Lectures 9 and 10: Constrained optimization problems and their optimality conditions

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C6.2/B2: Continuous Optimization

Problems and solutions

$$\text{minimize } f(x) \quad \text{subject to } x \in \Omega \subseteq \mathbb{R}^n.$$

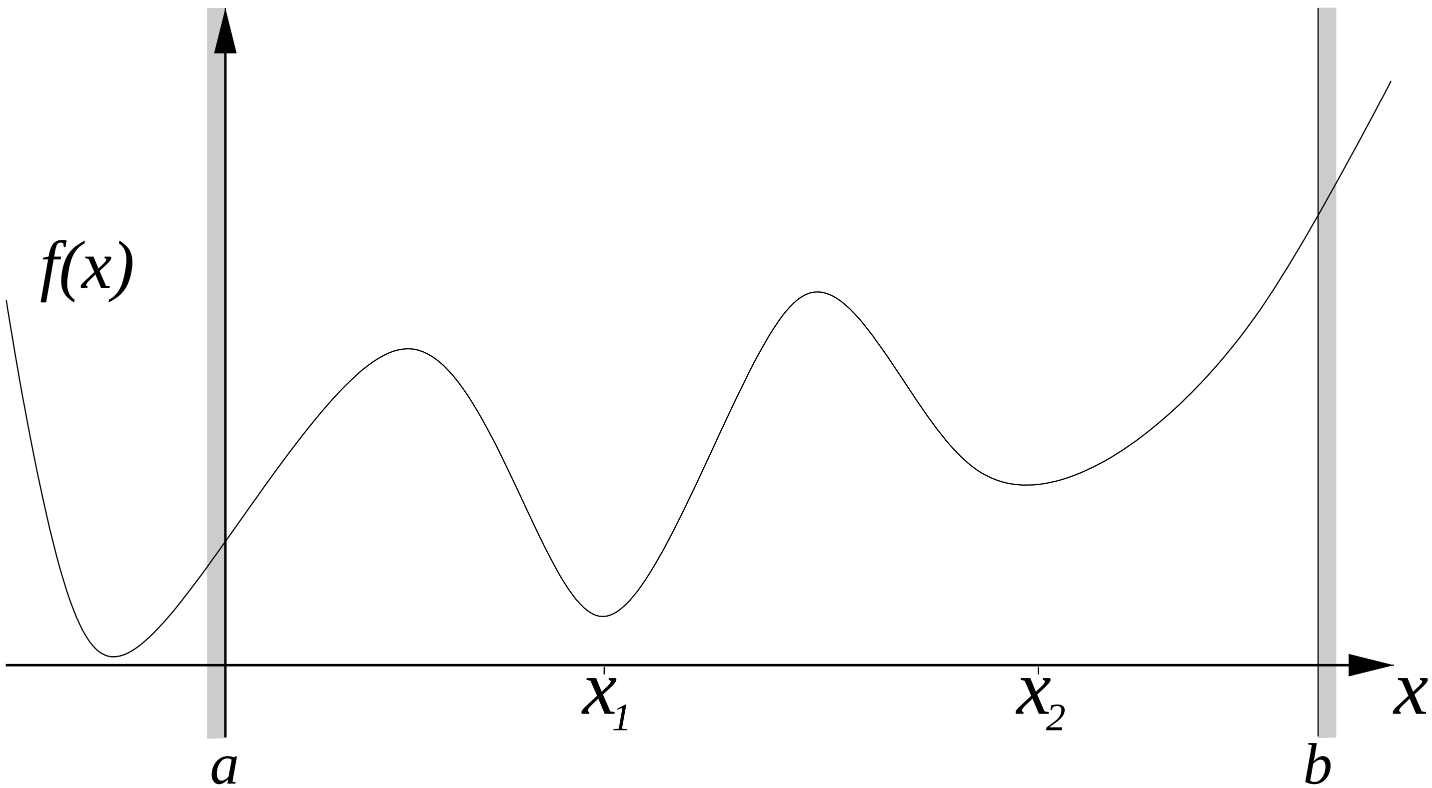
- $f : \Omega \to \mathbb{R}$ is (sufficiently) smooth.
- f objective; x variables.
- Ω feasible set determined by finitely many (equality and/or inequality) constraints.

x^* global minimizer of f over $\Omega \implies f(x) \geq f(x^*), \forall x \in \Omega$.

x^* local minimizer of f over $\Omega \implies$
$\exists N(x^*, \delta)$ such that $f(x) \geq f(x^*)$, for all $x \in \Omega \cap N(x^*, \delta)$.
• $N(x^*, \delta) := \{x \in \mathbb{R}^n : \|x - x^*\| \leq \delta\}$.

