- i^0 endpoint of spacelike geodesics
- $\mathcal{I}^\pm$ future/past endpoint of future/past directed null geodesics.

E.g.: • radial null geodesics in 3+1-dim Minkowski are lines at 45° in above Penrose diagram
• $t \longmapsto (t, r_0, \theta_0, \varphi_0)$ are the $r = const$ lines above.

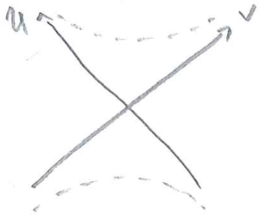
iii) Maximal analytic Schwarzschild spacetime

Recall Kruskal coordinates (U, V, θ, φ), $g = -\frac{16M^3}{r} e^{-\frac{r}{2M}} (dU \otimes dV + dV \otimes dU)$
$+ r^2 d\sigma^2$

with $U \cdot V = (1 - \frac{r}{2M}) e^{r/2M}$, $\frac{V}{U} = -e^{t/2M}$

and thus $\{(U,V) \in \mathbb{R}^2 \mid U \cdot V < 1\}$

Recall:



1) Let $\tilde{u} = \arctan U$ $\quad \tilde{v} = \arctan V$

$$\rightsquigarrow \quad g = \frac{1}{\cos^2\tilde{u}\cos^2\tilde{v}}\left(- \frac{16M^3}{r} e^{-\frac{r}{2M}} \left(d\tilde{u} \otimes d\tilde{v} + d\tilde{v} \otimes d\tilde{u} \right) + r^2 \cos^2\tilde{u}\cos^2\tilde{v} \, d\sigma^2 \right)$$

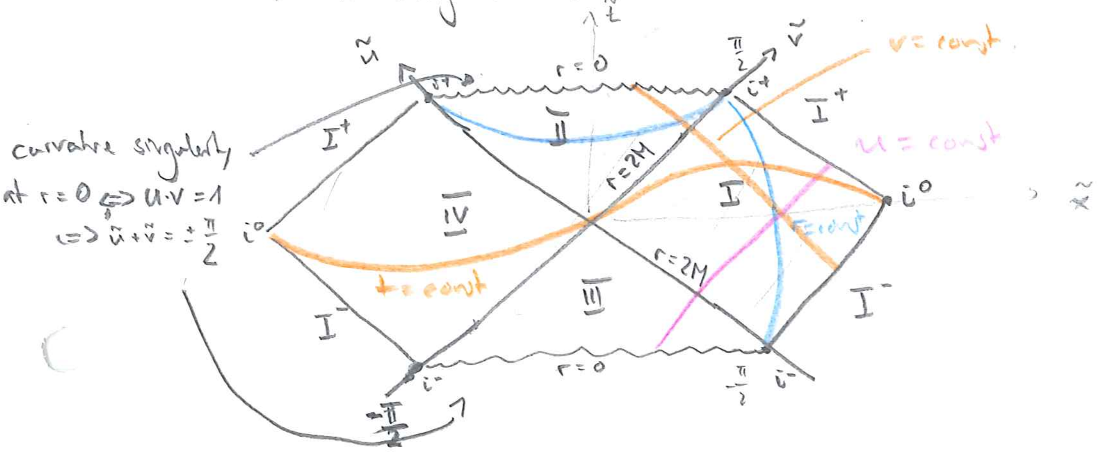
& $\tan\tilde{u} \cdot \tan\tilde{v} = U \cdot V < 1 \quad \Leftrightarrow \quad \sin\tilde{u}\sin\tilde{v} < \cos\tilde{u}\cos\tilde{v} \quad \Leftrightarrow \quad \cos(\tilde{u}+\tilde{v}) > 0$

Range of coordinates $\left\{ (\tilde{u},\tilde{v}) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \;\middle|\; -\frac{\pi}{2} < \tilde{u}+\tilde{v} < \frac{\pi}{2} \right\}$

2) Conformal factor $\Omega^2 = \cos^2\tilde{u}\cos^2\tilde{v}$

$$\longrightarrow \quad \tilde{g} = - \frac{16M^3}{r} e^{-\frac{r}{2M}} \left(d\tilde{u} \otimes d\tilde{v} + d\tilde{v} \otimes d\tilde{u} \right) + r^2 \cos^2\tilde{u}\cos^2\tilde{v} \, d\sigma^2$$

3 & 4) Let $\tilde{t} = \tilde{v} + \tilde{u}$, $\tilde{x} = \tilde{v} - u$, quotient out spheres of symmetry, draw Penrose diagram of Schwarzschild



($t, r, U, V = const.$ lines exercises)

Remarks:
- Metric $\tilde{g}$ extends continuously to future / past null infinity $\mathscr{I}^+/\mathscr{I}^-$ and to spacelike infinity i^0 – so these infinities can again be added as conformal boundaries – but not to future / past timelike infinity i^+/i^- (Think, because they meet singularity at $r=0$ in Penrose topology).
- Timelike geodesics asymptote either to $\{r=0\}$ (falling into the black hole) or to i^+/i^-
- Null geodesics asymptote either to $\{r=0\}$ (falling into BH), i^+/i^- (if they asymptote towards the photon sphere at $\{r=3M\}$ or lie on horizons $\{r=2M\}$), or $\mathscr{I}^+/\mathscr{I}^-$.

