Solutions sheet 4 General Relativity II, Hilary Term 2020

Questions marked with a star have lowest priority to be discussed during class. Any comments or corrections please to Jan.Sbierski@maths.ox.ac.uk.

1) Let (M, g) be a Lorentzian manifold and let $\tilde{g} = \Omega^2 g$ be a Lorentzian metric on M that is conformal to g, where Ω is a smooth function with $\Omega(x) \neq 0$ for all $x \in M$.

   (a) Show that the Christoffel symbols $\tilde{\Gamma}^\mu_{\nu\kappa}$ of $\tilde{g}$ are given by

   $$\tilde{\Gamma}^\mu_{\nu\kappa} = \Gamma^\mu_{\nu\kappa} + \partial_\kappa \log\Omega \cdot \delta^\mu{}_\nu + \partial_\nu \log\Omega \cdot \delta^\mu{}_\kappa - \partial_\lambda \log\Omega \cdot g^{\mu\lambda} g_{\nu\kappa} \;.$$

   (b) Let $\gamma : \mathbb{R} \supseteq I \to M$ be a null geodesic with respect to g. Show that it is also a null geodesic with respect to $\tilde{g}$ (but not necessarily affinely parametrised).

   (c) * Give a counterexample to the above for timelike/spacelike geodesics, i.e., give an explicit example of a Lorentzian manifold (M, g) together with a conformal metric $\tilde{g}$ and a timelike/spacelike geodesic $\gamma : I \to \mathbb{R}$ with respect to g which, however, is not a timelike/spacelike geodesic with respect to $\tilde{g}$.

Solution:

(a) Note that $\tilde{g}^{\mu\lambda} = \frac{1}{\Omega^2} g^{\mu\lambda}$. Then

$$\begin{aligned}
\tilde{\Gamma}^\mu_{\nu\kappa} &= \frac{1}{2}\tilde{g}^{\mu\lambda}(\tilde{g}_{\lambda\nu,\kappa} + \tilde{g}_{\lambda\kappa,\nu} - \tilde{g}_{\nu\kappa,\lambda}) \\
&= \frac{1}{2\Omega^2} g^{\mu\lambda}(\partial_\kappa(\Omega^2 g_{\lambda\nu}) + \partial_\nu(\Omega^2 g_{\lambda\kappa}) - \partial_\lambda(\Omega^2 g_{\nu\kappa})) \\
&= \Gamma^\mu_{\nu\kappa} + \frac{1}{2}\Big(\partial_\kappa \log\Omega^2 \cdot \delta^\mu{}_\nu + \partial_\nu \log\Omega^2 \cdot \delta^\mu{}_\kappa - \partial_\lambda \log\Omega^2 \cdot g^{\mu\lambda} g_{\nu\kappa}\Big) \;.
\end{aligned}$$

(b) Assume that γ is affinely parametrised and let $V = \dot{\gamma}(s)$. We then have $\nabla_V V = 0$. We now compute

$$\begin{aligned}
(\tilde{\nabla}_V V)^\mu &= V^\nu \partial_\nu V^\mu + \tilde{\Gamma}^\mu_{\nu\kappa} V^\nu V^\kappa \\
&= V^\nu \partial_\nu V^\mu + \Gamma^\mu_{\nu\kappa} V^\nu V^\kappa + \Big(2\partial_\kappa \log\Omega \cdot V^\kappa V^\mu - \partial_\lambda \log\Omega \cdot g^{\mu\lambda} g(V,V)\Big) \\
&= 0 + 2\partial_\kappa \log\Omega \cdot V^\kappa V^\mu - \partial_\lambda \log\Omega \cdot g^{\mu\lambda} g(V,V) \;.
\end{aligned}$$

Thus, if V is null we have $\tilde{\nabla}_V V \sim V$, thus γ is also a null geodesic with respect to $\tilde{g}$, but in general not affinely parametrised.

(c) A possible counterexample is $M = \mathbb{R}^2$ with $g = -dt^2 + dx^2$, and $\Omega = e^x$. Then $\gamma(s) = (s, x_0)$ for some $x_0 \in \mathbb{R}$ is an affinely parametrised timelike geodesic with respect to g, we have $\dot{\gamma}(s) = \partial_t$. And we compute with the formula from above

$$(\tilde{\nabla}_V V)^x = -\partial_x \log e^x \cdot (-1) = 1 \;,$$

thus $\tilde{\nabla}_V V$ is not proportional to ∂_t.