Example problem in one dimension

$$\textbf{Example}: \quad \min f(x) \quad \text{subject to} \quad a \le x \le b.$$

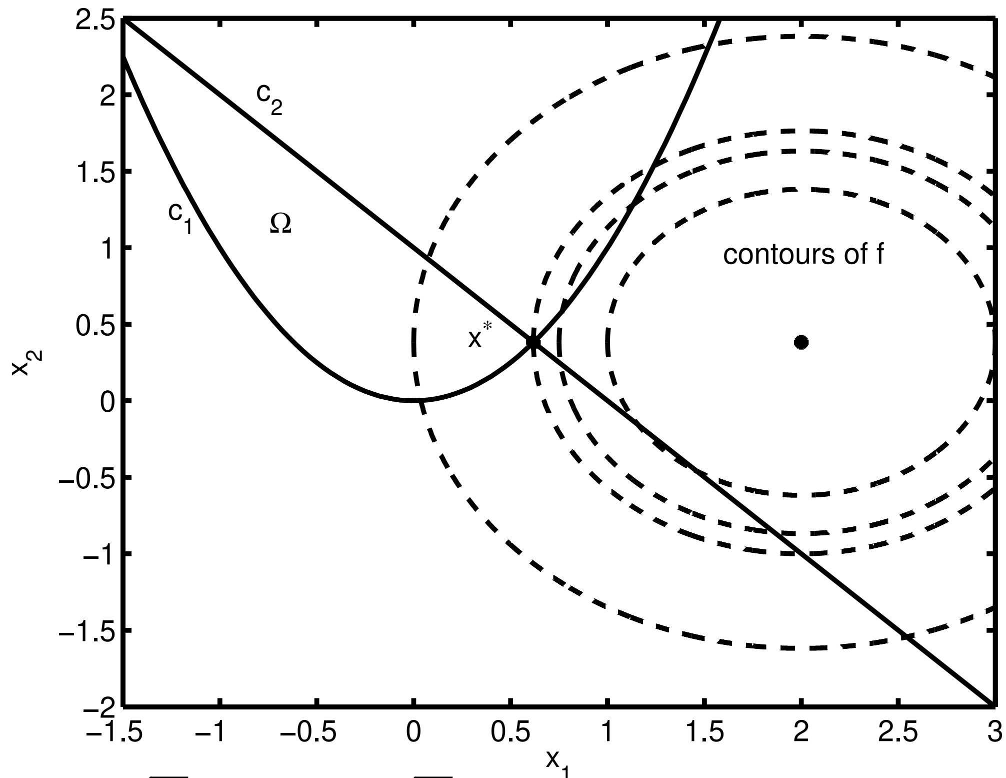


- The feasible region Ω is the interval $[a, b]$.
- The point x_1 is the global minimizer; x_2 is a local (non-global) minimizer; $x = a$ is a constrained local minimizer.

An example of a nonlinear constrained problem

$$\min_{x \in \mathbb{R}^2} (x_1 - 2)^2 + (x_2 - 0.5(3 - \sqrt{5}))^2 \quad \text{subject to}$$

$$-x_1 - x_2 + 1 \geq 0, \;\; x_2 - x_1^2 \geq 0.$$



$x^* = 0.5(-1 + \sqrt{5}, 3 - \sqrt{5})$; Ω feasible set.

Optimality conditions for constrained problems

== algebraic characterizations of solutions $\longrightarrow$ suitable for computations.

- provide a way to guarantee that a candidate point is optimal (sufficient conditions)
- indicate when a point is not optimal (necessary conditions)

$$\text{minimize}_{x\in\mathbb{R}^n} \quad f(x) \quad \text{subject to} \quad c_E(x) = 0, \quad c_I(x) \geq 0. \tag{CP}$$

- $f : \mathbb{R}^n \to \mathbb{R}$, $c_E : \mathbb{R}^n \to \mathbb{R}^m$ and $c_I : \mathbb{R}^n \to \mathbb{R}^p$ (suff.) smooth;

• $c_I(x) \geq 0 \Leftrightarrow c_i(x) \geq 0$, $i \in I$.

• $\Omega := \{x : \ c_E(x) = 0, \ c_I(x) \geq 0\}$ feasible set of the problem.

Optimality conditions for constrained problems

unconstrained problem $\longrightarrow \hat{x}$ stationary point ($\nabla f(\hat{x}) = 0$).

constrained problem $\longrightarrow \hat{x}$ Karush-Kuhn-Tucker (KKT) point.