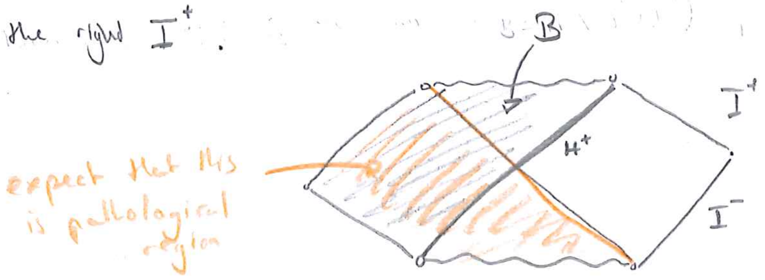
(spacelike geodesics inside BH can probably also go to i^+/i^-

IV. Black holes

IV.1 Recap of Schwarzschild spacetime, null hypersurfaces & event horizons

Let (M, g) be maximal analytic Schwarzschild

We now define the black hole region $\mathcal{B}$ by $\mathcal{B} := M \setminus J^-(\mathcal{I}^+)$, where we use the right $\mathcal{I}^+$.



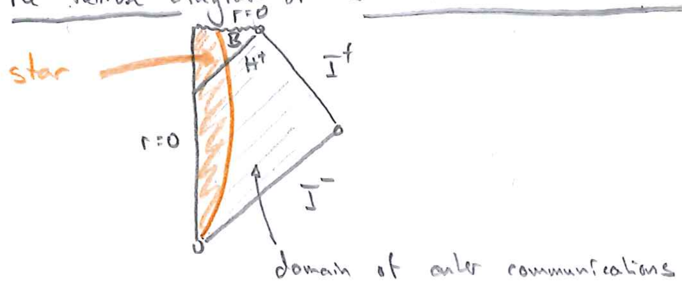
By definition this is the set of spacetime points from which one cannot send future directed signals to $\mathcal{I}^+$.

The boundary $\mathcal{H}^+ := \partial(J^-(\mathcal{I}^+))$ of $J^-(\mathcal{I}^+)$ in M is called the (future) event horizon.

The region $J^+(\mathcal{I}^-) \cap J^-(\mathcal{I}^+)$ is called the domain of outer communications.

The event horizon separates the black hole region from the domain of outer communications.

The Penrose diagram of a more realistic black hole spacetime: gravitational collapse of a star



→ only one asymptotically flat end.

Hyperspaces

Let (M, g) be a $\overbrace{}^{(n+1\text{-dim})}$ Lorentzian manifold, Σ hypersurface. Recall that this means that for every $p \in \Sigma$ there exist coord. nbhd $(\mathcal{U}, x^0, \dots, x^n)$ s.t. $\Sigma \cap \mathcal{U} = \{x^0 = 0\}$.

Tangent space $T\Sigma$ of Σ is locally given by $\operatorname{span}\{\partial_1, \dots, \partial_n\} \subseteq TM$ in these coords, i.e. for all $p \in \Sigma$, $T_p\Sigma$ is an n-dim subspace of T_pM.

We say that Σ is a
- spacelike hypersurface $:\Leftrightarrow$ $g|_{T_p\Sigma}$ is positive definite (Riemannian) $\forall p \in \Sigma$
- timelike hypersurface $:\Leftrightarrow$ $g|_{T_p\Sigma}$ is Lorentzian $\forall p \in \Sigma$
- null hypersurface $:\Leftrightarrow$ $g|_{T_p\Sigma}$ is degenerate $\forall p \in \Sigma$

Since $T_p\Sigma$ is an n-dim subspace of T_pM, there exist a covector $n \in T_p^*M$ s.t. $\ker n = T_p\Sigma$. This covector is unique up to multiplication by $\lambda \neq 0$ ~~and~~ is called a <u>normal covector</u> to Σ at p.
We have $n(X) = 0$ $\forall X \in T_p\Sigma$. In the local coordinates we have $n = \lambda dx^0$, $\lambda \neq 0$.

(i.e. $N^\mu = g^{\mu\nu} n_\nu$)

We can also define $N := n^\#$, a <u>normal vector</u> to Σ at p. We have $g(N, X) = 0$ $\forall X \in T_p\Sigma$. Again, N is unique up to multiplication by $\lambda \neq 0$.

<u>Proposition:</u> Σ is a

- spacelike hypersurface $\iff$ N is timelike $\forall p \in \Sigma$
- timelike hypersurface $\iff$ N is spacelike $\forall p \in \Sigma$
- null hypersurface $\iff$ N is null $\forall p \in \Sigma$

<u>Proof:</u> Let Σ be a hypersurface and let N be a normal vector at p. We distinguish the two cases that $N \in T_p\Sigma$ and $N \notin T_p\Sigma$.

i) $N \notin T_p\Sigma$. Then let $E_1, \ldots, E_n$ (e.g. $E_i = \partial_i$) be a basis of $T_p\Sigma$. Thus $\{N, E_1, \ldots, E_n\}$ is a basis of T_pM. Wrt this basis, g has the matrix

$$\begin{pmatrix} g(N,N) & 0 & \cdots & 0 \\ 0 & g(E_1,E_1) & \cdots & g(E_1,E_n) \\ \vdots & \vdots & g(E_i,E_j) & \vdots \\ 0 & g(E_n,E_1) & \cdots & g(E_n,E_n) \end{pmatrix}$$

It thus follows that $g(N,N) \neq 0$, since otherwise $g|_{T_pM}$ would be degenerate, and that $g|_{T_p\Sigma}$ is non-degenerate.

ii) $N \in T_p\Sigma$. Thus $\forall X \in T_p\Sigma$ we have $g|_{T_p\Sigma}(N, X) = 0$, i.e. $g|_{T_p\Sigma}$ is degenerate and choosing $X = N$ we have $g(N,N) = 0$.

The proof now follows easily from this. E.g. Σ spacelike hypersurface $\Rightarrow$ $g|_{T_p\Sigma}$ is non-degenerate $\Rightarrow$ we are in case i). Using $g|_{T_pM}$ Lorentzian & $g|_{T_p\Sigma}$ Riemannian gives $g(N,N) < 0$.
For the reverse, let N be timelike. Then we must be in case i) and $g|_{T_p\Sigma}$ is pos definite. Etc.