Changing the metric to $g = dt^2 - dx^2$, the same is a counterexample for spacelike geodesics.

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2) This question introduces the deSitter spacetime. Consider $4+1$-dimensional Minkowski spacetime, i.e., $\mathbb{R}^5$ with standard Cartesian coordinates $\{v, w, x, y, z\}$ and metric $m = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2$. Let $M \subseteq \mathbb{R}^5$ denote the level set

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2 ,$$

with $\alpha > 0$. Check that this is a timelike hypersurface. Can you sketch it (suppressing some dimensions)?

By restricting the Minkowski metric to the tangent spaces of M we obtain a Lorentzian metric g on M. In fact, the Ricci curvature of the Lorentzian metric g on M satisfies $R_{\mu\nu} = \frac{3}{\alpha^2} g_{\mu\nu}$. The Einstein equations with cosmological constant Λ read

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab} .$$

It thus follows that (M, g) is a solution to the Einstein equations with cosmological constant $\Lambda = \frac{3}{\alpha^2}$ and $T_{ab} = 0$. It is called the *deSitter* spacetime.

We now introduce coordinates on M by $(t, \chi, \theta, \varphi) \overset{\iota}{\mapsto} \big(v, w, x, y, z\big)$ with

$$\begin{aligned} v &= \alpha \sinh(\frac{t}{\alpha}) \\ w &= \alpha \cosh(\frac{t}{\alpha}) \cos \chi \\ x &= \alpha \cosh(\frac{t}{\alpha}) \sin \chi \cos \theta \\ y &= \alpha \cosh(\frac{t}{\alpha}) \sin \chi \sin \theta \cos \varphi \\ z &= \alpha \cosh(\frac{t}{\alpha}) \sin \chi \sin \theta \sin \varphi . \end{aligned}$$

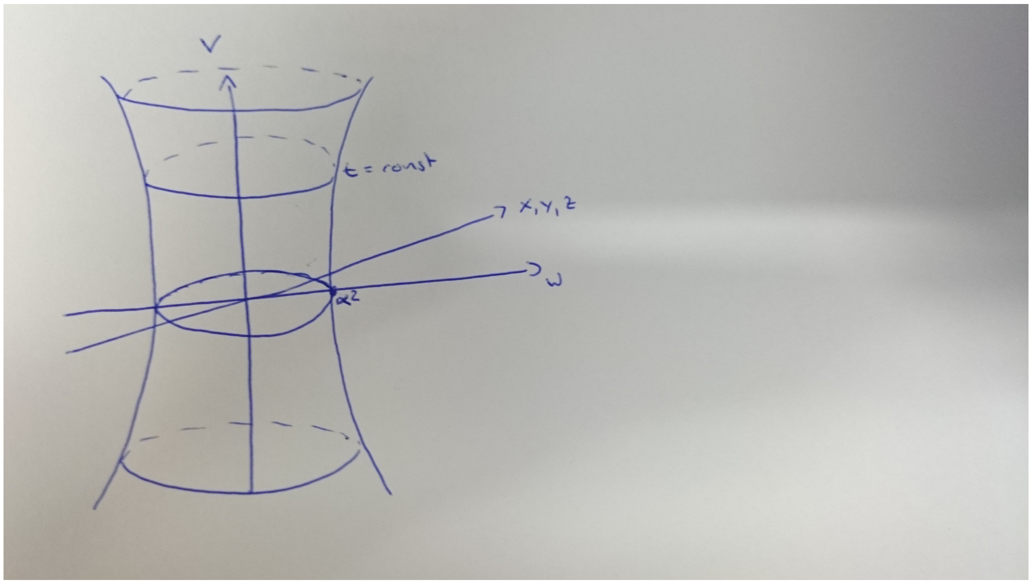
What is the range of these coordinates? Do they cover all of M? Show that in these coordinates the metric g is given by

$$g = -dt^2 + \alpha^2 \cosh^2(\frac{t}{\alpha})(d\chi^2 + \sin^2 \chi [d\theta^2 + \sin^2 \theta \, d\varphi^2]) .$$

Draw the hypersurfaces of constant t in your above sketch. What is their topology, how does their geometry change with coordinate time t?