Definition: $\hat{x}$ KKT point of (CP) if there exist $\hat{y} \in \mathbb{R}^m$ and $\hat{\lambda} \in \mathbb{R}^p$ such that $(\hat{x}, \hat{y}, \hat{\lambda})$ satisfies

$$\nabla f(\hat{x}) = \sum_{j \in E} \hat{y}_j \nabla c_j(\hat{x}) + \sum_{i \in I} \hat{\lambda}_i \nabla c_i(\hat{x}),$$

$$c_E(\hat{x}) = 0, \quad c_I(\hat{x}) \geq 0,$$

$$\hat{\lambda}_i \geq 0, \quad \hat{\lambda}_i c_i(\hat{x}) = 0, \quad \text{for all } i \in I.$$

- Let $\mathcal{A} := E \cup \{i \in I : c_i(\hat{x}) = 0\}$ index set of active constraints at $\hat{x}$; $c_j(\hat{x}) > 0$ inactive constraint at $\hat{x} \Rightarrow \hat{\lambda}_j = 0$. Then $\sum_{i \in I} \hat{\lambda}_i \nabla c_i(\hat{x}) = \sum_{i \in I \cap \mathcal{A}} \hat{\lambda}_i \nabla c_i(\hat{x})$.
- $J(x) = \left(\nabla c_i(x)^T\right)_i$ Jacobian matrix of constraints c. Thus $\sum_{j \in E} \hat{y}_j \nabla c_j(\hat{x}) = J_E(x)^T \hat{y}$ and $\sum_{i \in I} \hat{\lambda}_i \nabla c_i(\hat{x}) = J_I(x)^T \hat{\lambda}$.

Optimality conditions for constrained problems ...

$\hat{x}$ KKT point $\longrightarrow$ $\hat{y}$ and $\hat{\lambda}$ Lagrange multipliers of the equality and inequality constraints, respectively.

$\hat{y}$ and $\hat{\lambda}$ $\longrightarrow$ sensitivity analysis.

$\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ Lagrangian function of (CP),

$$\mathcal{L}(x, y, \lambda) := f(x) - y^\top c_E(x) - \lambda^\top c_I(x), \quad x \in \mathbb{R}^n.$$

Thus $\nabla_x \mathcal{L}(x, y, \lambda) = \nabla f(x) - J_E(x)^\top y - J_I(x)^\top \lambda$,

and $\hat{x}$ KKT point of (CP) $\Longrightarrow$ $\nabla_x \mathcal{L}(\hat{x}, \hat{y}, \hat{\lambda}) = 0$
(i. e., $\hat{x}$ is a stationary point of $\mathcal{L}(\cdot, \hat{y}, \hat{\lambda})$).

- duality theory...

An illustration of the KKT conditions

$$\min_{x\in\mathbb{R}^2} (x_1-2)^2 + (x_2 - 0.5(3-\sqrt{5}))^2 \quad \text{subject to}$$

$$-x_1 - x_2 + 1 \geq 0, \; x_2 - x_1^2 \geq 0. \qquad (*)$$

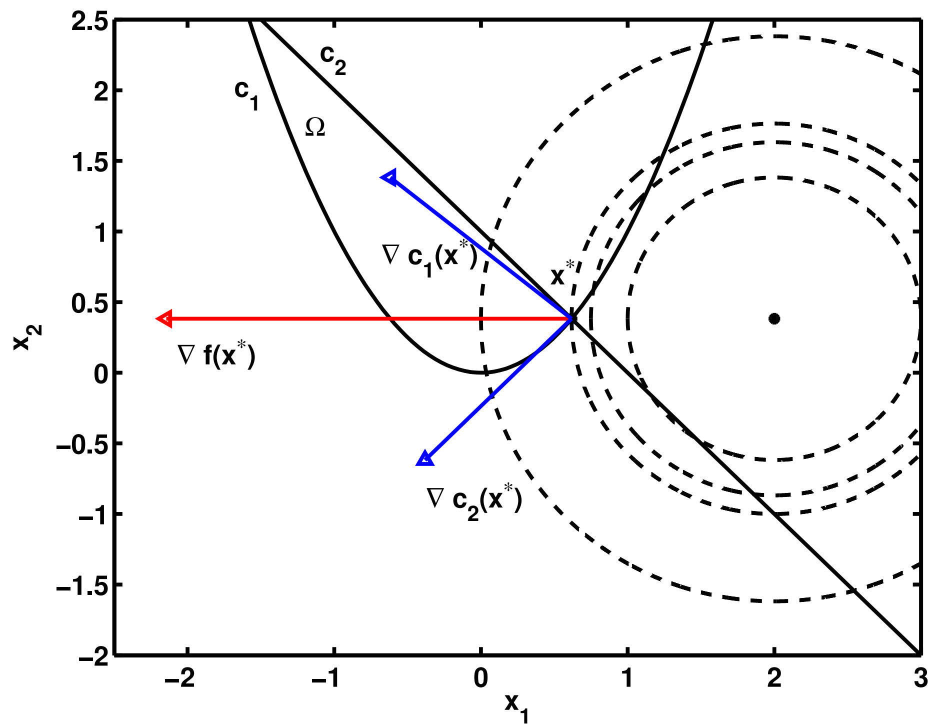
$x^* = \frac{1}{2}(-1+\sqrt{5}, 3-\sqrt{5})^\top$:

