Schwarzschild and Birkhoff a la Weyl

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Abstract

We provide a simple derivation of the Schwarzschild solution in General Relativity, generalizing an early approach by Weyl, to include Birkhoff's theorem: constancy of the mass; its deeper, Hamiltonian, basis is also given. Our procedure is illustrated by a parallel derivation of the Coulomb field and constancy of electric charge, in electrodynamics.

I. INTRODUCTION

Deriving and understanding some of the basic properties of the fundamental – Schwarzschild – geometry is a significant hurdle in elementary expositions of General Relativity (GR). Indeed, no less distinguished an investigator than Hermann Weyl proudly discovered an enticing shortcut¹, totally unjustified at the time, but legitimized much later^{2,3}. More recently, it has been used for more complicated gravity models as well⁴. Weyl's result is, however, incomplete: He got the famous $\left(1 - \frac{2m}{r}\right)$ factor but assumed a priori, rather than derived, the constancy of m. The latter is almost as important a property as the factor itself, and of course, a consequence of Einstein's equations. This property is Birkhoff's theorem⁵ – absence of monopole radiation in GR. Our aim here is to retain the attractiveness of Weyl's shortcut, while simultaneously proving the absence of the $\dot{m} \neq 0$ "non-solutions". In order to clarify the physics of this approach, we first establish it in the simpler, but quite relevant, context of the Coulomb field in electrodynamics. We will also briefly discuss the theorem's basis in the deeper context of the theories' Hamiltonian forms.

II. ELECTRODYNAMICS A LA WEYL

We derive the Coulomb field in Maxwell theory, in order to introduce and suitably extend the Weyl method to include the vector Birkhoff's theorem – constancy of electric charge. Weyl's general approach was to exploit the special symmetries of the desired solution by using suitable coordinates and gauges, then insert the simplified field variables into the action, and vary only these remaining functions instead of the original set of variables.

Spherical symmetry means that \mathbf{r} is the only vector, hence the (\mathbf{A}, A_0) are restricted to the form

$$\mathbf{A} = A_r(r, t)\hat{\mathbf{r}} \qquad A_0 = A_0(r, t)$$
$$\mathbf{B} = 0 \qquad \mathbf{E} = \left(A'_0 - \dot{A}_r\right)\hat{\mathbf{r}},\tag{1}$$

primes and overdots respectively indicate radial and temporal derivatives. The vector potential is necessarily a pure gauge, so can be removed. As we shall see, this seemingly attractive step loses the Birkhoff part of Maxwell's equations and hence requires the additional assumption of time-independence, thereby missing the fact that the latter is implied by the theory. Let us now insert (1) into the Maxwell action,

$$I_{Max} = \frac{1}{2} \int d^4x \left(\mathbf{E}^2 - \mathbf{B}^2 \right) \,, \tag{2}$$

to obtain the reduced form

$$I_{Max} \to 2\pi \int \left(\dot{A}_r - A_0'\right)^2 r^2 dr dt; \tag{3}$$

we consider only source-free regions throughout. [This approach is valid in arbitrary dimensions, with $r^2 \to r^{D-2}$.] If we impose Coulomb gauge, $A_r = 0$ before varying, we immediately obtain the single field equation $(r^2A'_0)' = 0$, whose solution is of course $A_0 = q(t)/r$. However, we cannot then infer $\dot{q} = 0$, the subset that solves Maxwell's equations. If instead, we only set $A_r = 0$ after varying (2), we learn that the time derivative of the field equation also vanishes – the variation of the gauge part gives

$$\frac{\delta I}{\delta A_r}\Big|_{A_r=0} = 4\pi r^2 \dot{A}_0'(r,t) = 0 = \dot{q}$$
(4)

Of course, we need not set $A_r = 0$ at all, the gauge invariant content of (2) being

$$\nabla \cdot \mathbf{E} = 0 = \nabla \cdot \dot{\mathbf{E}},\tag{5}$$

since varying A_r and A_0 manifestly yields the respective time and space derivatives of the same quantity, namely **E**. Note that the second equation

$$\nabla \cdot \dot{\mathbf{E}} = \nabla \cdot (\nabla \times \mathbf{B}) = 0, \tag{6}$$

reflects the Bianchi identities $\partial_{\mu} (\partial_{\nu} F^{\mu\nu}) = 0$, i.e. $\partial_0 (\partial_i F^{0i}) + \partial_j (\partial_{\mu} F^{\mu j}) = 0$, as the last term is a field equation.

