C3.10 Additive and Combinatorial Number Theory, Hilary 2020
Exercises 3

Q 1. Let $S \subset \mathbb{Z}$ be a finite set, and let $a_1, \dots, a_k$ be nonzero integers. Give a formula for the number of tuples $(x_1, \dots, x_k) \in S$ such that $a_1 x_1 + \cdots + a_k x_k = 0$ in terms of the Fourier transform $\hat{1}_S$.

Q 2. By considering points on spheres, or otherwise, show that there is a set $A \subset [N]^d$, free of 3-term arithmetic progressions, with $|A| \geqslant \frac{1}{d} N^{d-2}$.

By considering maps of the form $(x_1, \dots, x_d) \to x_1 + L x_2 + \cdots + L^{d-1} x_d$ for appropriate M, show that if $\varepsilon > 0$, then for infinitely many values of M there is a set $A \subset [M]$, free of 3-term progressions, with $|A| \geqslant M^{1-\varepsilon}$.

Q 3. Let $G = \mathbb{F}_2^n$, and let $e_1, \dots, e_n$ be basis vectors for this vector space. Let Σ_r be the Hamming ball of radius r, that is to say the set of sums of at most r of the basis vectors e_i.

Show that G may be covered by at most

$$\frac{2^n}{1 + \binom{n}{1} + \cdots + \binom{n}{\lfloor r/2 \rfloor}}$$

translates of Σ_r.

Q 4. Suppose that $A \subset \mathbb{F}_2^n$ has doubling constant at most K. Show that $3A \subset 2A + X$, where $|X| \leqslant K^4$. Deduce that the subspace generated by A has size at most $K 2^{K^4} |A|$.

Q 5. Give an example of a set A of integers with $|A + A| < |A - A|$. Is it true that there is an absolute constant K such that $|A - A| \leqslant K |A + A|$ for all sets of integers?

Q 6. Suppose that A is a finite set of integers with $|2A| \geqslant 100|A|$. Is it true that $|3A| \geqslant 1000|A|$?

Q 7. By considering sets of the form

$$A = [N]^3 \cup \{(n, 0, 0), (0, n, 0), (0, 0, n) : n \leqslant LN\},$$

for appropriate L, show that for all K there are arbitrarily large sets $A \subset \mathbb{Z}^3$ with $|2A| \leqslant K|A|$ and $|3A| \geqslant \frac{1}{100} K^3 |A|$.

Q 8. Let p be a prime, and suppose that $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ are nonempty sets containing 0.

(i) For $e \in \mathbb{Z}/p\mathbb{Z}$, write $A_e := A \cup (B + e)$ and $B_e := B \cap (A - e)$. Show that $|A_e| + |B_e| = |A| + |B|$ and that $A_e + B_e \subseteq A + B$.

(ii) Suppose that $A \neq \mathbb{Z}/p\mathbb{Z}$ and that $|B| > 1$. Show that there is some $e \in A$ such that $|A_e| > |A|$.

(iii) Prove the Cauchy-Davenport Theorem: if $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ are nonempty then $|A + B| \geqslant \min(|A| + |B| - 1, p)$.

Q 9. Let $\alpha > 0$, and let N be sufficiently large in terms of α. Show that there is a constant $\alpha' > 0$ such that the following is true: if $A \subseteq [N]$ is a set of size αN, then A contains at least $\alpha' N^2$ 3-term APs.

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