- global solution of $(*)$,
- KKT point of $(*)$.

$\nabla f(x^*) = (-5+\sqrt{5}, 0)^\top$,

$\nabla c_1(x^*) = (1-\sqrt{5}, 1)^\top$,

$\nabla c_2(x^*) = (-1, -1)^\top$.



$\nabla f(x^*) = \lambda_1^* \nabla c_1(x^*) + \lambda_2^* \nabla c_2(x^*)$, with $\lambda_1^* = \lambda_2^* = \sqrt{5} - 1 > 0$.
$c_1(x^*) = c_2(x^*) = 0$: constraints are active at x^*.

An illustration of the KKT conditions ...

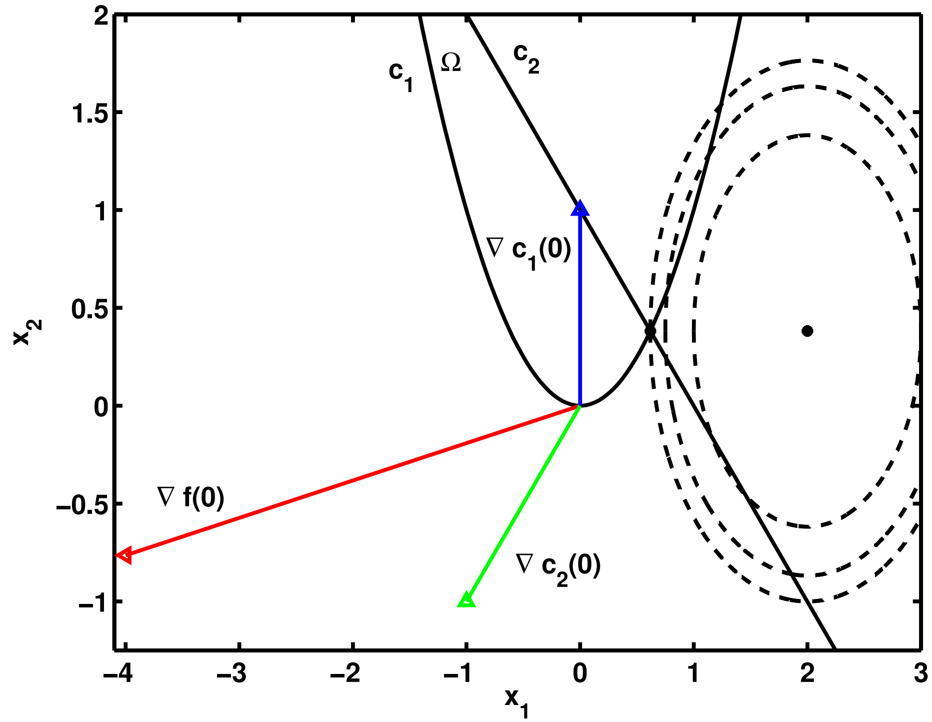
$$\min_{x \in \mathbb{R}^2} (x_1 - 2)^2 + (x_2 - 0.5(3 - \sqrt{5}))^2 \quad \text{subject to}$$

$$-x_1 - x_2 + 1 \geq 0, \;\; x_2 - x_1^2 \geq 0. \qquad (*)$$

$x := (0, 0)^\top$
is NOT a KKT point of $(*)$!

$c_1(x) = 0$: active at x.
$c_2(x) = 1$: inactive at x.
$\Longrightarrow \lambda_2 = 0$ and
$\nabla f(x) = \lambda_1 \nabla c_1(x)$,
with $\lambda_1 \geq 0$.

$\Downarrow$



Contradiction with $\nabla f(x) = (-4, \sqrt{5} - 3)^\top$ and
$\nabla c_1(x) = (0, 1)^\top$.

Optimality conditions for constrained problems ...

In general, need constraints/feasible set of (CP) to satisfy regularity assumption called constraint qualification in order to derive optimality conditions.

