C3.10 Additive and Combinatorial Number Theory, Hilary 2020

Exercises 4

Comment. This aim of this sheet is to prove the following result of Furstenberg and Sárközy: if A is a set of integers of positive density, then A contains two elements differing by a square. I have divided the proof of the theorem up into exercises which all have something to do with other parts of the course, and which can hopefully be attempted more-or-less independently of one another. As the main purpose of this sheet is to practice technique, there is some redundancy.

There is some commentary and definitions in addition to the questions. Therefore, to avoid confusion, the end of each question is denoted with a box.

The first set of questions concern the following theorem.

Theorem 1: uniformly for all real numbers θ, we have $\inf_{1 \leqslant n \leqslant N} \|n^2\theta\|_{\mathbb{R}/\mathbb{Z}} \leqslant \varepsilon(N)$, where $\varepsilon(N) \to 0$ as $N \to \infty$.

Question 1. By considering sets of the form $\{n : \frac{n^2\theta}{2} \pmod 1 \in I\}$ for appropriate intervals I, deduce Theorem 1 from Roth's theorem. $\square$

If $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is an integrable function, we define the Fourier transform

$$\hat{f}(m) := \int_0^1 f(\theta)e(-\theta m).$$

A version of the inversion formula states that if $\sum_m |\hat{f}(m)| < \infty$ then

$$f(\theta) = \sum_m \hat{f}(m)e(\theta m).$$

In the next three questions, $\varepsilon > 0$ is fixed and the dependence of implied constants on ε is not indicated.

Question 2. By considering the convolution of $1_{[-\varepsilon/2,\varepsilon/2]}$ with itself, show that there is a function $\psi : \mathbb{R}/\mathbb{Z} \to [0, \infty)$ with the following properties:

- ψ is supported on $[-\varepsilon, \varepsilon]$;
- $\hat{\psi}(0) = \int \psi = 1$;
- $\sum_{m\in\mathbb{Z}} |\hat{\psi}(m)| < \infty$. $\square$

Question 3. With ψ as in Q2, prove the *inversion formula*

$$\psi(\theta) = \sum_m \hat{\psi}(m)e(\theta m).$$

(*You may use the uniqueness property of Fourier series in the following form: if $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is continuous and if $\hat{f}(n) = 0$ for all n, then f is identically zero.*) Deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Question 4. Suppose that there is no $n \leqslant N$ such that $\|n^2\theta\|_{\mathbb{R}/\mathbb{Z}} \leqslant \varepsilon$.

(i) Using the result of Question 2, or otherwise, show that there is some $m = O(1)$, $m \neq 0$, such that

$$|\sum_{n \leqslant N} e(m\theta n^2)| \gg N.$$

(the implied constants here should be uniform in θ).

(ii) Using an appropriate result from the course, show that there is some nonzero $q = O(1)$ such that $\|q\theta\|_{\mathbb{R}/\mathbb{Z}} \ll N^{-2}$.

(iii) Give a second proof of Theorem 1. □

Question 5. Sketch a proof of the following result. There is a function $\omega(N)$, $\omega(N) \to \infty$ as $N \to \infty$, with the following property. There is a partition $[N] = P_1 \cup \cdots \cup P_m$ into progressions with square common difference, with $|P_i| \geqslant \omega(N)$ for all i, and such that $\mathrm{diam}_{P_i}(e(\theta \cdot)) \leqslant \omega(N)^{-1}$ for all i. □

Given two functions $f_1, f_2 : [N] \to \mathbb{R}$, define

$$T(f_1, f_2) := \sum_{x,d} f_1(x) f_2(x+d) 1_X(d),$$

where $X = \{n^2 : n \leqslant N^{1/2}\}$ (as in the course, specialised to $k = 2$).

Question 6. Write an expression for $T(f_1, f_2)$ in terms of the Fourier transforms of f_1, f_2 and 1_X. □

Write $f_A = 1_A - \alpha 1_{[N]}$ for the balanced function of A.

Question 7. Suppose that A does not have any pair of elements differing by a square. Show that there are two 1-bounded[1] functions g_1, g_2, at least one of which is f_A, such that $|T(g_1, g_2)| \gg \alpha^2 N^{3/2}$. □

Question 8. Using any results from the course that you like, explain why there is a positive integer s such that

$$\int_0^1 |\hat{1}_X(\theta)|^{2s} d\theta \ll N^{s-1}.$$

□

Question 9. Suppose that $g_1, g_2 : [N] \to \mathbb{R}$ are two 1-bounded functions. Suppose that $T(g_1, g_2) \geqslant \delta N^{3/2}$. Show that for $i = 1, 2$ we have $\sup_\theta |\hat{g}_i(\theta)| \gg_\delta N$. *Hint:* you probably want to recall Questions 6 and 8, and you may wish to use Hölder's inequality, which states that

$$\int_0^1 \prod_{i=1}^t \phi_i(\theta) d\theta \leqslant \prod_{i=1}^t \Big(\int_0^1 |\phi_i(\theta)|^{p_i} \Big)^{1/p_i}$$

whenever $p_1, \dots, p_t > 1$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_t} = 1$. □

[1] That is, bounded pointwise by 1.

Question 10. Outline how a complete proof of the Furstenberg-Sárközy theorem follows by assembling the above ingredients. *Hint:* modify the proof of Roth's theorem. □

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