We now construct the Penrose diagram. Choose a new time-coordinate $\lambda(t)$ which satisfies $\frac{d\lambda}{dt} = \frac{1}{\alpha \cosh(\frac{t}{\alpha})}$. Write the metric in the coordinates $(\lambda, \chi, \theta, \varphi)$ and show that the deSitter spacetime is conformal to part of the Einstein static universe. Which boundary surfaces would you call past/future null infinity? Draw the Penrose diagram. Explain why an observer, even if she observes for an infinite time, cannot observe the entire spacetime. How does this compare to the situation in Minkowski spacetime?

Solution: It is a hypersurface since the differential of the level set function is $2(-vdv + wdw + xdx + ydy + zdz)$, which is non-vanishing on M. It is a timelike hypersurface, since the m-norm of the differential of the level set function equals $4\alpha^2 > 0$ on M, thus the normal is spacelike and the hypersurface is timelike.



The range of the coordinates is $t \in \mathbb{R}$, $0 < \chi < \pi$, $0 < \theta < \pi$, $0 < \varphi < 2\pi$. They cover the whole space apart from the usual points on the $\mathbb{S}^3$ in the spherical coordinates (χ, θ, φ). To compute the metric $g = \iota^* m$ in these coordinates we compute

$$\begin{aligned} dz &= d(\alpha \cosh(\frac{t}{\alpha}) \sin\chi \sin\theta) \sin\varphi + \alpha \cosh(\frac{t}{\alpha}) \sin\chi \sin\theta \cos\varphi d\varphi \\ dy &= d(\alpha \cosh(\frac{t}{\alpha}) \sin\chi \sin\theta) \cos\varphi - \alpha \cosh(\frac{t}{\alpha}) \sin\chi \sin\theta \sin\varphi d\varphi \\ \implies dz^2 + dy^2 &= \big[d(\alpha \cosh(\frac{t}{\alpha}) \sin\chi \sin\theta)\big]^2 + \big(\alpha \cosh(\frac{t}{\alpha}) \sin\chi \sin\theta\big)^2 d\varphi^2 \ . \end{aligned}$$

$$\begin{aligned} dx &= d(\alpha \cosh(\frac{t}{\alpha}) \sin\chi) \cos\theta - \alpha \cosh(\frac{t}{\alpha}) \sin\chi \sin\theta \, d\theta \\ d(\alpha \cosh(\frac{t}{\alpha}) \sin\chi \sin\theta) &= d(\alpha \cosh(\frac{t}{\alpha}) \sin\chi) \sin\theta + \alpha \cosh(\frac{t}{\alpha}) \sin\chi \cos\theta d\theta \\ \implies dx^2 + dy^2 + dz^2 &= \big[d(\alpha \cosh(\frac{t}{\alpha}) \sin\chi)\big]^2 + \big[\alpha \cosh(\frac{t}{\alpha}) \sin\chi\big]^2 d\theta^2 + \big[\alpha \cosh(\frac{t}{\alpha}) \sin\chi \sin\theta\big]^2 d\varphi^2 \ . \end{aligned}$$

$$\begin{aligned} dw &= d(\alpha \cosh(\frac{t}{\alpha}) \cos\chi - \alpha \cosh(\frac{t}{\alpha}) \sin\chi \, d\chi \\ d(\alpha \cosh(\frac{t}{\alpha}) \sin\chi) &= d(\alpha \cosh(\frac{t}{\alpha}) \sin\chi + \alpha \cosh(\frac{t}{\alpha}) \cos\chi d\chi \\ \implies dw^2 + dx^2 + dy^2 + dz^2 &= \big[d(\alpha \cosh(\frac{t}{\alpha}))\big]^2 + \alpha^2 \cosh^2(\frac{t}{\alpha})\big(d\chi^2 + \sin^2\chi\big[d\theta^2 + \sin^2\theta \, d\varphi^2\big]\big) \ . \end{aligned}$$

$$\begin{aligned} dv &= \cosh(\frac{t}{\alpha}) dt \\ d(\alpha \cosh(\frac{t}{\alpha}) &= \sinh(\frac{t}{\alpha}) dt \\ \implies -dv^2 + dw^2 + dx^2 + dy^2 + dz^2 &= -dt^2 + \alpha^2 \cosh^2(\frac{t}{\alpha})\big(d\chi^2 + \sin^2\chi\big[d\theta^2 + \sin^2\theta \, d\varphi^2\big]\big) \ . \end{aligned}$$

The surfaces of constant t are round 3-spheres of radius $\alpha \cosh(\frac{t}{\alpha})$. They have minimum radius at time $t = 0$ and expand infinitely for $t \to \pm\infty$.