Theorem 16 (First order necessary conditions) Under suitable constraint qualifications,

x^* local minimizer of (CP) $\Longrightarrow$ x^* KKT point of (CP).

Proof of Theorem 16 (for equality constraints only): Let $I = \emptyset$ and so we must show that $c_E(x^*) = 0$ (which is trivial as x^* feasible) and $\nabla f(x^*) = J_E(x^*)^T y^*$ for some $y^* \in \mathbb{R}^m$. Consider feasible perturbations/paths $x(\alpha)$ around x^*, where α (small) scalar, $x(\alpha) \in \mathcal{C}^2(\mathbb{R}^n)$ and

$$x(0) = x^* \text{ and } c(x(\alpha)) = 0^{(\dagger)}.$$

($\dagger$) requires constraint qualifications

Then by Taylor expansion, $x(\alpha) = x^* + \alpha s + \frac{1}{2}\alpha^2 p + \mathcal{O}(\alpha^3)^{(\dagger\dagger)}$.

($\dagger\dagger$) [α^2 and higher order terms not needed here; only for 2nd order conditions later]

Optimality conditions for constrained problems ...

Proof of Theorem 16 (for equality constraints only): (continued)
For any $i \in E$, by Taylor's theorem for $c_i(x(\alpha))$ around x^*,

$$\begin{aligned} 0 &= c_i(x(\alpha)) = c_i(x^* + \alpha s + \tfrac{1}{2}\alpha^2 p + \mathcal{O}(\alpha^3)) \\ &= c_i(x^*) + \nabla c_i(x^*)^T(\alpha s + \tfrac{1}{2}\alpha^2 p) + \tfrac{1}{2}\alpha^2 s^T \nabla^2 c_i(x^*) s + \mathcal{O}(\alpha^3) \\ &= \alpha \nabla c_i(x^*)^T s + \tfrac{1}{2}\alpha^2 \left[\nabla c_i(x^*)^T p + s^T \nabla^2 c_i(x^*) s\right]^{(*)} + \mathcal{O}(\alpha^3). \end{aligned}$$

where we used $c_i(x^*) = 0$. Thus for all $i \in E$,

$$\nabla c_i(x^*)^T s = 0 \quad \text{and} \quad \nabla c_i(x^*)^T p + s^T \nabla^2 c_i(x^*) s = 0^{(*)},$$

and so $J_E(x^*)s = 0$. Now expanding f, we deduce

$$\begin{aligned} f(x(\alpha)) &= f(x^*) + \nabla f(x^*)^T(\alpha s + \tfrac{1}{2}\alpha^2 p) + \tfrac{1}{2}\alpha^2 s^T \nabla^2 f(x^*) s + \mathcal{O}(\alpha^3) \\ &= f(x^*) + \alpha \nabla f(x^*)^T s + \tfrac{1}{2}\alpha^2 \left[\nabla f(x^*)^T p + s^T \nabla^2 f(x^*) s\right]^{(*)} + \mathcal{O}(\alpha^3). \end{aligned}$$

$(*)$[these terms are only needed for 2nd order optimality conditions later]

As $x(\alpha)$ feasible, f is unconstrained along $x(\alpha)$ and so $f'(x(0)) = \nabla f(x^*)^T s = 0$ since x^* is a local minimizer along $x(\alpha)$. Thus $\nabla f(x^*)^T s = 0$ for all s such that $J_E(x^*)s = 0^{(1)}$.

Optimality conditions for constrained problems ...

Proof of Theorem 16 (for equality constraints only): (continued)
If we let Z be a basis for the null space of $J_E(x^*)$, we deduce there exists y^* and s^* such that

$$\nabla f(x^*) = J_E(x^*)^T y^* + Z s^*. \qquad (2)$$

From (1), $Z^T \nabla f(x^*) = 0$ and so from (2),

$$0 = Z^T J_E(x^*)^T y^* + Z^T Z s^*,$$

and furthermore, since $J_E(x^*)Z = 0$, we must have $Z^T Z s^* = 0$. As Z is a basis, it is full rank and so $s^* = 0$. We conclude from (2) that $\nabla f(x^*) = J_E(x^*)^T y^*$. $\square$

- Let (CP) with equalities only ($I = \emptyset$). Then feasible descent direction s at $x \in \Omega$ if $\nabla f(x)^T s < 0$ and $J_E(x) s = 0$.
- Let (CP). Then feasible descent direction s at $x \in \Omega$ if $\nabla f(x)^T s < 0$, $J_E(x) s = 0$ and $\nabla c_i(x)^T s \geq 0$ for all $i \in I \cap \mathcal{A}(x)$.