In particular we have seen that if Σ is a null hypersurface, then the normal vector field N is tangent to Σ.

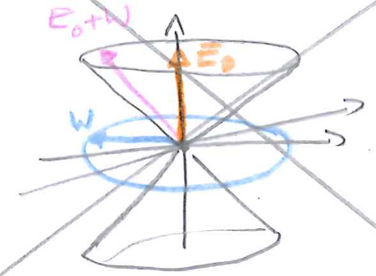
<u>Proposition</u>: Let Σ be a null hypersurface and N a normal vector field. Then the integral curves of N are null geodesics, but not necessarily affinely parametrised. They are called the <u>generators</u> of the null hypersurface.

~~<u>Pf</u>: Let $p \in \Sigma$.~~

~~Also let $E_0, E_1, \dots, E_n$ be an orthonormal basis for T_pM with~~

~~$g(E_\mu, E_\nu) = \eta_{\mu\nu} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{pmatrix}$ the Minkowski metric.~~

~~Note that we can write all null vectors L at p in the form $L = \lambda(E_0 + W)$ where W is a vector of unit length in $\operatorname{span}\{E_1, \dots, E_n\}$ and $\lambda \in \mathbb{R}$~~



~~After rotation of the basis $E_1, \dots, E_n$ and rescaling of $N(p)$ we can assume wlog that $N(p) = E_0 + E_1$.~~

~~Then $T_p\Sigma = \ker(g(N(p), \cdot)) = \operatorname{span}\{E_0 + E_1, E_2, \dots, E_n\}$~~

<u>Pf</u>: Let N be a normal vector field. Locally Σ is given as the level set of a function f ($= x^0$ coord.). Then $df = n$ is a normal covector field on Σ and we have $N = \lambda (df)^\#$, $\lambda \neq 0$. Since the integral curves of N and $\lambda^{-1} N$ are the same $\underbrace{\text{, wlog}}_{\text{(up to parametrisation)}}$ we can assume $N = (df)^\#$.

$$\text{Now} \quad (\nabla_N n)_a = N^b \nabla_b \nabla_a f \overset{\nabla\nabla f \text{ sym}}{\Downarrow} N^b \nabla_a \nabla_b f = N^b \nabla_a n_b = \frac{1}{2} \partial_a (N^b n_b)$$

But $N^b n_b = g(N, N)$ is constant on Σ (equal to zero), and thus

$$d(N^b n_b) = \mu \cdot n \quad \text{with } \mu \in \mathbb{R}.$$

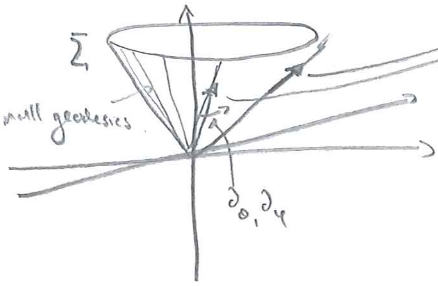
$$\Rightarrow (\nabla_N n)_a = \frac{1}{2} \mu n_a \quad \Rightarrow \nabla_N N = \frac{1}{2} \mu N$$

■

(if $N^b n_b \equiv 0$ everywhere, as is the case if all level sets of f are null hypersurfaces, then $\mu = 0$)

Examples: i) 3+1 - dim. Minkowski spacetime : a) $t = \text{const.}$ are spacelike hypersurfaces ($g^{-1}(dt, dt) = -1$), b) $x_i = \text{const.}$ are nullike hypersurfaces ($g^{-1}(dx_i, dx_i) = 1$)

c) Σ = The future light cone of the origin in Minkowski spacetime with the origin removed



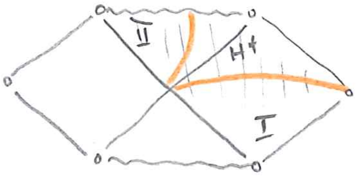
$\partial_t + \partial_r$ & $\partial_\theta, \partial_\varphi$ span the tangent space.

$\partial_t + \partial_r$ is null and it is also orthogonal to $\partial_\theta, \partial_\varphi$ $\rightsquigarrow$ $\partial_t + \partial_r$ is the normal of Σ.

$\Rightarrow$ Σ is a null hypersurface.

It is generated by the null geodesics which are the straight lines in the cone.

ii) Schwarzschild $g = -(1 - \frac{2M}{r}) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\sigma^2$



a) $t = \text{const.}$ are $\begin{cases} \text{spacelike hypersurfaces for } r > 2m \\ \text{timelike hypersurfaces for } r < 2m \end{cases}$

follows from $g^{-1}(dt, dt) = -\frac{1}{1 - \frac{2M}{r}} = \begin{cases} < 0 & \text{for } r > 2m \\ > 0 & \text{for } r < 2m \end{cases}$

b) $r = \text{const.}$ are $\begin{cases} \text{timelike hypersurfaces for } r > 2m \\ \text{spacelike hypersurfaces for } r < 2m \end{cases}$

follows from $g^{-1}(dr, dr) = 1 - \frac{2M}{r} = \begin{cases} > 0 & \text{for } r > 2m \\ < 0 & \text{for } r < 2m \end{cases}$

c) Let $v = t + r^*$, $r^* = r + 2M \log\left(\frac{r - 2M}{2M}\right)$, $r > 2M$

Then (v, r, θ, φ) are ingoing Eddington - Finkelstein coordinates, they cover regions I & II and $g = -(1 - \frac{2M}{r}) dv^2 + dv \otimes dr + dr \otimes dv + r^2 d\sigma^2$

$$g^{-1} = \partial_v \otimes \partial_r + \partial_r \otimes \partial_v + (1 - \frac{2M}{r}) \partial_r \otimes \partial_r + \frac{1}{r^2}\left(\partial_\theta \otimes \partial_\theta + \frac{1}{\sin^2\theta} \partial_\varphi \otimes \partial_\varphi\right)$$

and thus $g^{-1}(dr, dr)\big|_{r = 2M} = 0$

$\Rightarrow$ $\{r = 2M\} = H^+$ is a null hypersurface.