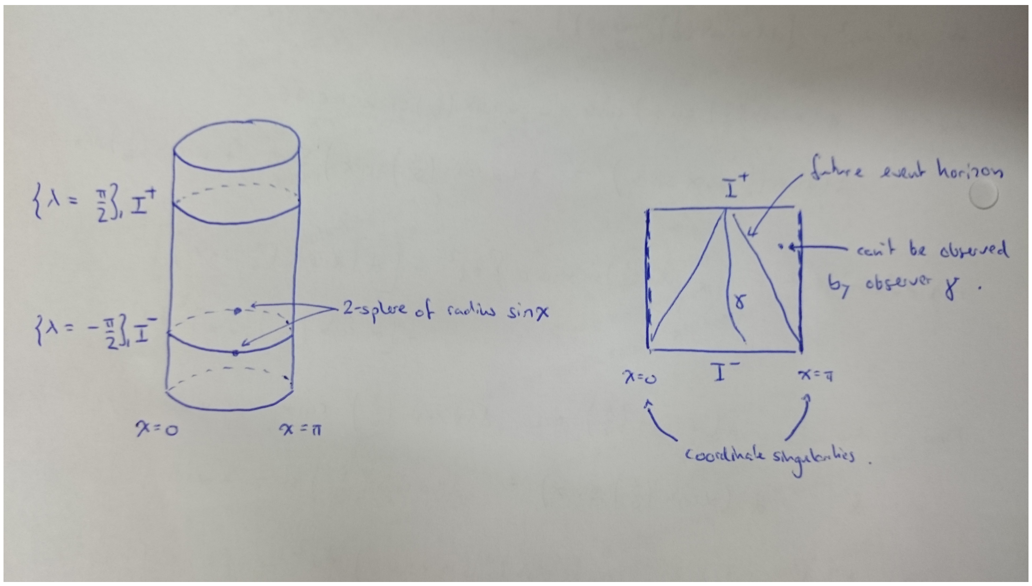
Integrating $\frac{d\lambda}{dt} = \frac{1}{\alpha \cosh(\frac{t}{\alpha})}$ we obtain

$$\lambda = 2 \arctan(\exp(\frac{t}{\alpha})) + c \,.$$

We choose $c = -\frac{1}{2}\pi$, so that $\lambda \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The metric becomes

$$g = \alpha^2 \cosh^2(\frac{t}{\alpha}) \underbrace{\left[-d\lambda^2 + d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta \, d\varphi^2)\right]}_{\text{Einstein static universe}} \,.$$

Recall that the Einstein static universe is $\mathbb{R}\times\mathbb{S}^3$ with coordinates $\{\lambda, \chi, \theta, \varphi\}$ together with the underbraced metric. Thus, deSitter is conformal to the part $(-\frac{\pi}{2}, \frac{\pi}{2})\times\mathbb{S}^3$ of the Einstein static universe. We can attach the past and the future boundary – here, there is no differentiation between timelike and null infinities. Spacelike infinity is not present.



It is clear from the Penrose diagram that given an observer γ there are always events which she will never be able to see. This is different to Minkowski spacetime (which is immediate from its Penrose diagram) – at least if the observer does not accelerate infinitely so that she asymptotes to future null infinity.

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3) Let (M, g) be a Lorentzian manifold and let $\Sigma \subseteq M$ be a Killing horizon of a Killing vector field T. Show that the surface gravity κ, given by $\nabla_T T|_\Sigma = \kappa T|_\Sigma$, satisfies

$$\kappa^2 = -\frac{1}{2}\big[(\nabla_a T_b)(\nabla^a T^b)\big]|_\Sigma \,.$$

Hint: Use that T is hypersurface orthogonal on Σ.

Solution:

We use that T is hypersurface orthogonal on Σ, i.e., <u>on Σ</u> we have

$$\begin{aligned} 0 &= T_{[a}\nabla_b T_{c]} \\ &= \frac{1}{6}\Big(T_a\nabla_b T_c - T_a\nabla_c T_b + T_c\nabla_a T_b - T_c\nabla_b T_a + T_b\nabla_c T_a - T_b\nabla_a T_c\Big) \\ &= \frac{1}{6}\Big(2T_a\nabla_b T_c + 2T_c\nabla_a T_b + 2T_b\nabla_c T_a\Big) \,, \end{aligned}$$

where we have used Killing's equation $\nabla_a T_b = -\nabla_b T_a$. Thus we have

