C3.10 Additive and Combinatorial Number Theory, Hilary 2020
Exercises 1

Question 1. Let $p = 5101$ (this is a prime). Starting from the observation that $2p = 101^2 + 1$, find a representation of p as a sum of two squares.

Question 2. Show that if $n = 4^a(8k + 7)$ then n is *not* a sum of three squares.

Question 3. Show that a number is the sum of two squares if and only if all primes that are $3 \pmod 4$ occur in the prime factorisation of n with an even exponent.

Question 4. As in lectures, write $r_{k,s}(N)$ for the number of representations of N as a sum of s kth powers. Prove from first principles that there are positive constants c, C (depending on k, s but not on X) such that for sufficiently large X we have
$$cX^{s/k} \leqslant \sum_{N \leqslant X} r_{k,s}(N) \leqslant CX^{s/k}.$$

Explain why it follows that $G(k) \geqslant k$. Can one have $G(k) = k$ for $k \geqslant 2$?

Question 5. Show that every positive integer is the sum of at most 53 fourth powers. *Hint: you may wish to consider the identity*

$$6(\sum_{i=1}^{4} a_i^2)^2 = \sum_{1 \leqslant i < j \leqslant 4} (a_i \pm a_j)^4,$$

where there are twelve terms on the right corresponding to the possible choices of signs $\pm$.

Question 6.   (i) Show that $G(4) \geqslant 15$.

 (ii) By considering numbers of the form $31 \cdot 2^{4k}$, or otherwise, show that $G(4) \geqslant 16$.

Question 7. Show that every integer is the sum of five cubes of integers (not necessarily positive).

Question 8. Does $r_{2,4}(N) \to \infty$ as $N \to \infty$?

Question 9. Show that every $n \leqslant 3^{100}$ is the sum of at most 20 cubes. (You should be able to do this with just a pocket calculator.)

ben.green@maths.ox.ac.uk