A choice of normal vector field is $N = \partial_v = (dr)^\#$.

Thus the integral curves of ∂_v are null geodesics by the last proposition.

Indeed we have $T^v_{vv}\big|_{r=2M} = \frac{1}{2} g^{vr}(-g_{vv,r})\big|_{r=2M} = \frac{1}{2}(1 - \frac{2M}{r})_{,r}\big|_{r=2M} = \frac{M}{(2M)^2} = \frac{1}{4M}$

and $T^r_{vv}\big|_{r=2M} = T^\theta_{vv}\big|_{r=2M} = T^\varphi_{vv}\big|_{r=2M} = 0$ as easily seen.

Thus $\nabla_{\partial_v}\partial_v = \frac{1}{4M}\partial_v$. Hence the integral curves of ∂_v are <u>not</u> <u>affinely</u> parametrised null geodesics. But let $\partial_V = e^{-\frac{1}{4M}v}\partial_v$.

Then $\nabla_{\partial_V}\partial_V = e^{-\frac{1}{4M}v}\nabla_{\partial_v}\left(e^{-\frac{1}{4M}v}\partial_v\right) = e^{-\frac{1}{2M}v}\nabla_{\partial_v}\partial_v - \frac{1}{4M}e^{-\frac{1}{2M}v}\partial_v = 0$

Thus the integral curves of ∂_V are affinely parametrised. Note:/recall that ∂_V is exactly the coordinate vector field in Kruskal coordinates (V, U, θ, φ)

■

<u>Definition:</u> (M,g) Lorentzian manifold with Killing vector field T. A null hypersurface Σ is a <u>Killing horizon</u> of T iff T is normal to Σ on Σ.

Since T is a normal, by the last proposition we have $\nabla_T T\big|_\Sigma = \kappa T\big|_\Sigma$ for some (a priori) function κ on Σ. κ is called the <u>surface gravity</u> of Σ (wrt T).

<u>Remarks:</u> 1) If Σ is a Killing horizon of T, then so it is also of $\tilde{T} = c\cdot T$ with $c \in \mathbb{R}\setminus\{0\}$. Then $\nabla_{\tilde{T}}\tilde{T}\big|_\Sigma = \tilde{\kappa}\tilde{T}\big|_\Sigma$ with $\tilde{\kappa} = c\kappa$.

Hence, the surface gravity depends on the normalisation of the KVF T.

For asymptotically flat spacetimes, normalise T at infinity; e.g. if T is a the translation require $g(T,T) \longrightarrow -1$ for $r \longrightarrow \infty$ and fix sign of κ by requiring T to be future directed.

2) Using Killing's eq. $\nabla_\mu T_\nu + \nabla_\nu T_\mu = 0$ we obtain

$$(\nabla_T T)_\mu = T^\nu \nabla_\nu T_\mu = -T^\nu \nabla_\mu T_\nu = -\frac{1}{2}\partial_\mu(g(T,T))$$

and thus $\underline{d(g(T,T))\big|_\Sigma = -2\kappa T^\flat\big|_\Sigma}$ $(*)$

3) Take Lie derivative $\mathcal{L}_T$ of $(*)$. Note, that since T is tangent to Σ we have for any tensor field E that $(\mathcal{L}_T E)\big|_\Sigma = \mathcal{L}_T(E|_\Sigma)$.

(follows from $\mathcal{L}_T E = \frac{d}{dt}\Big|_{t=0}\underset{\uparrow \text{ all on } \Sigma}{\Phi_t^* E}$)

Thus

$$\begin{array}{ccc} \mathcal{L}_T\left(d(g(T,T))\big|_\Sigma\right) & = & -2\mathcal{L}_T\left(\kappa T^\flat\big|_\Sigma\right) \\ \| & & \| \\ \mathcal{L}_T(d(g(T,T)))\big|_\Sigma & & -2\mathcal{L}_T(\kappa T^\flat)\big|_\Sigma \\ \| & & \| \\ d(\mathcal{L}_T(g(T,T)))\big|_\Sigma & & -2T(\kappa)\cdot T^\flat \\ \| & & \\ d(0)\big|_\Sigma & & \\ \| & & \\ 0 & & \end{array}$$

$\mathcal{L}_T df = d\mathcal{L}_T f \longrightarrow$

$\longleftarrow \begin{vmatrix} \mathcal{L}_T T = 0 \\ \mathcal{L}_T g = 0 \end{vmatrix}$

Using $\mathcal{L}_T g = 0$, $\mathcal{L}_T T = 0$ $\longrightarrow$

And hence $T(\kappa) = 0$. Thus κ is constant along the generators of Σ.

4) Indeed, one can show that if (M, g) is a solution of $G_{ab} = 8\pi T_{ab}$ where the matter T_{ab} satisfies the so-called dominant energy condition (i.e. $T^{ab} W_a W_b \geq 0$ for all W timelike & $T^{ab} W_b$ is a causal vector) and if Σ is a Killing horizon of a KVF K, then the surface gravity κ is constant on Σ.