$$0 = T_a \nabla_b T_c + T_c \nabla_a T_b + T_b \nabla_c T_a \ .$$

We now contract this with $\nabla^b T^c$ and we use that $\partial_a(T_b T^b) = -2\kappa T_a$ (derived in lectures) which implies $\nabla_a T_b \cdot T^b = -\kappa T_a$, to obtain

$$\begin{aligned} 0 &= T_a(\nabla_b T_c)(\nabla^b T^c) + T_c(\nabla^b T^c)(\nabla_a T_b) + (\nabla_T T)^c(\nabla_c T_a) \\ &= T_a(\nabla_b T_c)(\nabla^b T^c) - \kappa T^b \nabla_a T_b + \kappa T^c \nabla_c T_a \\ &= T_a(\nabla_b T_c)(\nabla^b T^c) + \kappa^2 T_a + \kappa^2 T_a \ , \end{aligned}$$

which yields the result.

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4) This problem guides you through the derivation of the laws of geometric optics in curved spacetime. Let (M, g) be a Lorentzian manifold and $F \in \Omega^2(M)$ a smooth two-form, the Faraday tensor. The source-free Maxwell equations read

$$dF = 0 \qquad \text{and} \qquad \nabla^\mu F_{\mu\nu} = 0 \ . \tag{1}$$

Since $dF = 0$, one can locally[1] find a potential $A \in \Omega^1(M)$ such that $dA = F$.

(a) Show that F satisfies (1) iff A satisfies

$$\nabla^\mu \nabla_\mu A_\nu - \nabla_\nu \nabla^\mu A_\mu - R_{\kappa\nu} A^\kappa = 0 \ . \tag{2}$$

(b) Recall the gauge freedom $\tilde{A}_\mu = A_\mu + \partial_\mu \chi$. Show that any solution A_μ can be put into the Lorentz gauge $\nabla^\mu \tilde{A}_\mu = 0$ by solving an inhomogeneous wave equation for χ (note that $\Box_g \chi := \nabla^\mu \nabla_\mu \chi$ is the wave operator in curved spacetimes).

(c) We now construct approximate solutions of (2) in the Lorentz gauge, i.e., of

$$\nabla^\mu \nabla_\mu A_\nu - R_{\mu\nu} A^\mu = 0 \qquad \text{and} \qquad \nabla^\mu A_\mu = 0 \ . \tag{3}$$

We make the *geometric optics ansatz*

$$A_\nu^{\text{approx}} = \frac{1}{\lambda} a_\nu e^{i\lambda\phi} \ , \tag{4}$$

where $a_\nu \in \Omega^1(M)$, $\phi \in C^\infty(M)$, and $\lambda > 0$ is a large parameter. Compute $\nabla^\mu \nabla_\mu A_\nu^{\text{approx}} - R^\mu{}_\nu A_\mu^{\text{approx}}$ and $\nabla^\mu A_\mu^{\text{approx}}$, group the terms according to their power in λ, and show that the equations (3) are satisfied by (4) up to order $\mathcal{O}(\frac{1}{\lambda})$ iff a_μ and ϕ satisfy

$$\nabla^\mu \phi \cdot a_\mu = 0 \ , \qquad \nabla^\mu \phi \cdot \nabla_\mu \phi = 0 \ , \qquad \nabla^\mu \phi \cdot \nabla_\mu a_\nu + \frac{1}{2} \Box_g \phi \cdot a_\nu = 0 \ . \tag{5}$$

Also infer that if the large parameter λ is large compared to covariant derivatives of a_ν and the spacetime curvature $R_{\mu\nu}$, then (4) with a_ν and ϕ satisfying (5) is a good approximate solution of (3).

(d) The vector $k := (d\phi)^\sharp$ is called the *wave vector*. Can you justify this terminology?

Consider an observer following a timelike curve γ parametrised by proper time who carries with himself an orthonormal basis $\{E_0 = \dot{\gamma}, E_1, \dots, E_n\}$ of the tangent space which forms his local reference frame. Show that he would interpret the quantity $-\frac{1}{2\pi}\lambda \cdot E_0 \phi|_p = -\frac{1}{2\pi}\lambda \cdot g(E_0, k)|_p$ as the frequency of the electromagnetic wave (4) at a point p on his worldline.

[1] Or in fact in any *simply connected* domain – so for example in particular in all of the Schwarzschild spacetime.