Constraint qualifications

- Proof of Th 16: used (first-order) Taylor to linearize f and c_i along feasible paths/perturbations $x(\alpha)$ etc. Only correct if linearized approximation covers the essential geometry of the feasible set. CQs ensure this is the case.

  Examples:

- (CP) satisfies the Slater Constraint Qualification (SCQ) $\Longleftrightarrow$ if $\exists\, x$ s.t. $c_E(x) = Ax - b = 0$ and $c_I(x) > 0$ (i.e., $c_i(x) > 0$, $i \in I$).
- (CP) satisfies the Linear Independence Constraint Qualification (LICQ) $\Longleftrightarrow$ $\nabla c_i(x)$, $i \in \mathcal{A}(x)$, are linearly independent (at relevant x).

  Both SCQ and LICQ fail for

$$\Omega = \{(x_1, x_2) : c_1(x) = 1 - x_1^2 - (x_2 - 1)^2 \geq 0;\ c_2(x) = -x_2 \geq 0\}.$$

$T_\Omega(x) = \{(0,0)\}$ and $\mathcal{F}(x) = \{(s_1, 0) : s_1 \in \mathbb{R}\}$. Thus $T_\Omega(x) \neq \mathcal{F}(x)$.

Constraint qualifications...

Tangent cone to Ω at x: [See Chapter 12, Nocedal & Wright]

$T_\Omega(x) = \{s : \text{limiting direction of feasible sequence}\}$ ['geometry' of Ω]

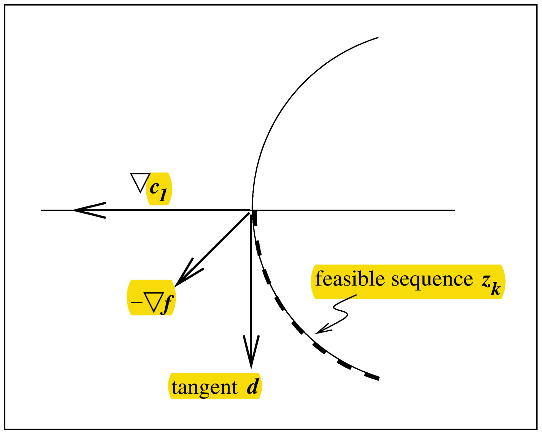
$s = \lim_{k\to\infty} \frac{z^k - x}{t^k}$ where $z^k \in \Omega$, $t^k > 0$, $t^k \to 0$ and $z^k \to x$ as $k \to \infty$.

Set of linearized feasible directions: ['algebra' of Ω]

$\mathcal{F}(x) = \{s : s^T \nabla c_i(x) = 0, i \in E;\ s^T \nabla c_i(x) \geq 0, i \in I \cap \mathcal{A}(x)\}$

Want $T_\Omega(x) = \mathcal{F}(x) \longleftarrow$[ensured if a CQ holds]

$\min_{(x_1,x_2)} x_1 + x_2$
s.t. $x_1^2 + x_2^2 - 2 = 0.$



Optimality conditions for constrained problems ...

If the constraints of (CP) are linear in the variables, no constraint qualification is required.

Theorem 17 (First order necessary conditions for linearly constrained problems) Let $(c_E, c_I)(x) := Ax - b$ in (CP). Then x^* local minimizer of (CP) $\Longrightarrow$ x^* KKT point of (CP).

Let $A = (A_E, A_I)$ and $b = (b_E, b_I)$ corresponding to equality and inequality constraints.
KKT conditions for linearly-constrained (CP): x^* KKT point $\Leftrightarrow$ there exists (y^*, λ^*) such that

$$\nabla f(x^*) = A_E^T y^* + A_I^T \lambda^*,$$

$$A_E x^* - b_E = 0, \quad A_I x^* - b_I \geq 0,$$

$$\lambda^* \geq 0, \quad (\lambda^*)^T (A_I x^* - b_I) = 0.$$

Optimality conditions for convex problems

(CP) is a convex programming problem if and only if $f(x)$ is a convex function, $c_i(x)$ is a concave function for all $i \in I$ and $c_E(x) = Ax - b$.

- c_i is a concave function $\Leftrightarrow$ $(-c_i)$ is a convex function.
- (CP) convex problem $\Rightarrow$ Ω is a convex set.
- (CP) convex problem $\Rightarrow$ any local minimizer of (CP) is global.

First order necessary conditions are also sufficient for optimality when (CP) is convex.