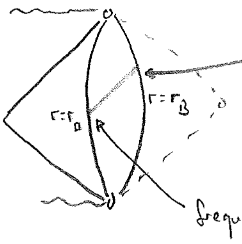
5) The event horizon $\{r = 2m\}$ in Schwarzschild is a Killing horizon of the KVF $\underset{\substack{\uparrow \\ \text{Boyer-Lindquist}}}{\partial_t} = \underset{\substack{\uparrow \\ \text{ingoing EF}}}{\partial_v}$, see example ii). We have $\nabla_{\partial_v} \partial_v = \frac{1}{4M} \partial_v$, and thus the surface gravity is $\kappa = \frac{1}{4M}$. Note that $\partial_v = \partial_t$ is normalised at infinity.

6) One can show that the event horizon H^+ of an asymptotically flat black hole spacetime (i.e. the complement $J^-(\mathcal{I}^+)$ is non-empty) is a null hypersurface. If the spacetime is in addition stationary (i.e. there exists a KVF T that is timelike in the asymp. flat region), then one can show that H^+ is a Killing horizon (but not necessarily of T).

Physical interpretation of the surface gravity κ

Consider maximal analytic Schwarzschild

Already familiar with gravitational redshift outside black hole region



frequency of received light signal is $f_B = \sqrt{\dfrac{1 - \frac{2M}{r_A}}{1 - \frac{2M}{r_B}}}\, f_A < f_A$

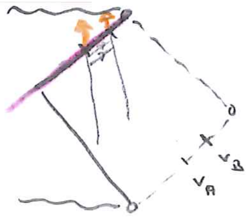
for $r_A < r_B$

(see Example sheet 4)

frequency of light here f_A

Positive surface gravity implies that there is also a gravitational redshift at the surface of the black hole, i.e. along the event horizon:

From last example: integral curves of $\partial_V = \overbrace{e^{-\frac{1}{4M} v}}^{= e^{-\kappa v}} \cdot \partial_v$ are affinely parametrised null geodesics (light rays) along the event horizon.



Consider an observer A crossing the event horizon at $(v_A, 2M, \theta_0, \varphi_0)$ with unit velocity vector $E_0^{(A)} = \frac{1}{\sqrt{2}}(\partial_v - \partial_r)$ and sending a light signal along the event horizon that is received by another observer B crossing $\mathcal{H}^+$ at $(v_B, 2M, \theta_0, \varphi_0)$, $v_B > v_A$ with unit velocity vector $E_0^{(B)} = \frac{1}{\sqrt{2}}(\partial_v - \partial_r)$ . .

Recall : $g = -(1 - \frac{2M}{r})dv^2 + dv \otimes dr + dr \otimes dv + r^2 d\sigma^2$

Angular frequency of light ray given by ∂_V as observed by A : $-g(\partial_V, E_0^{(A)}) = \frac{1}{\sqrt{2}}e^{-\kappa v_A}$

– " – by B : $-g(\partial_V, E_0^{(B)}) = \frac{1}{\sqrt{2}}e^{-\kappa v_B}$

$= \frac{1}{\sqrt{2}}e^{-\kappa v_A} \cdot e^{-\kappa(v_B - v_A)}$

$\Rightarrow$ light is red-shifted by factor $e^{-\kappa(v_B - v_A)}$ with $\kappa = \frac{1}{4M}$

$\rightsquigarrow$ exponential redshift in affine time v along event horizon with exponential factor given by surface gravity.

Remark: 1) The observed frequency depends of course on the 4-velocity of the observer. However note that $[\partial_v - \partial_r, \underset{\partial_v}{\underset{\|}{\partial_t}}] = 0$, so the observer B arises from Lie-transporting the observer A to some later time along the flow-lines of the stationary KVF ∂_t (i.e. really the 'same' observer as A, just at later time)

2) Generalises to other BH spacetimes. Black holes with $\kappa = 0$ are called extremal (no red-shift along $\mathcal{H}^+$).

Aside:
Another interpretation via Hawking radiation : $T_H = \frac{\kappa}{2\pi}$ temperature of BH .

IV.2 The Kerr solution

Consider $g = g_{tt}dt^2 + g_{t\varphi}(dt\otimes d\varphi + d\varphi \otimes dt) + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + g_{\varphi\varphi}d\varphi^2$ (*)

where $g_{tt} = -1 + \frac{2Mr}{\rho^2}$, $g_{t\varphi} = -\frac{2Mra\sin^2\theta}{\rho^2}$

$g_{\varphi\varphi} = [r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}]\sin^2\theta$

with $\rho^2 = r^2 + a^2\cos^2\theta$ & $\Delta = r^2 - 2Mr + a^2$.

Assume $0 < a < M$. ($a = M$ corresponds to extremal BH, $a > M$ to naked singularity. Both are not discussed in this course)

Then Δ has the two roots $r_\pm = M \pm \sqrt{M^2 - a^2}$

First define g on $M = \underset{t}{\mathbb{R}} \times \underset{r}{(r_+, \infty)} \times \underset{\theta,\varphi}{S^2}$

$$g = \left(\begin{array}{ccccc} t & \varphi & r & \theta & \\ g_{tt} & g_{t\varphi} & 0 & 0 & t \\ g_{t\varphi} & g_{\varphi\varphi} & 0 & 0 & \varphi \\ 0 & 0 & \frac{\rho^2}{\Delta} & 0 & r \\ 0 & 0 & 0 & \rho^2 & \theta \end{array}\right)$$

Computation (exercise): $g_{tt} g_{\varphi\varphi} - (g_{t\varphi})^2 = -\Delta \sin^2\theta$ and $\Delta > 0$ in $r > r_+$.

