Geometric Group Theory

Problem Sheet 4

We use the notation from Lecture Notes, $X \sim Y$, for two metric spaces that are quasi-isometric.

1. i) Show that the relation of quasi-isometry of metric spaces $\sim$ is an equivalence relation.

ii) Let S_1, S_2 be finite generating sets of a group G. Show that $\Gamma(S_1, G) \sim \Gamma(S_2, G)$.

2. *Given $\epsilon, \delta > 0$ a subset N of a metric space X is called an (ϵ, δ)*-net *(or simply a* net*) if for every $x \in X$ there is some $n \in N$ such that $d(x, n) \leq \epsilon$ and for every $n_1, n_2 \in N$, $d(n_1, n_2) \geq \delta$.*

A set N that satisfies only the second condition (i.e. for every $n_1, n_2 \in N$, $d(n_1, n_2) \geq \delta$) is called δ-separated.

i) Show that any metric space X has a $(1, 1)$-net.

ii) Show that if $N \subset X$ is a net then $X \sim N$.

iii) Show that $X \sim Y$ if and only if there are nets $N_1 \subset X, N_2 \subset Y$ and a bilipschitz map $f : N_1 \to N_2$.

iv) Let G be a f.g. group. Show that $H < G$ is a net in G if and only if H is a finite index subgroup of G.

3. Prove that for every $K \geq 1$ and $A \geq 0$ there exists $\lambda \geq 1$, $\mu \geq 0$ and $D \geq 0$ such that the following is true. Given a (K, A)-quasi-geodesic $q : I \to X$ of endpoints x, y in a geodesic metric space X there exists a (continuous) path $\alpha : I' \to X$ of endpoints x, y such that:

1. for all $t, s \in I$,
$$\textit{length}\,(\alpha([t, s])) \leq \lambda d(\alpha(t), \alpha(s)) + \mu;$$
2. for every $x \in I$, $d(q(x), \alpha(I')) \leq D$;
3. for every $t \in I'$, $d(\alpha(t), q(I)) \leq D$.

4. Let X be a δ-hyperbolic geodesic metric space. If L is a geodesic in X and $a \in X$ we say that $b \in L$ is a projection of a to L if

$$d(a, b) = \inf\{d(a, x) : x \in L\}.$$

Show that if b_1, b_2 are projections of a to L then $d(b_1, b_2) \leq 2\delta$.

5. Let X be a geodesic metric space.

If $\Delta = [x, y, z]$ is a geodesic triangle in X, then there is a metric tree (a 'tripod' if Δ is not degenerate) T_Δ with vertices x', y', z' (the endpoints when T_Δ is not a segment) such that there is an onto map $f_\Delta : \Delta \to T_\Delta$ that restricts to an isometry from each side $[x, y], [y, z], [x, z]$ to the corresponding segments $[x', y'], [y', z'], [x', z']$ in the tree. We denote by c_Δ the point $[x', y'] \cap [y', z'] \cap [x', z']$ of T_Δ.

We say that a geodesic triangle $\Delta = [x, y, z]$ in a geodesic metric space is δ-*thin* if for every $t \in T_\Delta = [x', y', z']$, $diam(f_\Delta^{-1}(t)) \leq \delta$.

Prove that the following are equivalent:

1. There is a $\delta \geq 0$ such that all geodesic triangles in X are δ-slim.
2. There is a $\delta' \geq 0$ such that all geodesic triangles in X are δ'-thin.

6. Let $G = \langle S \rangle$ be δ-hyperbolic for some $\delta \in \mathbb{N}$, $\delta \geq 1$.

1. Assume that for some $g \in G, x \in \Gamma(S, G)$ with $d(x, gx) > 100\delta$ we have that $d(x, g^2x) \geq 2d(x, gx) - 12\delta$.

   Prove that
   $$d(x, g^n x) \geq nd(x, gx) - 16n\delta$$
   for all $n \in \mathbb{N}$.

2. Assume that g is an element of infinite order in G. Prove that there are constants $c > 0, d \geq 0$ such that
   $$d(1, g^n) \geq cn - d$$
   for all $n \in \mathbb{N}$.

3. Show that G has no subgroup isomorphic to $< x, t | txt^{-1} = x^2 >$.

7. Let $G =< S|R >$ be a Dehn presentation of a of a δ-hyperbolic group. Show that we can decide whether a word w on S represents an infinite order element.

8. Let $G =< S|R >$ be a Dehn presentation of a δ-hyperbolic group. Show that we can decide whether a word w on S lies in the subgroup $< v >$.