Theorem 18. (Sufficient optimality conditions for convex problems: Let (CP) be a convex programming problem.
$\hat{x}$ KKT point of (CP) $\implies$ $\hat{x}$ is a (global) minimizer of (CP). $\Box$

Optimality conditions for convex problems

Proof of Theorem 18.

f convex $\Longrightarrow f(x) \geq f(\hat{x}) + \nabla f(\hat{x})^\top (x - \hat{x})$, for all $x \in \mathbb{R}^n$. (1)

(1)+$[\nabla f(\hat{x}) = A^\top \hat{y} + \sum_{i \in I} \hat{\lambda}_i \nabla c_i(\hat{x})] \Longrightarrow$

$f(x) \geq f(\hat{x}) + (A^\top \hat{y})^\top (x - \hat{x}) + \sum_{i \in I} \hat{\lambda}_i (\nabla c_i(\hat{x})^\top (x - \hat{x}))$,

$f(x) \geq f(\hat{x}) + \hat{y}^\top A(x - \hat{x}) + \sum_{i \in I} \hat{\lambda}_i (\nabla c_i(\hat{x})^\top (x - \hat{x}))$ (2).

Let $x \in \Omega$ arbitrary $\Longrightarrow Ax = b$ and $c(x) \geq 0$.

$Ax = b$ and $A\hat{x} = b \Longrightarrow A(x - \hat{x}) = 0$. (3)

c_i concave $\Longrightarrow c_i(x) \leq c_i(\hat{x}) + \nabla c_i(\hat{x})^\top (x - \hat{x})$.

$\Longrightarrow \nabla c_i(\hat{x})^\top (x - \hat{x}) \geq c_i(x) - c_i(\hat{x})$.

$\Longrightarrow \hat{\lambda}_i (\nabla c_i(\hat{x})^\top (x - \hat{x})) \geq \hat{\lambda}_i (c_i(x) - c_i(\hat{x})) = \hat{\lambda}_i c_i(x) \geq 0$,

since $\hat{\lambda} \geq 0$, $\hat{\lambda}_i c_i(x) = 0$ and $c(x) \geq 0$.

Thus, from (2), $f(x) \geq f(\hat{x})$. $\square$

Optimality conditions for nonconvex problems

- When (CP) is not convex, the KKT conditions are not in general sufficient for optimality
$\longrightarrow$ need positive definite Hessian of the Lagrangian function along “feasible” directions.

- More on second-order optimality conditions later on.

Example: Optimality conditions for QP problems

A Quadratic Programming (QP) problem has the form

$$\text{minimize}_{x \in \mathbb{R}^n} \; c^\top x + \tfrac{1}{2} x^\top H x \;\; \text{s. t.} \;\; Ax = b, \;\; \tilde{A}x \geq \tilde{b}. \quad \text{(QP)}$$

H symm. pos. semidefinite $\Longrightarrow$ (QP) convex problem.

The KKT conditions for (QP):

$\hat{x}$ KKT point of (QP) $\iff$ $\exists\,(\hat{y}, \hat{\lambda}) \in \mathbb{R}^m \times \mathbb{R}^p$ such that

$$\begin{aligned} &H\hat{x} + c = A^\top \hat{y} + \tilde{A}^\top \hat{\lambda}, \\ &A\hat{x} = b, \;\; \tilde{A}\hat{x} \geq \tilde{b}, \\ &\hat{\lambda} \geq 0, \;\; \hat{\lambda}^\top (\tilde{A}\hat{x} - \tilde{b}) = 0. \end{aligned}$$

- "An example of a nonlinear constrained problem" is convex; removing the constraint $x_2 - x_1^2 \geq 0$ makes it a convex (QP).

Example: Duality theory for QP problems

For simplicity, let $A := 0$ and $H \succ 0$ in (QP): primal problem:

$$\text{minimize}_{x\in\mathbb{R}^n}\ c^\top x + \frac{1}{2}x^\top H x \ \text{ s. t. }\ \tilde{A}x \geq \tilde{b}. \quad \text{(QP)}$$

The KKT conditions for (QP):

$$\begin{aligned} &H\hat{x} + c = \tilde{A}^\top \hat{\lambda}, \\ &\tilde{A}\hat{x} \geq \tilde{b}, \\ &\hat{\lambda} \geq 0,\ \hat{\lambda}^\top(\tilde{A}\hat{x} - \tilde{b}) = 0. \end{aligned}$$

Dual problem:

$$\text{maximize}_{(x,\lambda)} - \frac{1}{2}x^T H x + \tilde{b}^T \lambda \text{ s.t. } - Hx + \tilde{A}^\top \lambda = c \text{ and } \lambda \geq 0.$$

Optimal value of primal pb=optimal value of dual pb (provided they exist).