$\Rightarrow$ g is a Lorentzian metric on M. One can show that it is a solution of the vacuum EE. (It is called the <u>Kerr solution</u>.) The coords (t, r, θ, φ) are called <u>Boyer - Lindquist coordinates</u>.

For later:

$$g^{-1} = \left(\begin{array}{ccccc} t & \varphi & r & \theta & \\ -\frac{g_{\varphi\varphi}}{\Delta \sin^2\theta} & \frac{g_{t\varphi}}{\Delta \sin^2\theta} & 0 & 0 & t \\ \frac{g_{t\varphi}}{\Delta \sin^2\theta} & -\frac{g_{tt}}{\Delta \sin^2\theta} & 0 & 0 & \varphi \\ 0 & 0 & \frac{\Delta}{\rho^2} & 0 & r \\ 0 & 0 & 0 & \frac{1}{\rho^2} & \theta \end{array}\right) \qquad (**)$$

<u>Far field:</u>

Set $\tilde{r} = \frac{1}{2}\left(r - M + r\left(1 - \frac{2M}{r}\right)^{1/2}\right)$ (as we did for Schwarzschild) and define

$x = \tilde{r} \sin\theta \cos\varphi$
$y = \tilde{r} \sin\theta \sin\varphi$
$z = \tilde{r} \cos\theta$

then

$$g = -\left(1 - \frac{2M}{\tilde{r}} + \mathcal{O}\left(\frac{1}{\tilde{r}^2}\right)\right) dt^2 - \frac{4Ma}{\tilde{r}^3} dt\,[-y\,dx + x\,dy] + \mathcal{O}\left(\frac{1}{\tilde{r}^3}\right) dt\,(dx, dy, dz)$$
$$+ \left(1 + \frac{2M}{\tilde{r}}\right)(dx^2 + dy^2 + dz^2) + \mathcal{O}\left(\frac{1}{\tilde{r}^2}\right)(dx, dy, dz)(dx, dy, dz)$$

Thus M = total mass, $J = Ma$ = angular momentum in z-direction.

$a = \frac{J}{M}$ is <u>angular momentum per unit mass</u>

$\rightsquigarrow$ metric of a rotating body.

<u>Remark:</u> When $a = 0$, can show that g reduces to the Schwarzschild metric with mass M, and when $M = 0$ (but not necessarily $a = 0$), g equals the Minkowski metric in spheroidal coordinates (problem sheet).

Now: Page 58½

Global structure

Metric degenerates at $r = r_+$. We show that this is a coord singularity similar to Schwarzschild.

Let $r^*(r)$ satisfy $\frac{dr^*}{dr} = \frac{r^2+a^2}{\Delta}$ and $\bar{r}(r)$ satisfy $\frac{d\bar{r}}{dr} = \frac{a}{\Delta}$

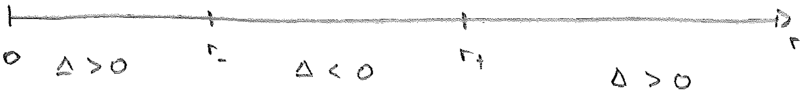
Define $v_\pm := t \pm r^*$, $\varphi_\pm := \varphi \pm \bar{r} \mod 2\pi$.

$(v_\pm, r, \theta, \varphi_\pm)$ are Eddington-Finkelstein-like coordinates ("+" for ingoing, "-" for outgoing).

In $(v_+, r, \theta, \varphi_+)$ coords

$$g = g_{tt} dv_+^2 + g_{t\varphi}(dv_+ \otimes d\varphi_+ + d\varphi_+ \otimes dv_+) + g_{\varphi\varphi} d\varphi_+^2 + \rho^2 d\theta^2 + (dv_+ \otimes dr + dr \otimes dv_+)$$
$$- a\sin^2\theta\,(d\varphi_+ \otimes dr + dr \otimes d\varphi_+)$$

which is a non-deg. Lorentzian metric on $\tilde{M} = \underset{v_+}{\mathbb{R}} \times \underset{r}{(0,\infty)} \times \underset{\theta,\varphi_+}{S^2}$.



In $r \in (r_-, r_+)$ choose function $r^*_{int}(r)$ with $\frac{dr^*_{int}}{dr} = \frac{r^2+a^2}{\Delta}$ and $\bar{r}_{int}(r)$ with $\frac{d\bar{r}_{int}}{dr} = \frac{a}{\Delta}$ and set $t = v_+ - r^*_{int}$ & $\varphi = \varphi_+ - \bar{r}_{int}$ to obtain again the form (**) of the metric g in the region $\mathbb{R} \times (r_-, r_+) \times S^2$ in Boyer-Lindquist coords (t, r, θ, φ).

Want to compute $g^{-1}(dr, dr)$. For $r \in (r_-, r_+)$ & $r \in (r_+, \infty)$ can use Boyer-Lindquist coords (and (**)) to compute it easily to get $g^{-1}(dr, dr) = \frac{\Delta}{\rho^2}$ for $r_- < r < r_+$ & $r_+ < r < \infty$.

By continuity also get $g^{-1}(dr, dr) = \frac{\Delta}{\rho^2}$ for $r_- < r < \infty$

$$\Rightarrow \{r = r_0\} \text{ is } \begin{cases} \text{timelike hypersurface for } r_+ < r_0 < \infty \\ \text{null hypersurface for } r_0 = r_+ \\ \text{spacelike hypersurface for } r_- < r_0 < r_+ \end{cases}$$

[The region $r \leq r_-$ is not discussed in this course]

<u>Proposition</u>: The hypersurface $\{r = r_+\}$ is a black hole event horizon.

$$(dr)^\# = g^{rr} \frac{\partial}{\partial r}\Big|_{BL} = \frac{\Delta}{\rho^2} \frac{\partial}{\partial r}\Big|_{BL}$$

The Kerr solution has two Killing vectors $\partial_t, \partial_\varphi$ (metric independent of t & φ)

which commute $[\partial_t, \partial_\varphi] = 0$.

