C3.10 Additive and Combinatorial Number Theory, Hilary 2020
Sheet 0

This example sheet will not be marked. It provides some exercises on the basic notation for the course, as well as on the Fourier transform.

Question 1. Which of the following statements are true, as $x \to \infty$?

(i) $100x + 1000 = O(x)$;

(ii) $\sqrt{x} = o(x/\log x)$;

(iii) $ax^2 + bx + c \ll_{a,b,c} x^2$;

(iv) $e^{\sqrt{\log x}} = x^{o(1)}$;

(v) $x^{1/\log\log x} = O(e^{(\log x)^{2/3}})$;

(vi) $\log^A x \ll_A x^{1/10}$.

Question 2. Give an expression for the Fourier transform $\hat{f}(\xi)$, where $f : \mathbb{R} \to \mathbb{R}$ is the function taking value 1 for $|x| \leqslant 1$ and 0 otherwise. Show that $|f(\xi)| \ll |\xi|^{-1}$ as $|\xi| \to \infty$.

Question 3. With f as in the previous question, draw a picture of the convolution $g = f * f$. Show that $\int_{-\infty}^{\infty} |\hat{g}(\xi)| d\xi < \infty$.

Question 4. Let $A \subset \mathbb{Z}/q\mathbb{Z}$, and suppose that $|A| = \alpha q$. Let 1_A be the characteristic function of A, that is to say the function taking value 1 on A, and 0 outside A. Show that $|\hat{1}_A(r)| \leqslant \alpha$ for all r. Show additionally that the number of $r \in \mathbb{Z}/q\mathbb{Z}$ with $|\hat{1}_A(r)| \geqslant \eta\alpha$ is at most $\eta^{-2}\alpha^{-1}$.

Question 5. Let $f : \mathbb{Z} \to \mathbb{C}$ be a compactly supported function. Show that $\int_0^1 |\hat{f}(\theta)|^4 d\theta$ is the sum of $f(n_1)f(n_2)\overline{f(n_3)f(n_4)}$ over all quadruples (n_1, n_2, n_3, n_4) with $n_1 + n_2 = n_3 + n_4$.

Question 6. Let q be large, and let $A \subset \mathbb{Z}/q\mathbb{Z}$ be the set $\{1, \dots, \lfloor q/10 \rfloor\}$. Show that $|\hat{1}_A(1)| \geqslant 1/100$.

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