The time-constancy of spherical solutions is manifest in the action's Hamiltonian form, where $(-\mathbf{E}, \mathbf{A})$ are independent variables, the canonical "(p, q)" pairs:

$$I_{Max} = -\int d^4x \left[\mathbf{E}^T \cdot \dot{\mathbf{A}}^T + \mathbf{E}^L \left(\dot{\mathbf{A}}^L - \boldsymbol{\nabla} A_0 \right) + \frac{1}{2} \left\{ \mathbf{E}^2 + \left(\boldsymbol{\nabla} \times \mathbf{A}^T \right)^2 \right\} \right].$$
(7)

We have used the orthogonal decomposition of a vector,

$$\mathbf{V} = \mathbf{V}^T + \mathbf{V}^L , \quad \nabla \cdot \mathbf{V}^T \equiv 0 \equiv \nabla \times \mathbf{V}^L , \quad \int d^3 r \, V^T \cdot W^L = 0 . \tag{8}$$

Since time-dependence only appears in the " $p\dot{q}$ " terms, and there are no transverse spherically symmetric vectors, we learn immediately from varying the surviving component, \mathbf{A}^L , that $\dot{\mathbf{E}}^L = 0$, the rest of the action being \mathbf{A}^L -independent. The lesson, one that will carry over unaltered to GR, is that Weyl's approach, using as few functions as gauge choice allows, lulls one into the unjustified belief that all is timeindependent just because only spatial derivatives remain. This point is relevant because, even if one does not assume time-independence but prematurely drops A_r , the resulting equation for A_0 has no explicit time-derivatives.

This is a good place to discuss the validity of the Weyl procedure itself. For the linear Maxwell theory, it is easy enough to understand the "commutativity" between first inserting a symmetric ansatz in an action before varying, or only doing so after full variation. Clearly, $\dot{\mathbf{E}} = \nabla \times \mathbf{B}$ and $\nabla \cdot \mathbf{E} = 0$ immediately degenerate, with the spherical symmetry requirement that $\mathbf{E} = \nabla \phi$, $\mathbf{B} = 0$, into $\nabla^2 \phi = 0$, $\dot{\phi} = 0$, which can then be variously decomposed in different gauges. So no information is lost by varying the action if (and only if) both functions are kept.

More generally, it is intuitively pretty clear that, since the solutions are extrema also within the set of spherically symmetric trial variables, we will not get any false ones. That we will also not miss any true solutions in this way is pretty reasonable as well. We will, however, say no more on this deep and difficult problem nor on its extensive fine print; for this, we refer to the original work² and to a later exeges specifically in GR³. Some of the perils involved are illustrated in a recent note⁶.

III. GR WEYL – STATIC

The stage has now been set for our GR target. We approach it in two steps. The first still adheres to the original Weyl line, losing time-independence. That will be followed by the full Birkhoff treatment.

We begin with the general form of a spherically symmetric metric tensor $g_{\mu\nu}$ or its corresponding interval $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$. For completeness, we first set to rest the misplaced worry that spherical symmetry cannot mean anything in a theory with general coordinate invariance; this is a misunderstanding of coordinates, having nothing to do with geometry. The fancy answer is that symmetries are characterized by the existence of (one or more) Killing vectors $X^{(a)}_{\mu}$ obeying the invariant equation $D_{\mu}X^{(a)}_{\nu} + D_{\nu}X^{(a)}_{\mu} \equiv$ $\partial_{\mu}X^{(a)}_{\nu} + \partial_{\nu}X^{(a)}_{\mu} - (\partial_{\nu}g_{\mu\alpha} + \partial_{\mu}g_{\nu\alpha} - \partial_{\alpha}g_{\mu\nu})X^{\alpha(a)} = 0$. For example, if there is an $X^{(a)}_{\mu}$ which takes the non-invariant form $X^{(a)}_{\mu} = g_{\mu a}$ in some coordinates, then the Killing equation immediately reduces to the statement that (in that frame) $\partial_a g_{\mu\nu} = 0$. But it suffices simply to remember what spherical symmetry means in Cartesian coordinates: given that x^i is the only vector and δ_{ij} the only 2-tensor that can appear, then