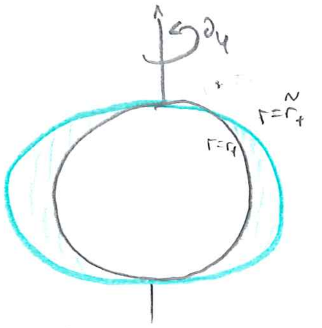
∂_t is timelike for large r $\rightarrow$ stationary (but not static)

∂_φ rotation around z-axis $\rightarrow$ axisymmetric (∂_φ always spacelike)

Note that

$$g(\partial_t, \partial_t) = -1 + \frac{2Mr}{\rho^2} = \begin{cases} < 0 & \text{for } 2Mr < \rho^2 \\ = 0 & \text{for } 2Mr = \rho^2 \\ > 0 & \text{for } 2Mr > \rho^2 \end{cases}$$

$\rho^2 = r^2 + a^2\cos^2\theta$, thus $r^2 + a^2\cos^2\theta - 2Mr = 0$ $\Leftrightarrow$ $r = \tilde{r}_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta}$



The region $r_+ < r \le \tilde{r}_+$ is called the <u>ergoregion</u> and $r = \tilde{r}_+$ the <u>ergosphere</u>

Given a stationary observer A with velocity $\sim (\partial_t + \Omega\partial_\varphi)$, then Ω is the angular frequency of this observer <u>as seen by an observer B with velocity ∂_t at infinity</u> (problem sheet).

Thus A appears <u>static</u> to B iff $\Omega = 0$.

In order for an observer with velocity $(\partial_t + \Omega\partial_\varphi)$ to exist at radius r and latitude θ, we need

$$0 > g(\partial_t + \Omega\partial_\varphi, \partial_t + \Omega\partial_\varphi) = g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi}$$

$$\Rightarrow \quad \Omega \in (\Omega_{min}, \Omega_{max}) \quad \text{with} \quad \Omega_{min} = \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}}$$

$$\Omega_{max} = \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}}$$

$$\text{and} \quad \omega = \frac{1}{2}(\Omega_{min} + \Omega_{max}) = -\frac{g_{\varphi t}}{g_{\phi\phi}}$$

i) Since $g_{\varphi t}^2 - g_{tt}g_{\varphi\varphi} = \Delta\sin^2\theta > 0$ $\forall r > r_+$, we indeed have two roots $\Omega_{min} < \Omega_{max}$

and since $g_{tt} < 0$ for large r and $g_{\varphi\varphi} > 0$, we have $\Omega_{min} < 0 < \Omega_{max}$ for r large.

ii) At the ergosphere $g_{tt} = 0$, and thus $\Omega_{min} = 0$. In the ergoregion $g_{tt} > 0$, and

so $0 < \Omega_{min} < \Omega_{max}$.

Thus in the ergoregion stationary observers have to rotate in the φ-direction as seen from infinity.

$\rightarrow$ gravitational dragging of frames by rotating bodies in GR.

(see MTW §33.4)

In $(v_+, r, \theta, \varphi_+)$ coords have

$$\frac{\partial}{\partial r}\Big|_{BL} = \frac{\partial v_+}{\partial r}\Big|_{BL}\frac{\partial}{\partial v_+} + \frac{\partial r}{\partial r}\Big|_{BL}\frac{\partial}{\partial r} + \frac{\partial \varphi_+}{\partial r}\Big|_{BL}\frac{\partial}{\partial \varphi_+} = \frac{r^2+a^2}{\Delta}\frac{\partial}{\partial v_+} + \frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial \varphi_+}$$

Thus by continuity $$(dr)^\#\Big|_{r=r_+} = \frac{r^2+a^2}{\rho^2}\frac{\partial}{\partial v_+} + \frac{\Delta}{\rho^2}\frac{\partial}{\partial r} + \frac{a}{\rho^2}\frac{\partial}{\partial \varphi_+}\Big|_{r=r_+}$$

$$= \frac{r_+^2+a^2}{r_+^2+a^2\cos^2\theta}\frac{\partial}{\partial v_+} + \frac{a}{r_+^2+a^2\cos^2\theta}\frac{\partial}{\partial \varphi_+}$$

$\Rightarrow$ $\frac{\partial}{\partial v_+} + \frac{a}{r_+^2+a^2}\frac{\partial}{\partial \varphi_+}\Big| = \frac{r_+^2+a^2\cos^2\theta}{r_+^2+a^2}(dr)^\#$ is future directed null normal to $\{r=r_+\}$.

__Lemma:__ Let $X \in T_pM$ be future directed timelike, $Y \in T_pM$ future directed causal. Then

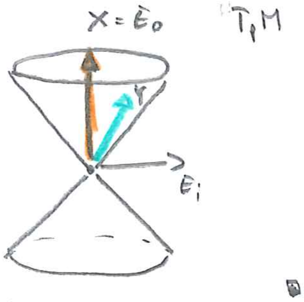
$$g(X,Y) < 0 \, .$$

__Pf:__ wlog assume $g(X,X) = -1$ and let $E_0 = X, E_1, \dots, E_n$ be an ONB, etc.

Then $Y = a((E_0 + \sum_i b^i E_i)$ with with $a > 0$ (future directed)

$\sum_i (b^i)^2 \leq 1$ (causal).

$\Rightarrow$ $g(X,Y) = -a < 0$.



□

We have shown that for $r_- < r \leq r_+$ we have $(dr)^\#$ future directed causal.

