

Prelims Probability

Sheet 4 — MT22

1. Let $\mathbb{P}(X = k) = 1/n$ for $k = 1, 2, \dots, n$. Find the mean and variance of X .
2. Suppose that the discrete random variables X and Y have joint probability mass function given by

	X	-1	0	1
Y				
-1		$\frac{1}{27}$	$\frac{6}{27}$	$\frac{2}{27}$
0		$\frac{2}{27}$	$\frac{6}{27}$	$\frac{1}{27}$
1		$\frac{3}{27}$	$\frac{2}{27}$	$\frac{4}{27}$

Find the marginal distributions of X and Y . What is the covariance of X and Y ? Are X and Y independent?

3. Suppose that X and Y are independent Poisson random variables with parameters λ and μ respectively. Find
 - (a) the joint probability mass function $\mathbb{P}(X = k, Y = m)$;
 - (b) $\mathbb{P}(X + Y = n)$ (what is this distribution?);
 - (c) $\mathbb{P}(X = k | X + Y = n)$ (what is this distribution?);
 - (d) $\mathbb{E}[X | X + Y = n]$.
4. Let X and Y be independent random variables, both with Geometric(p) distribution.
 - (a) Find $\mathbb{P}(X = k | X + Y = n + 1)$, for $k \in \{1, 2, \dots, n\}$.
 - (b) Find the distribution of $\min\{X, Y\}$. [*Hint: consider $\mathbb{P}(\min\{X, Y\} > k$), and see Question 3(a) of sheet 3.*]
5.
 - (a) A set of lecture notes has n pages. The number of typos on each page is a Poisson random variable with parameter λ , and is independent of the number of typos on all other pages. What is the expected number of pages with no typos?
 - (b) When reading the notes, you detect each typo with probability p , independently of detecting others. Let M denote the number of typos on a particular page and let D denote the number that you detect on that page. Write down $\mathbb{P}(D = k | M = m)$. Hence, for each $k \geq 0$, find $\mathbb{P}(D = k)$.

6. Let X and Y be discrete random variables. Show that the following two definitions of independence of X and Y are equivalent:

(i) For all $x, y \in \mathbb{R}$,

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y);$$

(ii) For all $A, B \subseteq \mathbb{R}$,

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B).$$

Show that if X and Y are independent, then also $f(X)$ and $g(Y)$ are independent for any functions $f, g: \mathbb{R} \rightarrow \mathbb{R}$.

7. Solve the following recurrence relations:

1. $u_{n+1} = 3u_n + 2$ with $u_0 = 0$.

2. $u_{n+1} = 2u_n + n$ with $u_0 = 1$.

3. $u_{n+1} - 5u_n + 6u_{n-1} = 2$ with $u_0 = u_1 = 1$.

4. $u_{n+1} - 3u_n + 2u_{n-1} = 1$ with $u_0 = u_1 = 0$.