(e) The equation $\nabla^\mu\phi \cdot \nabla_\mu \phi = 0$ is known as the *Eikonal equation.* It can be always solved locally. Show that it implies that the wave vector k is null and that it satisfies $\nabla_k k = 0$, i.e., it is propagated affinely along null geodesics.

(f) Let us now decompose the covector amplitude a_ν in (4) as $a_\nu = \alpha \cdot f_\nu$, with the *amplitude* $\alpha \in C^\infty(M)$ and the *polarisation covector* $f_\nu \in \Omega^1(M)$. It follows from the first equation in (5) that $f_\nu k^\nu = 0$, i.e., the polarisation vector is orthogonal to the wave vector, i.e., it must be tangent to the null hypersurfaces $\phi = \mathit{const}$. Show that to leading order in λ the electric and magnetic fields do not change by adding a multiple of k_ν to f_ν.

Thus, only if f is spacelike do we have a non-vanishing electromagnetic field. Without loss of generality we can thus normalise the polarisation covector by $f_\nu f^\nu = 1$. Show that the third equation in (5) implies the propagation equation

$$\nabla_k \alpha + \frac{1}{2}\nabla^\mu k_\mu \cdot \alpha = 0 \tag{6}$$

for the amplitude along the integral curves of k and that the polarisation covector is parallely propagated along k, i.e.,

$$\nabla_k f = 0\ .$$

Note that (6) in particular implies that if α vanishes on some point on an integral curve of k (which are null geodesics by $\nabla_k k = 0$), then it vanishes along the whole curve. *This makes precise in which sense and under what conditions 'light propagates along null geodesics in general relativity'.*

(g) Consider now the Schwarzschild spacetime with an observer γ_A following a timelike curve of constant $r = r_A > 2M$, $\theta = \theta_0$, $\varphi = \varphi_0$ and another observer γ_B following a timelike curve of constant $r = r_B > r_A$, $\theta = \theta_0$, $\varphi = \varphi_0$. Make precise, using the laws of geometric optics derived in this exercise, that a high-frequency light signal of frequency f_A as measured by observer A, sent from A to B, arrives red-shifted at observer B with a frequency $f_B = \sqrt{\frac{1-\frac{2M}{r_A}}{1-\frac{2M}{r_B}}} f_A$.

Solution:

(a) Using $F_{\mu\nu} = dA_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and $\nabla_\nu \nabla_\mu A^\kappa - \nabla_\mu \nabla_\nu A^\kappa = R^\kappa{}_{\sigma\nu\mu} A^\sigma$ this follows immediately from the second of Maxwell's equations.

(b) $\nabla^\mu \tilde{A}_\mu = \nabla^\mu A_\mu + \nabla^\mu \nabla_\mu \chi$. Thus $\nabla^\mu \tilde{A}_\mu = 0$ follows from solving $\Box_g \chi = -\nabla^\mu A_\mu$. Comments: 1) Note that this does not require A to be a solution. 2) The wave equation can be solved on a large class of Lorentzian manifolds, namely globally hyperbolic ones.

(c) We compute

$$\nabla^\mu A_\mu^{\text{approx}} = i\nabla^\mu\phi \cdot a_\mu e^{i\lambda\phi} + \frac{1}{\lambda}\nabla^\mu a_\mu \cdot e^{i\lambda\phi}\ ,$$

and

$$\begin{aligned}\nabla^\mu\nabla_\mu A_\nu^{\text{approx}} - R^\mu{}_\nu A_\mu^{\text{approx}} &= \nabla^\mu\Big(i\nabla_\mu\phi\cdot a_\nu e^{i\lambda\phi} + \frac{1}{\lambda}\nabla_\mu a_\nu \cdot e^{i\lambda\phi}\Big) - R^\mu{}_\nu A_\mu^{\text{approx}} \\ &= -\lambda\nabla^\mu\phi\nabla_\mu\phi\cdot a_\nu e^{i\lambda\phi} + i(2\nabla^\mu\phi\nabla_\mu a_\nu + \Box_g\phi\cdot a_\nu)e^{i\lambda\phi} + \frac{1}{\lambda}\Big(\nabla^\mu\nabla_\mu a_\nu - R^\mu{}_\nu a^\nu\Big)e^{i\lambda\phi}\ .\end{aligned}$$

The rest follows directly from this.

Comment: Note that while $\frac{|\nabla a|}{\lambda}$ is assumed small, this does not imply that $\frac{|a|}{\lambda}$ is small, i.e., the solution is strictly larger than the error term. In other words, one assumes that the amplitude varies slowly compared to the frequency.