$$g_{ij} = A\delta_{ij} + Bx^i x^j \quad g_{0i} = Cx^i \quad g_{00} = -D$$
(9)

where (A, B, C, D) depend only on $r^2 \equiv (x^2 + y^2 + z^2)$ and any other "irrelevant" parameters such as time. The corresponding interval is:

$$ds^{2} = -Ddt^{2} + (A + Br^{2})dr^{2} + Ar^{2}d\Omega + 2Crdrdt$$
(10)

where $d\Omega$ is the usual unit 2-sphere element, since $x^i x^j dx^i dx^j \equiv r^2 dr^2$, $\delta_{ij} dx^i dx^j = dr^2 + r^2 d\Omega$. This four-function parametrization really consists of two physical, plus two gauge, components – double the (A_0, A_r) set of vector theory. Weyl's choice was to diagonalize away the drdt term, and use Schwarzschild coordinates, A = 1, leaving just one spatial and one temporal metric component. We begin with the more instructive choice in which all three functions (A, B, D) are kept, but still dropping the off-diagonal C. The latter is in fact precisely the analog of the first pass in Maxwell theory, so we will not yet achieve Birkhoff's theorem; indeed, this pinpoints where the original Weyl ansatz is insufficient and requires the redundant assumption of time-independence.

Our starting point then is the 3-function interval

$$ds^{2} = -ab^{2}dt^{2} + a^{-1}dr^{2} + c^{2}d\Omega$$
(11)

where we have made things a lot easier to calculate by the above (a, b) parametrization. There is no loss of generality in this, just looking ahead to the b = 1 result for Schwarzschild.

Calculation of the curvature cannot be avoided, even here, but it is mercifully short and yields

$$I_E = \int d^4x \sqrt{-g} R \Rightarrow I_E(a, b, c) = I_r + I_t$$
(12)

$$I_r = 8\pi \int dt dr \left(ab'(c^2)' + b \left(1 + c'(ac)' \right) \right)$$
(13)

$$I_t = 8\pi \int dt dr \frac{\dot{c}}{a^2 b} \left(c\dot{a} - a\dot{c} \right). \tag{14}$$

The two parts I_r and I_t of the action contain either space or time derivatives, but not both. The original 2-function Weyl ansatz was to set c = r before varying, which is why he would never see I_t but only I_r ; this leaves

$$I_W(a,b) = 8\pi \int dr \left(b + rab'\right) \tag{15}$$

which immediately yields the "Schwarzschild" result $a = 1 - \frac{2m}{r}$, and $b = b_0$, but with possibly time-dependent (b_0, m) . Time-dependence of b_0 is irrelevant as it can be absorbed into dt by fixing the remaining $t \to t'(t)$ gauge freedom.

If we keep all three functions, but drop the time-dependence, then (a, b) are parametrized by c, which stays undetermined:

$$a = \frac{1}{c^{\prime 2}} \left(1 - \frac{2m}{c} \right)$$

$$b = b_0 c^{\prime},$$
(16)

corresponding to the interval

$$ds^{2} = -b_{0}^{2} \left(1 - \frac{2m}{c}\right) dt^{2} + \frac{1}{1 - \frac{2m}{c}} dc^{2} + c^{2} d\Omega,$$
(17)

using $dr^2 = dc^2/c'^2$. This result shows the very special role played by Schwarzschild coordinates; they are not so much a gauge as the natural parametrization of the interval in terms of the 2-sphere "orbits". Indeed, writing c = r in (17) is more an exercise in penmanship than a choice of gauge!

IV. BIRKHOFF'S THEOREM

The Maxwell example linked absence of monopole radiation to that of "scalar" – helicity zero – modes. Let us first turn to linearized gravity, its direct counterpart. Here the Hamiltonian form is expressed in terms of the conjugate pair of spatial tensors (π^{ij} , $h_{ij} \equiv g_{ij} - \delta_{ij}$). The tensorial orthogonal transverse-longitudinal decomposition can be written as

$$h_{ij} = h_{ij}^{TT} + \left(\partial_i h_j^T + \partial_j h_i^T\right) + \nabla^{-2} \partial_{ij}^2 h^L + \frac{1}{2} \left(\partial_{ij}^2 - \frac{\delta_{ij}}{\nabla^2}\right) h^T.$$
(18)