Second-order optimality conditions

• When (CP) is not convex, the KKT conditions are not in general sufficient for optimality.

- Assume some CQ holds. Then at a given point x^*: the set of feasible directions for (CP) at x^*:

$$\mathcal{F}(x^*) = \left\{ s : J_E(x^*)s = 0,\ s^T \nabla c_i(x^*) \geq 0, i \in \mathcal{A}(x^*) \cap I \right\}.$$

- If x^* is a KKT point, then for any $s \in \mathcal{F}(x^*)$, either

$$s^T \nabla f(x^*) > 0$$

$\longrightarrow$ so f can only increase and stay feasible along s

or $s^T \nabla f(x^*) = 0$

$\longrightarrow$ cannot decide from 1st order info if f increases or not along such s.

$$F(\lambda^*) = \{ s \in \mathcal{F}(x^*) :\ s^T \nabla c_i(x^*) = 0,\ \forall i \in \mathcal{A}(x^*) \cap I \text{ with } \lambda_i^* > 0 \},$$

where λ^* is a Lagrange multiplier of the inequality constraints.
Then note that $s^T \nabla f(x^*) = 0$ for all $s \in F(\lambda^*)$.

Second-order optimality conditions ...

Theorem 19 (Second-order necessary conditions)
Let some CQ hold for (CP). Let x^* be a local minimizer of (CP), and (y^*, λ^*) Lagrange multipliers of the KKT conditions at x^*. Then

$$s^T \nabla^2_{xx} \mathcal{L}(x^*, y^*, \lambda^*) s \geq 0 \text{ for all } s \in F(\lambda^*),$$

where $\mathcal{L}(x, y, \lambda) = f(x) - y^T c_E(x) - \lambda^T c_I(x)$ is the Lagrangian function.

Theorem 20 (Second-order sufficient conditions)
Assume that x^* is a feasible point of (CP) and (y^*, λ^*) are such that the KKT conditions are satisfied by (x^*, y^*, λ^*). If

$$s^T \nabla^2_{xx} \mathcal{L}(x^*, y^*, \lambda^*) s > 0 \text{ for all } s \in F(\lambda^*),\ s \neq 0,$$

then x^* is a local minimizer of (CP).

Second-order optimality conditions ...

Proof of Theorem 19 (for equality constraints only) [**NON-EXAMINABLE**]:
Let $I = \emptyset$ and so $\mathcal{F}(x^*) = F(\lambda^*)$. We have to show that

$$s^T \nabla^2_{xx} \mathcal{L}(x^*, y^*, \lambda^*) s \geq 0 \text{ for all } s \text{ such that } J_E(x^*) s = 0.$$

Recall the proof of Theorem 16: along any feasible path of the form $x(\alpha) = x^* + \alpha s + \frac{1}{2}\alpha^2 p + \mathcal{O}(\alpha^3)$ (for any s and p), we showed that

$$J_E(x^*) s = 0 \text{ and } \quad \nabla c_i(x^*)^T p + s^T \nabla^2 c_i(x^*) s = 0,\ i \in E,$$

and that

$$f(x(\alpha)) = f(x^*) + \tfrac{1}{2}\alpha^2 \left[\nabla f(x^*)^T p + s^T \nabla^2 f(x^*) s\right] + \mathcal{O}(\alpha^3).$$

As x^* is a local minimizer, we must have that

$$\nabla f(x^*)^T p + s^T \nabla^2 f(x^*) s \geq 0. \quad (*)$$

From the KKT conditions, $\nabla f(x^*) = J_E(x^*)^T y^*$ and so

$$\nabla f(x^*)^T p = (y^*)^T J_E(x^*) p = -\textstyle\sum_{i \in E} y_i^* s^T \nabla^2 c_i(x^*) s. \quad (**)$$

Second-order optimality conditions ...

Proof of Theorem 19 (for equality constraints only):(continued)
From (*) and (**), we deduce

$$\begin{aligned} 0 &\leq s^T \nabla^2 f(x^*) s - \sum_{i \in E} y_i^* s^T \nabla^2 c_i(x^*) s \\ &= s^T [\nabla^2 f(x^*) - \sum_{i \in E} \nabla^2 c_i(x^*)] s \\ &= s^T \nabla^2_{xx} \mathcal{L}(x^*, y^*) s. \qquad \square \end{aligned}$$

Some simple approaches for solving (CP)

Equality-constrained problems: direct elimination (a simple approach that may help/work sometimes; cannot be automated in general)

Method of Lagrange multipliers: using the KKT and second order conditions to find minimizers (again, cannot be automated in general)

[see Pb Sheet 4]