(d) Start by observing that $F_{\mu\nu}^{\text{approx}} = i(\partial_\mu \phi \cdot a_\nu - \partial_\nu \phi \cdot a_\mu)e^{i\lambda\phi} + \mathcal{O}(\frac{|\nabla a|}{\lambda})$.

The surfaces of constant ϕ are the surfaces of constant phase, i.e., of the wave fronts. In particular they give the crests and troughs of the wave. Thus $d\phi$ deserves the name of wave vector.

Introduce a local coordinate system centred at p with $E_0 = \partial_t$ and $E_i = \partial_i$. Then $\phi(t, \underline{x}) = \phi(0) + \partial_t \phi(0) t + \sum_i \partial_i \phi(0) x_i + \ldots$. Thus we have near p that $e^{i\lambda\phi} \approx e^{i\lambda\phi(0)} \cdot e^{i\lambda(\partial_t\phi(0)t + \sum_i \partial_i \phi(0) x_i)}$. This shows that the observer would interpret the quantity $-\frac{\lambda E_0 \phi|_p}{2\pi}$ as the frequency at the point p.

(e) The fact that the wave vector is null is immediate. Furthermore covariant differentiation of the Eikonal equation and using that the Hessian of a function is symmetric, we obtain

$$0 = \nabla_\mu (k_\nu k^\nu) = 2(\nabla_\mu k_\nu) k^\nu = 2(\nabla_\mu \nabla_\nu \phi) k^\nu = 2(\nabla_\nu \nabla_\mu \phi) k^\nu = 2k^\nu \nabla_\nu k_\mu \; .$$

(f) Consider $A_\nu^{\text{approx}} = \frac{1}{\lambda} \alpha k_\nu e^{i\lambda\phi}$. Then

$$\begin{aligned} F_{\mu\nu}^{\text{approx}} &= \frac{1}{\lambda} \nabla_\mu \alpha \cdot k_\nu e^{i\lambda\phi} + \frac{1}{\lambda} \alpha \nabla_\mu k_\nu \cdot e^{i\lambda\phi} + i\alpha k_\mu k_\nu e^{i\lambda\phi} - \frac{1}{\lambda} \nabla_\nu \alpha \cdot k_\mu e^{i\lambda\phi} - \frac{1}{\lambda} \alpha \nabla_\nu k_\mu e^{i\lambda\phi} - i\alpha k_\nu k_\mu e^{i\lambda\phi} \\ &= \mathcal{O}(\frac{1}{\lambda}) \end{aligned}$$

We compute from the third equation in (5) that

$$0 = k^\mu \nabla_\mu (\alpha f_\nu) + \frac{1}{2} (\nabla^\mu k_\mu) \alpha f_\nu = (\nabla_k \alpha) f_\nu + \frac{1}{2} (\nabla^\mu k_\mu) \alpha f_\nu + \alpha (\nabla_k f)_\nu \; .$$

Contracting with f^ν and noticing that $(\nabla_k f)_\nu f^\nu = \frac{1}{2} \nabla_k g(f,f) = 0$ because of normalisation, we obtain first $\nabla_k \alpha + \frac{1}{2} \nabla^\mu k_\mu \alpha = 0$ and then also $\nabla_k f = 0$.

(g) $g = -(1 - \frac{2M}{r})\, dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\sigma^2$. Use outgoing EF coordinates $u = t - r^*$, then

$$g = -(1 - \frac{2M}{r})\, du^2 - du \otimes dr - dr \otimes du + r^2 d\sigma^2 \; .$$

The inverse metric is

$$g^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 - \frac{2M}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{pmatrix} .$$

The curves $r \mapsto (u_0, r, \theta_0, \varphi_0)$ are <u>affinely</u> parametrised outgoing null geodesics. This follows easily from checking that $\Gamma^r_{rr} = 0$, using the above expressions of the metric and its inverse in outgoing EF coordinates.