For our purposes, it suffices to note that spherically symmetric tensors lack the transversetraceless (TT) tensor quadrupole, and transverse vector (h_i^T) dipole, modes. The action

$$I_{\ell in}[\pi, h] = \int d^3r dt [\pi^{ij} h_{ij} - H(\pi, h)]$$
(19a)

then reduces to^7

$$I_{\ell in}[\pi,h] \to \int d^3r dt \left(\pi^L \dot{h}^L + \frac{1}{2}\pi^T \dot{h}^T - H\right).$$
(19b)

The Hamiltonian's details are irrelevant: all that counts for us is its (abelian) gauge invariance, that it is independent of the two gauge functions (h^L, π^T) . Hence, just as in Maxwell theory, we may immediately conclude from their variation that

$$\partial_0 h^T = 0 = \partial_0 \pi^L. \tag{20}$$

This time-independence, equivalent to $\partial_0 (\nabla \cdot \mathbf{E}) = 0$ is also a direct consequence of the linearized Bianchi identities which state that (on shell) $\partial_0 G_{\ell in}^{0\mu} = 0$, precisely the same two statements as (6). Since our fields are tensorial, they have four (energy-momentum) conservation laws rather than the single one of electrodynamics, though here there is only radial momentum left. The full GR action can also be gauge-fixed to the simple (seeming!) form⁷.

$$I_E = \int d^4x \left(\pi_{TT}^{ij} \dot{g}_{ij}^{TT} - H(TT) \right) \,. \tag{21}$$

The only time dependence is in the "TT" modes. Thus Birkhoff's theorem holds also in full GR and indeed even rids one of dipole radiation since dipoles cannot construct "TT" tensors – in either the linearized or the full theory.

Let us now proceed to our concrete Weyl setting. Instead of keeping all four metric components in the generic interval (10), we just introduce the off-diagonal one, which is effectively the above gauge function π^T in the spherical case. In order to avoid pedantic overkill, let us use Schwarzschild coordinates *ab initio* here (since *c* just defines the "radial" coordinate, we lose nothing by doing so from the start) and concentrate on the three (r, t)dependent functions (a, b, f)

$$ds^{2} = -ab^{2}dt^{2} + a^{-1}dr^{2} + r^{2}d\Omega + 2bfdrdt;$$
(22)

writing the cross term as bf simplifies the calculation. The full three-function action is neither pretty nor useful: all we need are the new terms linear in f, since we will set f = 0after varying anyway (analogous to the electrodynamics case, where the gauge $A_r = 0$ is our choice). The action then simply reduces to the old Weyl term I(a, b) plus the f-term that will act as a Lagrange multiplier enforcing time-independence of m as follows:

$$I(a, b, t) = 8\pi \int dr dt \left(b \left(r - ar \right)' + a^{-1} r f \dot{a} \right).$$
(23)

The respective variations then yield (at f = 0) the desired constant mass geometry:

$$\frac{\delta I}{\delta a}\Big|_{f=0} = rb' = 0 \Rightarrow b = b(t)$$
(24)

$$\frac{\delta I}{\delta b}\Big|_{f=0} = (r-ar)' = 0 \Rightarrow a = 1 - \frac{2m(t)}{r}$$
(25)

$$\frac{\delta I}{\delta f}\Big|_{f=0} = ra^{-1}\dot{a} = 0 \Rightarrow \dot{m} = 0.$$
⁽²⁶⁾

As before, the time-dependence of $b_0(t)$ can be removed by a pure time redefinition, leaving us with the correct Birkhoff statement $\dot{m} = 0$ as the gauge-varied field equation, just as $\dot{q} = 0$ came from the analogous one in electrodynamics.

V. SUMMARY

We have attempted to present a logical and intuitive basis for deriving and understanding the Schwarzschild solution and its time-independence. We followed the Maxwell example to display the pattern of "true-plus-gauge" variables and the role played by the latter in ensuring constancy of the corresponding "charges".

Using the suitably extended Weyl method enabled us to avoid as much tensorial machinery as possible while still keeping all the implications of Einstein's equations. One obvious future application is to the considerably more complicated Kerr solution, the time-independent but rotating (stationary) dipole metric. Indeed, a useful exercise for the interested student would be to derive the linearized version of this geometry!

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