Let $\gamma : I \to M$ be a future directed timelike curve in $r_- < r \leq r_+$. Then

$$\dot\gamma^r = \dot\gamma(r) = dr(\dot\gamma) = g((dr)^\#, \dot\gamma) < 0 \quad \text{by lemma.}$$

Thus once a timelike curve has entered $r_- < r \leq r_+$, its r coordinate value can never increase beyond r_+. $\Rightarrow$ Black hole region.

Remark: By continuity the lemma and the above argument extend to future directed causal vector and future directed causal curves.

On the other hand it is easy to see that for every $r_0 > r_+$ there are future directed causal curves starting from r_0 which reach into the asymptotically flat region $r \gg r_+$.

For example consider stationary observer vector field $\partial_t + \omega(r)\partial_\varphi$ which is timelike $\forall r > r_+$ and add a bit of ∂_r so that it stays timelike, i.e., $\partial_t + \omega(r)\partial_\varphi + \varepsilon(r)\partial_r$. Integral curves reach $r = +\infty$.

$\Rightarrow$ $\{r = r_+\}$ is black hole event horizon.

$\{r \leq r_+\}$ is black hole interior.

Proposition: $\{r = r_+\}$ is a Killing horizon of the Killing vector field $T_H = \partial_{v_+} + \frac{a}{r_+^2 + a^2}\partial_{\varphi_+}$ with surface gravity $\kappa_+ = \frac{r_+ - r_-}{2(r_+^2 + a^2)}$.

Pf: First observe that $\left.\frac{\partial}{\partial t}\right|_{BL} = \frac{\partial}{\partial v_+}$ & $\left.\frac{\partial}{\partial \varphi}\right|_{BL} = \frac{\partial}{\partial \varphi_+}$, thus T_H is indeed a KVF. We have also shown that it is normal to the event horizon.

To compute, we use the formula $\partial_a\left((T_H)^b (T_H)_b\right) = -2\kappa_+ (T_H)_a$.

(exercise) (BH course Ex 19)

∎

Penrose diagram of Kerr

More difficult to draw 2-dim diagram since not spherically symmetric. Here we restrict to the axis $\theta = 0, \pi$.

Define Kruskal-like coords $\mathcal{U} = e^{-\kappa_+ v_-}$, $V = e^{\kappa_+ v_+}$, $\Phi = \varphi - \frac{at}{r_+^2 + a^2}$ in the region $r_+ < r < \infty$. (Recall $v_\pm = t \pm r^*$)

Then one can show that in the coords $(\mathcal{U}, V, \theta, \Phi)$ the metric extends analytically to $\mathbb{R}^2 \times S^2$ where r ranges from $r_- < r < \infty$. (O'Neill)

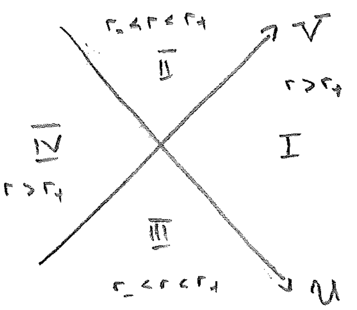
For $\theta = 0, \pi$, the metric takes the form

$$g = F(r)\, d\mathcal{U} dV$$

($\theta = 0, \pi$, a point, now φ, θ directions)

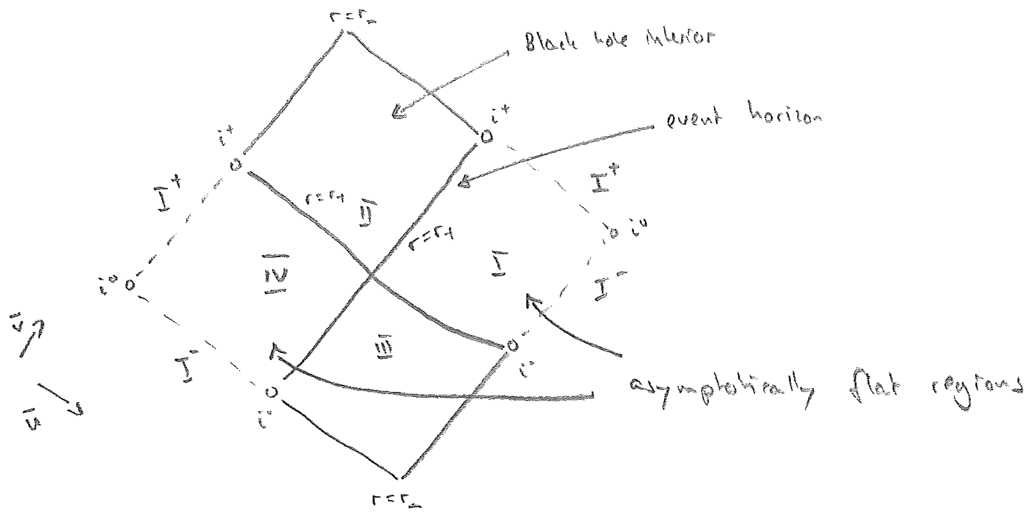
(For $\theta \neq 0, \pi$, the metric has a more complicated form, more off-diagonal terms)

where $F(r)$ is an analytic function in r and $r := r(\mathcal{U}V)$



each region is isometric to Kerr in Boyer Lindquist coords. with r restricted to this region.

Now can compactify $\bar{u} = \tan \mathcal{U}$, $\bar{v} = \tan \mathcal{V}$ and draw



The hypersurface $\{r=r_-\}$ in the black hole interior is called the "Cauchy horizon". The metric can be extended through it but the extension is not expected to correspond to anything physical. One expects that for small perturbations the Cauchy horizon becomes ~~sin~~ a singularity.