The observer at $r = r_A$ has normalised velocity $E_0^{(A)} = \frac{1}{\sqrt{1 - \frac{2M}{r_A}}} \partial_t$ and the observer at $r = r_B$ has normalised velocity $E_0^{(B)} = \frac{1}{\sqrt{1 - \frac{2M}{r_B}}} \partial_t$. Using that $\partial_t = \partial_u$, we compute $-g(E_0^{(A)}, \partial_r) = \frac{1}{\sqrt{1 - \frac{2M}{r_A}}}$ and $-g(E_0^{(B)}, \partial_r) = \frac{1}{\sqrt{1 - \frac{2M}{r_B}}}$. Thus, the light signal emitted by observer A with frequency f_A corresponds to a wave vector $2\pi f_A \cdot \sqrt{1 - \frac{2M}{r_A}} \partial_r$ and thus to the affinely parametrised null geodesics $r \mapsto (u_0, 2\pi f_A \cdot \sqrt{1 - \frac{2M}{r_A}} \cdot r, \theta_0, \varphi_0)$. It thus follows that

$$f_B = \sqrt{\frac{1 - \frac{2M}{r_A}}{1 - \frac{2M}{r_B}}} f_A \; .$$

5) Let $\mathcal{M} = \mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2$ with the standard $\{t, r, \theta, \varphi\}$ coordinates where $r_+ = M + \sqrt{M^2 - a^2}$, $M > 0$, and $0 < a < M$. We define the Kerr metric g on $\mathcal{M}$ by

$$g = -(1 - \frac{2Mr}{\rho^2})\, dt^2 - \frac{2Mra\sin^2\theta}{\rho^2}\, (dt \otimes d\varphi + d\varphi \otimes dt) + \frac{\rho^2}{\Delta}\, dr^2 + \rho^2\, d\theta^2 + \Big(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\Big) \sin^2\theta\, d\varphi^2 \,, \tag{7}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Consider a stationary observer A with velocity $u(\partial_t + \Omega\partial_\varphi)$ at some value of $r_0 \in (r_+, \infty)$ and some value of $\theta_0 \in (0, \pi)$, where $u > 0$ is chosen such that the velocity is normalised. Show that Ω corresponds to the angular frequency of A *as seen by an observer B with velocity ∂_t at infinity who is at rest with respect to the asymptotic Lorentz frame.*

Thus, an observer with $\Omega = 0$ appears static from infinity 'with respect to the fixed stars'.

(Hint: The movement of A as seen by B depends on the null geodesics connecting A's worldline with B's. Use the symmetries of the Kerr spacetime to answer this question without actually computing the null geodesics.)

Solution: Let A's worldline be $t \mapsto (t, r_0, \theta_0, \varphi_0 + \Omega t)$ (suitably parametrised so that it is unit speed, but this is not relevant here) and let B's worldline be $t \mapsto (t, r_1, \theta_1, \varphi_1)$ with $r_1 \gg 1$. Consider time t_{B_1} for observer B and consider all the past directed null geodesics emanating from $(t_{B_1}, r_1, \theta_1, \varphi_1)$. There will be at least one point of intersection with A's worldline. Thus, light emitted from this point by A, let us call it $(t_{A_1}, r_0, \theta_0, \varphi_0 + \Omega t_{A_1})$, is seen by observer B at time t_{B_1}. The exact configuration is not important. Then, when time increases for B, the direction from which this light ray is arriving from A's worldline is changing, since the mutual configuration is changing. Exactly how the direction is changing is quite a complicated thing to compute, but again it is not needed here. However, when observer A has moved for coordinate time $\Delta t = \frac{2\pi}{\Omega}$, he is at point $(t_{A_1} + \frac{2\pi}{\Omega}, r_0, \theta_0, \varphi_0 + \Omega t_{A_1})$. Since ∂_t is a KVF, the geometry is invariant if one translates everything by $t \mapsto t + \frac{2\pi}{\Omega}$. Thus the light emitted by observer A from point $(t_{A_1} + \frac{2\pi}{\Omega}, r_0, \theta_0, \varphi_0 + \Omega t_{A_1})$ arrives at observer B at time $t_{B_1} + \frac{2\pi}{\Omega}$ exactly from the same direction as it did at time t_{B_1}. Thus B will observe A rotating with a period of $\frac{2\pi}{\Omega}$. Also note that t is proper time for an observer at infinity.

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6) * Show that the Kerr metric (7) from the last problem reduces to

(a) the Schwarzschild metric for $a = 0$

(b) the Minkowski metric in spheroidal coordinates for $M = 0$, but $a \neq 0$. Here, the spheroidal coordinates in Minkowski spacetime are given by $x = (r^2 + a^2)^{\frac{1}{2}} \sin\theta\cos\varphi$, $y = (r^2 + a^2)^{\frac{1}{2}} \sin\theta\sin\varphi$, $z = r\cos\theta$. Note that the surfaces $r = const$ are spheroids $\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$.

Solution: This is direct computation.